Laboratory Work #8

FOURIER ANALYSIS AND SPECTRAL ANALYSIS OF DETERMINISTIC SIGNALS

1. Goal of the lab:

The goal of the lab is to develop an intuitive feel and an appreciation for the use of spectral density in identification.

2. Introduction

FOURIER SERIES

A continuous-time periodic signal (function) with fundamental period $Tp \ = \frac{1}{F0}$ is expressed as a (linear) weighted combination of (positive and negative) harmonics.

$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n F_0 t}$ (Synthesis equation)

This equation is the Fourier series expansion of $x(t)$ in terms of complex sinusoids that have the same period (not fundamental).

The coefficient c_n in the Synthesis equation is calculated as

$c_n = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi n F_0 t} dt$ (Analysis equation)

The result is predominantly useful in theoretical analysis of signals and systems.

FOURIER TRANSFORM

Aperiodic signals can be treated as a limiting case of periodic signal with period $Tp \to \infty$. Then, the spacing of harmonics on the frequency axis $\Delta F \ = \frac{1}{Tp} \ \to \ 0$. The family of basis functions then belongs to the continuous-domain.

The synthesis equation takes the integral form

$x(t) = \int_{-\infty}^{\infty} X(F) \, e^{j2\pi F t} \, dF$ (Fourier Synthesis)

Its dual, the analysis equation is now evaluated over the entire time axis,

$X(F) \ \triangleq \ \mathcal{F} \, \{x(t)\} = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi F t} \, dt$ (Fourier Transform)

The quantity $X \ (F)$ has a similar interpretation as that of c_n in the Analysis equation. It is a complex-valued continuous function of the frequency F.

SPECTRUM AND SPECTRAL DENSITY

A vast variety of signals that we encounter are either finite-energy aperiodic or stochastic (or mixed) signals, which are characterized by energy and power spectral density, respectively. However, the practical situation is that we have a finite-length signal $x_N = \{x[0], x[1], \ldots, x[N-1]\}$. Computing the N-point DFT amounts to treating the underlying infinitely long signal $\tilde{x}[k]$ as periodic with period N. Thus, strictly speaking we obtain a power spectrum (line spectrum) regardless of the nature of underlying signal.

In order to be able to use the spectrum as a tool for estimating spectral density of finite-length observations of random signals and to bring the deterministic and stochastic signals into a single framework of estimation, an artificial spectral density known as the periodogram is constructed as follows.

The power spectrum $P_{xx}(f_n)$ for the finite-length signal $\{x[k]\}_{k=0}^{N-1}$ is obtained by recalling the definition of power for a periodic signal

$$\Rightarrow P_{xx}(f_n) = |c_n|^2 = \frac{|X[n]|^2}{N^2}$$

Subsequently, an **empirical power spectral density** (power per unit cyclic frequency) for the finite-length sequence is introduced as,

$$P_{xx}(f_n) \triangleq PSD\ (f_n) = \frac{P_{xx}(f_n)}{\Delta f} = N|c^2| = \frac{|X[n]|^2}{N}$$

where the units of frequency is, as usual, in cycles/sample. The quantity $P_{xx}(f_n)$ is known as the periodogram, first introduced by Schuster (1897).

3. Tasks

3.1. Estimating signal with Fourier transform.

The series of interest consists of N = 500 samples of a simulated process

$$v[k] = \sin(0.2\pi k) + \sin(0.5\pi k) + e[k]$$

where $e[k]$ is the GWN process with variance $\sigma_e^2 = 2.25$.

a. Firstly, we need to generate the measurement. As the series of interest consists of 500 samples, define the discrete time series (**k**) as a vertical vector of *(0:499)*.
b. Generate the output signal using the equation above by representing the GWN process as normally distributed random numbers with the variance of $\sigma_e^2 = 2.25$. (***randn***)
c. Plot the generated output signal:
   Label for X: *'Series'*
   Label for Y: *'Amplitude'*
   Title: *'Generated signal'*

d. Compute periodogram (Use ***periodogram*** command). Specify the input and output parameters as below:
Outputs:
- Distribution of power per unit frequency: P_{xx}
- Vector of normalized frequencies: *W*

Inputs:
- Detrended output series: '*output – mean (output)' or use **detrend** command.*
- Window: *Default window ([ ])*
- The number of FFT points: *the length of simulated output series (N)*
- Sampling frequency: *Fs = 1 Hz*

e. Plot the periodogram output using the vector of normalized frequencies (W) and normalized PSD ($\frac{P_{xx}}{sum(P_{xx})}$)
Title: *'Periodogram'*
Label for X: *'Frequency (cycles/sample)'*
Label for Y: *'Spectral density'*

f. Take Fourier Transform of simulated output series (Use ***fft*** command).
g. Split the magnitude and phase part of the FFT result using ***abs*** and ***phase*** commands (**Hint:** Use the half of the data).
h. Zero out the contributions (assumed to be) due to noise.
i. Estimate the signal using the inverse Fourier transform function (***ifft*** command).
j. Plot the estimated signal against the true deterministic signal (the first 100 data). The true signal will be the simulated process represented above (part a.) removing the noise part.
Label for X: *'Series'*
Label for Y: *'Amplitude'*
Legend: *'True','Estimated'*

3.2. Repeat the procedure using different value of the $e[k]$ variance.

4. Questions

4.1. What is the basic difference between a Fourier series and a Fourier transform?

4.2. How is the Fourier transform useful in arriving at the frequency response functions for LTI systems?

4.3. Define periodogram and explain the reasons for introducing it.