Laboratory Work #7

MODELS FOR LINEAR STATIONARY PROCESSES

1. Goal of the lab:

The goal of the example is to estimate a moving average model and a parametric AR model for a simulated series. The idea is to illustrate the basic procedure.

2. Introduction

MOVING AVERAGE MODELS

The moving average representation of order M has the following form:

$$v[k] = \sum_{n=1}^{M} h_n e[k-n] + e[k]$$

which has a striking resemblance to the FIR representation. The MA(M) representation assumes that the process evolves as a linear combination of M past shock waves plus an inevitable uncertainty.

The corresponding transfer function (operator) representation is:

$$H^{(MA)}(q^{-1}) = 1 + \sum_{n=1}^{M} h_n q^{-n}$$

$$H^{(MA)}(z^{-1}) = 1 + \sum_{n=1}^{M} h_n z^{-n}$$

Just as FIR models represent stable processes as long as the coefficients of representation are finite, finite-order MA models always represent stationary processes as long as $|h_n| < \infty, \forall n$. On the other hand, the infinite-order MA process is stationary only when the sequence $\{h[n]\}$ is absolutely convergent.

AUTO-REGRESSIVE MODELS

The term regression is borrowed from health sciences where it is usually used to describe the influence of the past experiences on the present behavior (of a patient). The AR representation emerges whenever the best predictor for a process is a linear combination of its past.

$v[k] = \sum_{j=1}^{P}(-d_j)v[k-j] + e[k]$ or $(1 + \sum_{j=1}^{P} d_j\, q^{-j})\, v[k] = e[k]$

Observe that the uncertainty term represented by the white-noise signal $e[k]$ is indispensable to the model.

It is easy to show that any AR description is essentially the equivalent of a stationary process description under a parametrization of the IR coefficients $\{h[.]\}$.

The transfer function operator and the transfer function are given by

$$H^{(AR)}(q^{-1}) = \frac{1}{1 + \sum_{j=1}^{P} d_j q^{-j}}$$

$$H^{(AR)}(z^{-1}) = \frac{1}{1 + \sum_{j=1}^{P} d_j z^{-j}}$$

3. Tasks

3.1. Fitting an MA model to a simulated series:

a. Generate a random white-noise input which consists of 2000 (**2000** x **1**) samples (Use ***randn*** command).

b. Plot the white-noise process. **Title:** *'White-noise input'*.

c. Set up / initiate the coefficients for MA (2) model:
   a = 1 – *coefficients for the output*
   b = [1 1.3 0.4] – *coefficients for the input*

d. Find the output signal when the input signal (white noise) is filtered with the coefficients (MA model) defined above. (Use ***filter*** command)

e. Plot the output signal. **Title:** *'White-noise input, filtered with the MA model'*

f. Find and plot the ACF of the output signal using ***autocorr*** command.

   Number of lags = 20

   numMA - the number of lags beyond which the theoretical ACF is deemed to have died out = 1

   What is the order (M) of the MA process according to the ACF representation?

g. Estimate the MA(M) model for the generated output signal in **d)** using the coefficients below:

   $na = 0$

   $nc = M$

   Use the MATLAB command ***armax***.

h. Present the MA(M) model (Use ***present*** command).

i. Illustrate the ACF of residuals based on the estimated MA(M) model and computed output signal (part **d.**)

j. Make a conclusion about your results. Determine if the selected order for the MA model is satisfactory based on its ACF representation.

   What can be said about the residuals of the model?

3.2. Fitting an ARIMA model to a simulated series:

a. Generate the random signal with 1000 (**1000** x **1**) samples (Use ***randn*** command)

b. Based on the generated random signal (input), create an output using the ***filter*** command and coefficients below:
   a = [1 -1.51 0.5238] – *coefficients for the output*

   b = [1 0.3] – *coefficients for the input*

c. Plot the generated output series. **Title:** *'White-noise input, filtered with ARIMA model'*

d. Plot the ACF and PACF of the output data using the ***autocorr*** and ***parcorr*** commands, respectively. Provide the respective titles for each plot.
   Number of lags = 20.
   numMA - *the number of lags beyond which the theoretical ACF is deemed to have died out = 1*

   Determine the order (P) of the AR process according to the correlation function plots.

e. Estimate an AR(P) model with the Least Squares method for the original series (Use ***ar*** command):

   Order = P

   Approach = 'ls' (Least Squares)

   Estimate an AR(P) model for the signal using ***armax*** command and assigning proper coefficients.

f. Present both models and observe if there is any difference in their representation or accuracy (Use ***present*** command).

g. Illustrate the ACF of residuals based on the estimated AR (P) model and original signal computed in part **b.**

h. Find the differenced series of the original data series generated in part **b**. (Use ***diff*** command)

i. Plot the differential series. **Title:** *'Series difference'*.

j. Estimate an ARMA (1,1) model for the differenced series (Use ***armax*** command).

k. Present the ARMA(1,1) model (Use ***present*** command).

l. Show the ACF of residuals based on the estimated ARMA (1,1) model and differential series created in part **h**.

   Title: *'Residuals of ARMA (1,1) model for the differenced series'*

m. Plot the ACF and PACF of the differential series using the ***autocorr*** and ***parcorr*** commands respectively.

   Number of lags = 20.

   numMA **-** *the number of lags beyond which the theoretical ACF is deemed to have died out = 1*

   Determine the order (P) of the AR model and the order (M) for MA model according to the correlation function plots.

n. Estimate an AR(P) model for the differenced series. (Use ***ar*** command)

o. Estimate the MA (M) model for the differenced series. (Use ***armax*** command)

p. Present the AR(P) and MA (M) models (Use ***present*** command).

q. Show the ACF of residuals based on the estimated AR(P) and MA(M) models and differenced series created in part **h**.

   Title: *'Residuals of AR(P) model for the differenced series'.*

   Title: *'Residuals of MA(M) model for the differenced series'.*

Make conclusion about the results. Are the obtained models satisfactory in respect of reliability of parameter estimates as well as white prediction errors? Justify your answer with the generated models and plotted figures.

4. Questions

4.1. Describe a general moving average representation and its underlying philosophy.

4.2. Write the general auto-regressive model and conceptually explain the representation.

4.3. Write the difference equation representation for the AR (2) process:

$$\boldsymbol{H(q^{-1}) = \frac{1}{1 - 1.3q^{-1} + 0.4q^{-2}}}$$

Write the ACVF equations using the Yule-Walker equations for this process and find the theoretical ACFs ($\rho[0], \rho[1], \rho[2]$) for a given process.

4.4. The MA (3) process is represented by

$$v[k] = e[k] + 0.8\, e[k-1] + 0.15\, e[k-2] + 0.45\, e[k-3].$$

Use the ACVF generating function ($\boldsymbol{g_\sigma}\,(\boldsymbol{z})$) to arrive at the theoretical expressions for the ACVF of the MA (3) process.