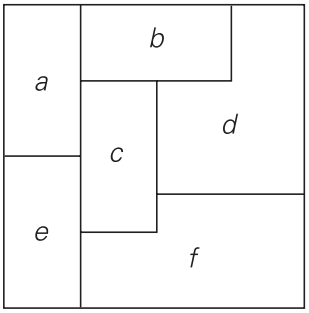
a. The problem's statement is somewhat vague, which is typical of real-life problems. In particular, what reasonable criterion can be used for defining the "best" route?

b. How would you model this problem by a graph?

7. a. Rephrase the traveling-salesman problem in combinatorial object terms.

b. Rephrase the graph-coloring problem in combinatorial object terms.

8. Consider the following map:



a. Explain how we can use the graph-coloring problem to color the map so that no two neighboring regions are colored the same.

b. Use your answer to part (a) to color the map with the smallest number of colors.

9. Design an algorithm for the following problem: Given a set of n points in the Cartesian plane, determine whether all of them lie on the same circumference.

10. Write a program that reads as its inputs the (x, y) coordinates of the endpoints of two line segments P_1Q_1 and P_2Q_2 and determines whether the segments have a common point.

1.4 Fundamental Data Structures

Since the vast majority of algorithms of interest operate on data, particular ways of organizing data play a critical role in the design and analysis of algorithms. A ***data structure*** can be defined as a particular scheme of organizing related data items. The nature of the data items is dictated by the problem at hand; they can range from elementary data types (e.g., integers or characters) to data structures (e.g., a one-dimensional array of one-dimensional arrays is often used for implementing matrices). There are a few data structures that have proved to be particularly important for computer algorithms. Since you are undoubtedly familiar with most if not all of them, just a quick review is provided here.

Linear Data Structures

The two most important elementary data structures are the array and the linked list. A (one-dimensional) ***array*** is a sequence of n items of the same data type that

are stored contiguously in computer memory and made accessible by specifying a value of the array's ***index*** (Figure 1.3).

In the majority of cases, the index is an integer either between 0 and $n - 1$ (as shown in Figure 1.3) or between 1 and n. Some computer languages allow an array index to range between any two integer bounds low and $high$, and some even permit nonnumerical indices to specify, for example, data items corresponding to the 12 months of the year by the month names.

Each and every element of an array can be accessed in the same constant amount of time regardless of where in the array the element in question is located. This feature positively distinguishes arrays from linked lists, discussed below.

Arrays are used for implementing a variety of other data structures. Prominent among them is the ***string***, a sequence of characters from an alphabet terminated by a special character indicating the string's end. Strings composed of zeros and ones are called ***binary strings*** or ***bit strings***. Strings are indispensable for processing textual data, defining computer languages and compiling programs written in them, and studying abstract computational models. Operations we usually perform on strings differ from those we typically perform on other arrays (say, arrays of numbers). They include computing the string length, comparing two strings to determine which one precedes the other in ***lexicographic*** (i.e., alphabetical) ***order***, and concatenating two strings (forming one string from two given strings by appending the second to the end of the first).

A ***linked list*** is a sequence of zero or more elements called ***nodes***, each containing two kinds of information: some data and one or more links called ***pointers*** to other nodes of the linked list. (A special pointer called "null" is used to indicate the absence of a node's successor.) In a ***singly linked list***, each node except the last one contains a single pointer to the next element (Figure 1.4).

To access a particular node of a linked list, one starts with the list's first node and traverses the pointer chain until the particular node is reached. Thus, the time needed to access an element of a singly linked list, unlike that of an array, depends on where in the list the element is located. On the positive side, linked lists do

FIGURE 1.3 Array of n elements.

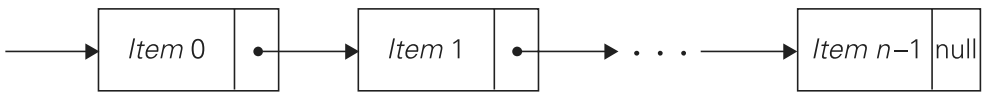


FIGURE 1.4 Singly linked list of n elements.

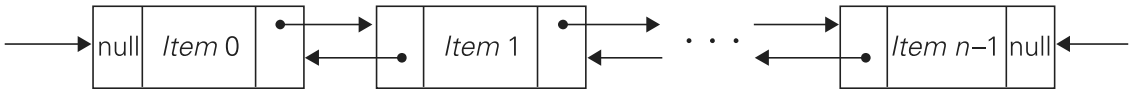


FIGURE 1.5 Doubly linked list of n elements.

not require any preliminary reservation of the computer memory, and insertions and deletions can be made quite efficiently in a linked list by reconnecting a few appropriate pointers.

We can exploit flexibility of the linked list structure in a variety of ways. For example, it is often convenient to start a linked list with a special node called the ***header***. This node may contain information about the linked list itself, such as its current length; it may also contain, in addition to a pointer to the first element, a pointer to the linked list's last element.

Another extension is the structure called the ***doubly linked list***, in which every node, except the first and the last, contains pointers to both its successor and its predecessor (Figure 1.5).

The array and linked list are two principal choices in representing a more abstract data structure called a linear list or simply a list. A ***list*** is a finite sequence of data items, i.e., a collection of data items arranged in a certain linear order. The basic operations performed on this data structure are searching for, inserting, and deleting an element.

Two special types of lists, stacks and queues, are particularly important. A ***stack*** is a list in which insertions and deletions can be done only at the end. This end is called the ***top*** because a stack is usually visualized not horizontally but vertically—akin to a stack of plates whose "operations" it mimics very closely. As a result, when elements are added to (pushed onto) a stack and deleted from (popped off) it, the structure operates in a "last-in–first-out" (LIFO) fashion—exactly like a stack of plates if we can add or remove a plate only from the top. Stacks have a multitude of applications; in particular, they are indispensable for implementing recursive algorithms.

A ***queue***, on the other hand, is a list from which elements are deleted from one end of the structure, called the ***front*** (this operation is called ***dequeue***), and new elements are added to the other end, called the ***rear*** (this operation is called ***enqueue***). Consequently, a queue operates in a "first-in–first-out" (FIFO) fashion—akin to a queue of customers served by a single teller in a bank. Queues also have many important applications, including several algorithms for graph problems.

Many important applications require selection of an item of the highest priority among a dynamically changing set of candidates. A data structure that seeks to satisfy the needs of such applications is called a priority queue. A ***priority queue*** is a collection of data items from a totally ordered universe (most often,

integer or real numbers). The principal operations on a priority queue are finding its largest element, deleting its largest element, and adding a new element. Of course, a priority queue must be implemented so that the last two operations yield another priority queue. Straightforward implementations of this data structure can be based on either an array or a sorted array, but neither of these options yields the most efficient solution possible. A better implementation of a priority queue is based on an ingenious data structure called the ***heap***. We discuss heaps and an important sorting algorithm based on them in Section 6.4.

Graphs

As we mentioned in the previous section, a graph is informally thought of as a collection of points in the plane called "vertices" or "nodes," some of them connected by line segments called "edges" or "arcs." Formally, a ***graph*** $G = \langle V, E \rangle$ is defined by a pair of two sets: a finite nonempty set V of items called ***vertices*** and a set E of pairs of these items called ***edges***. If these pairs of vertices are unordered, i.e., a pair of vertices (u, v) is the same as the pair (v, u), we say that the vertices u and v are ***adjacent*** to each other and that they are connected by the ***undirected edge*** (u, v). We call the vertices u and v ***endpoints*** of the edge (u, v) and say that u and v are ***incident*** to this edge; we also say that the edge (u, v) is incident to its endpoints u and v. A graph G is called ***undirected*** if every edge in it is undirected.

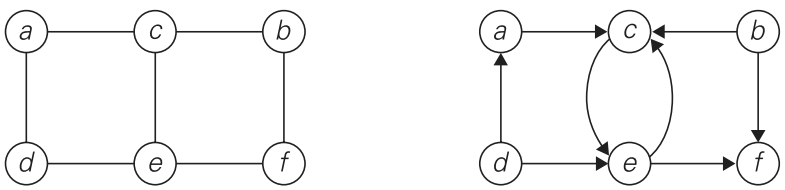
If a pair of vertices (u, v) is not the same as the pair (v, u), we say that the edge (u, v) is ***directed*** from the vertex u, called the edge's ***tail***, to the vertex v, called the edge's ***head***. We also say that the edge (u, v) leaves u and enters v. A graph whose every edge is directed is called ***directed***. Directed graphs are also called ***digraphs***.

It is normally convenient to label vertices of a graph or a digraph with letters, integer numbers, or, if an application calls for it, character strings (Figure 1.6). The graph depicted in Figure 1.6a has six vertices and seven undirected edges:

$$V = \{a, b, c, d, e, f\}, E = \{(a, c), (a, d), (b, c), (b, f), (c, e), (d, e), (e, f)\}.$$

The digraph depicted in Figure 1.6b has six vertices and eight directed edges:

$$V = \{a, b, c, d, e, f\}, E = \{(a, c), (b, c), (b, f), (c, e), (d, a), (d, e), (e, c), (e, f)\}.$$



(a) (b)

FIGURE 1.6 (a) Undirected graph. (b) Digraph.

Our definition of a graph does not forbid ***loops***, or edges connecting vertices to themselves. Unless explicitly stated otherwise, we will consider graphs without loops. Since our definition disallows multiple edges between the same vertices of an undirected graph, we have the following inequality for the number of edges $|E|$ possible in an undirected graph with $|V|$ vertices and no loops:

$$0 \le |E| \le |V|(|V| - 1)/2.$$

(We get the largest number of edges in a graph if there is an edge connecting each of its $|V|$ vertices with all $|V| - 1$ other vertices. We have to divide product $|V|(|V| - 1)$ by 2, however, because it includes every edge twice.)

A graph with every pair of its vertices connected by an edge is called ***complete***. A standard notation for the complete graph with $|V|$ vertices is $K_{|V|}$. A graph with relatively few possible edges missing is called ***dense***; a graph with few edges relative to the number of its vertices is called ***sparse***. Whether we are dealing with a dense or sparse graph may influence how we choose to represent the graph and, consequently, the running time of an algorithm being designed or used.

Graph Representations Graphs for computer algorithms are usually represented in one of two ways: the adjacency matrix and adjacency lists. The ***adjacency matrix*** of a graph with n vertices is an $n \times n$ boolean matrix with one row and one column for each of the graph's vertices, in which the element in the ith row and the jth column is equal to 1 if there is an edge from the ith vertex to the jth vertex, and equal to 0 if there is no such edge. For example, the adjacency matrix for the graph of Figure 1.6a is given in Figure 1.7a.

Note that the adjacency matrix of an undirected graph is always symmetric, i.e., $A[i, j] = A[j, i]$ for every $0 \le i, j \le n - 1$ (why?).

The ***adjacency lists*** of a graph or a digraph is a collection of linked lists, one for each vertex, that contain all the vertices adjacent to the list's vertex (i.e., all the vertices connected to it by an edge). Usually, such lists start with a header identifying a vertex for which the list is compiled. For example, Figure 1.7b represents the graph in Figure 1.6a via its adjacency lists. To put it another way,

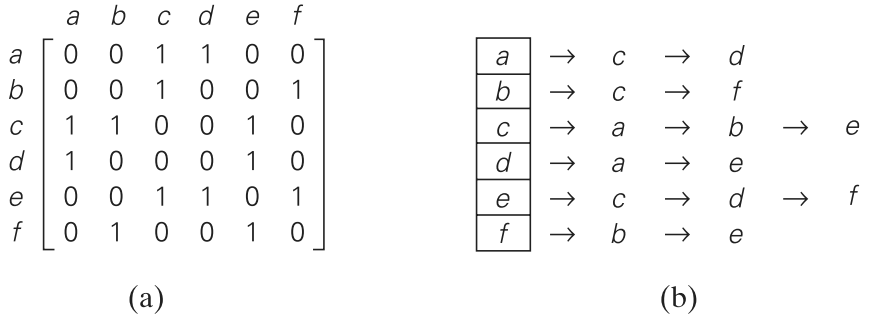


FIGURE 1.7 (a) Adjacency matrix and (b) adjacency lists of the graph in Figure 1.6a.

adjacency lists indicate columns of the adjacency matrix that, for a given vertex, contain 1's.

If a graph is sparse, the adjacency list representation may use less space than the corresponding adjacency matrix despite the extra storage consumed by pointers of the linked lists; the situation is exactly opposite for dense graphs. In general, which of the two representations is more convenient depends on the nature of the problem, on the algorithm used for solving it, and, possibly, on the type of input graph (sparse or dense).

Weighted Graphs A ***weighted graph*** (or weighted digraph) is a graph (or digraph) with numbers assigned to its edges. These numbers are called ***weights*** or ***costs***. An interest in such graphs is motivated by numerous real-world applications, such as finding the shortest path between two points in a transportation or communication network or the traveling salesman problem mentioned earlier.

Both principal representations of a graph can be easily adopted to accommodate weighted graphs. If a weighted graph is represented by its adjacency matrix, then its element $A[i, j]$ will simply contain the weight of the edge from the ith to the jth vertex if there is such an edge and a special symbol, e.g., ∞, if there is no such edge. Such a matrix is called the ***weight matrix*** or ***cost matrix***. This approach is illustrated in Figure 1.8b for the weighted graph in Figure 1.8a. (For some applications, it is more convenient to put 0's on the main diagonal of the adjacency matrix.) Adjacency lists for a weighted graph have to include in their nodes not only the name of an adjacent vertex but also the weight of the corresponding edge (Figure 1.8c).

Paths and Cycles Among the many properties of graphs, two are important for a great number of applications: ***connectivity*** and ***acyclicity***. Both are based on the notion of a path. A ***path*** from vertex u to vertex v of a graph G can be defined as a sequence of adjacent (connected by an edge) vertices that starts with u and ends with v. If all vertices of a path are distinct, the path is said to be ***simple***. The ***length*** of a path is the total number of vertices in the vertex sequence defining the path minus 1, which is the same as the number of edges in the path. For example, a, c, b, f is a simple path of length 3 from a to f in the graph in Figure 1.6a, whereas a, c, e, c, b, f is a path (not simple) of length 5 from a to f.

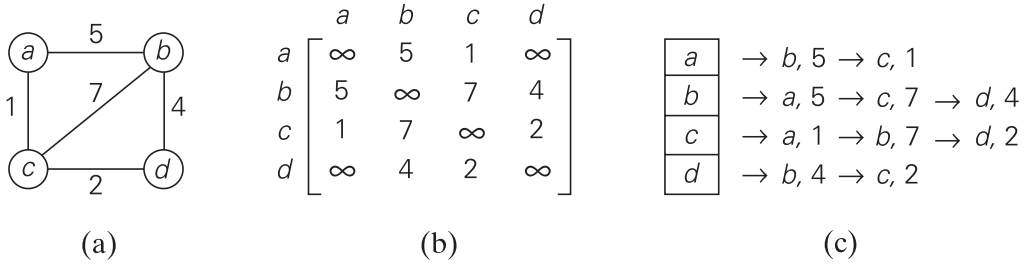


FIGURE 1.8 (a) Weighted graph. (b) Its weight matrix. (c) Its adjacency lists.

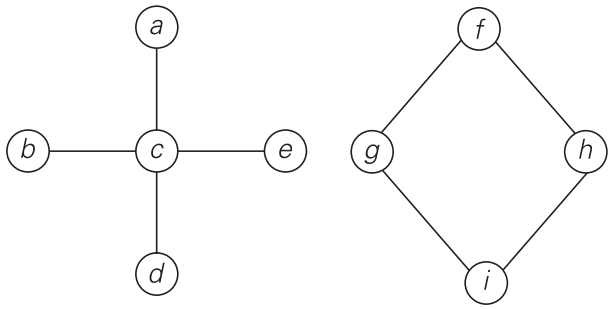


FIGURE 1.9 Graph that is not connected.

In the case of a directed graph, we are usually interested in directed paths. A ***directed path*** is a sequence of vertices in which every consecutive pair of the vertices is connected by an edge directed from the vertex listed first to the vertex listed next. For example, a, c, e, f is a directed path from a to f in the graph in Figure 1.6b.

A graph is said to be ***connected*** if for every pair of its vertices u and v there is a path from u to v. If we make a model of a connected graph by connecting some balls representing the graph's vertices with strings representing the edges, it will be a single piece. If a graph is not connected, such a model will consist of several connected pieces that are called connected components of the graph. Formally, a ***connected component*** is a maximal (not expandable by including another vertex and an edge) connected subgraph[2] of a given graph. For example, the graphs in Figures 1.6a and 1.8a are connected, whereas the graph in Figure 1.9 is not, because there is no path, for example, from a to f. The graph in Figure 1.9 has two connected components with vertices $\{a, b, c, d, e\}$ and $\{f, g, h, i\}$, respectively.

Graphs with several connected components do happen in real-world applications. A graph representing the Interstate highway system of the United States would be an example (why?).

It is important to know for many applications whether or not a graph under consideration has cycles. A ***cycle*** is a path of a positive length that starts and ends at the same vertex and does not traverse the same edge more than once. For example, f, h, i, g, f is a cycle in the graph in Figure 1.9. A graph with no cycles is said to be ***acyclic***. We discuss acyclic graphs in the next subsection.

Trees

A ***tree*** (more accurately, a ***free tree***) is a connected acyclic graph (Figure 1.10a). A graph that has no cycles but is not necessarily connected is called a ***forest***: each of its connected components is a tree (Figure 1.10b).

2. A ***subgraph*** of a given graph $G = \langle V, E \rangle$ is a graph $G' = \langle V', E' \rangle$ such that $V' \subseteq V$ and $E' \subseteq E$.

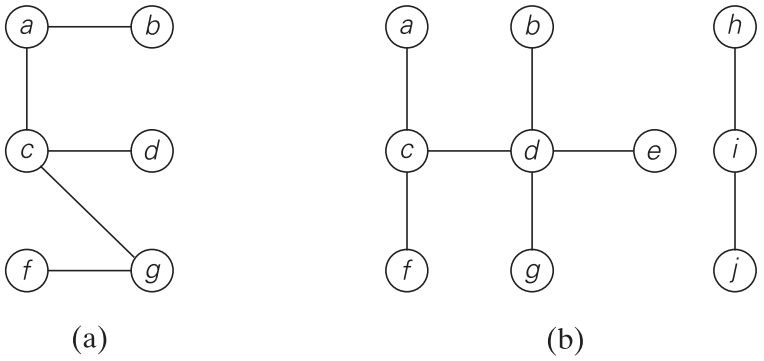


FIGURE 1.10 (a) Tree. (b) Forest.

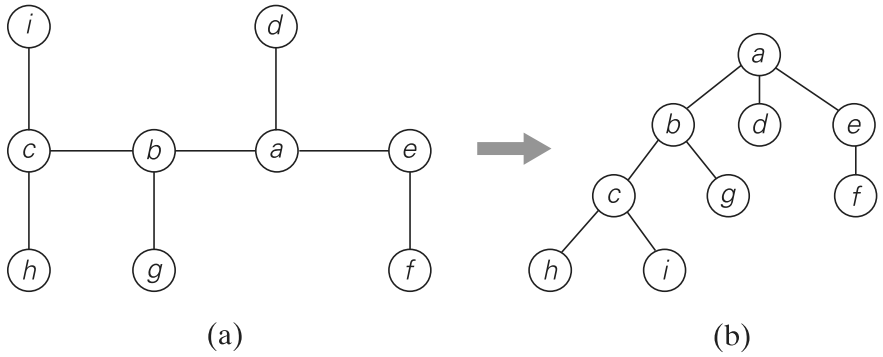


FIGURE 1.11 (a) Free tree. (b) Its transformation into a rooted tree.

Trees have several important properties other graphs do not have. In particular, the number of edges in a tree is always one less than the number of its vertices:

$$|E| = |V| - 1.$$

As the graph in Figure 1.9 demonstrates, this property is necessary but not sufficient for a graph to be a tree. However, for connected graphs it is sufficient and hence provides a convenient way of checking whether a connected graph has a cycle.

Rooted Trees Another very important property of trees is the fact that for every two vertices in a tree, there always exists exactly one simple path from one of these vertices to the other. This property makes it possible to select an arbitrary vertex in a free tree and consider it as the ***root*** of the so-called ***rooted tree***. A rooted tree is usually depicted by placing its root on the top (level 0 of the tree), the vertices adjacent to the root below it (level 1), the vertices two edges apart from the root still below (level 2), and so on. Figure 1.11 presents such a transformation from a free tree to a rooted tree.

Rooted trees play a very important role in computer science, a much more important one than free trees do; in fact, for the sake of brevity, they are often referred to as simply "trees." An obvious application of trees is for describing hierarchies, from file directories to organizational charts of enterprises. There are many less obvious applications, such as implementing dictionaries (see below), efficient access to very large data sets (Section 7.4), and data encoding (Section 9.4). As we discuss in Chapter 2, trees also are helpful in analysis of recursive algorithms. To finish this far-from-complete list of tree applications, we should mention the so-called ***state-space trees*** that underline two important algorithm design techniques: backtracking and branch-and-bound (Sections 12.1 and 12.2).

For any vertex v in a tree T, all the vertices on the simple path from the root to that vertex are called ***ancestors*** of v. The vertex itself is usually considered its own ancestor; the set of ancestors that excludes the vertex itself is referred to as the set of ***proper ancestors***. If (u, v) is the last edge of the simple path from the root to vertex v (and $u \neq v$), u is said to be the ***parent*** of v and v is called a ***child*** of u; vertices that have the same parent are said to be ***siblings***. A vertex with no children is called a ***leaf***; a vertex with at least one child is called ***parental***. All the vertices for which a vertex v is an ancestor are said to be ***descendants*** of v; the ***proper descendants*** exclude the vertex v itself. All the descendants of a vertex v with all the edges connecting them form the ***subtree*** of T rooted at that vertex. Thus, for the tree in Figure 1.11b, the root of the tree is a; vertices d, g, f, h, and i are leaves, and vertices a, b, e, and c are parental; the parent of b is a; the children of b are c and g; the siblings of b are d and e; and the vertices of the subtree rooted at b are $\{b, c, g, h, i\}$.

The ***depth*** of a vertex v is the length of the simple path from the root to v. The ***height*** of a tree is the length of the longest simple path from the root to a leaf. For example, the depth of vertex c of the tree in Figure 1.11b is 2, and the height of the tree is 3. Thus, if we count tree levels top down starting with 0 for the root's level, the depth of a vertex is simply its level in the tree, and the tree's height is the maximum level of its vertices. (You should be alert to the fact that some authors define the height of a tree as the number of levels in it; this makes the height of a tree larger by 1 than the height defined as the length of the longest simple path from the root to a leaf.)

Ordered Trees An ***ordered tree*** is a rooted tree in which all the children of each vertex are ordered. It is convenient to assume that in a tree's diagram, all the children are ordered left to right.

A ***binary tree*** can be defined as an ordered tree in which every vertex has no more than two children and each child is designated as either a ***left child*** or a ***right child*** of its parent; a binary tree may also be empty. An example of a binary tree is given in Figure 1.12a. The binary tree with its root at the left (right) child of a vertex in a binary tree is called the ***left*** (***right***) ***subtree*** of that vertex. Since left and right subtrees are binary trees as well, a binary tree can also be defined recursively. This makes it possible to solve many problems involving binary trees by recursive algorithms.

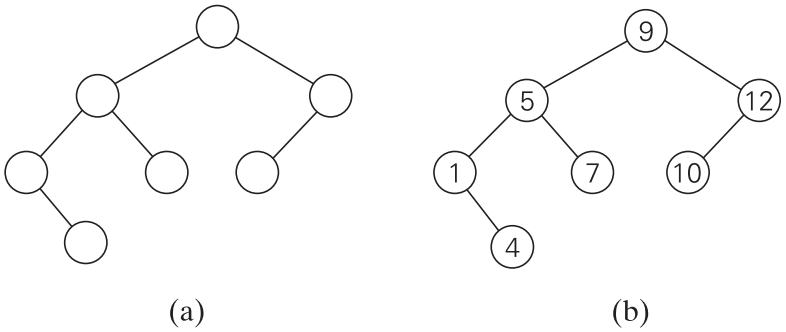


FIGURE 1.12 (a) Binary tree. (b) Binary search tree.

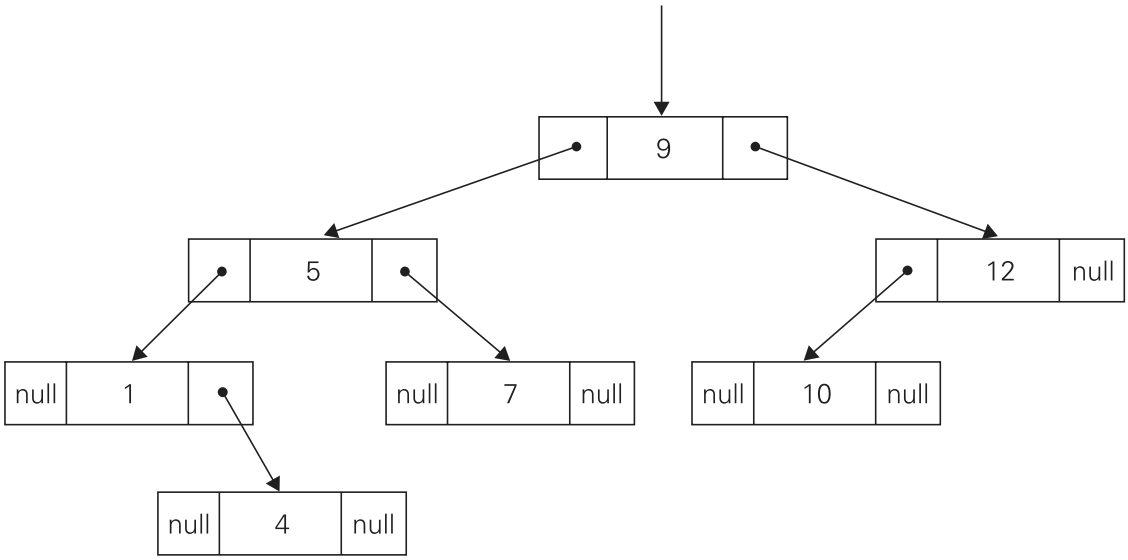


FIGURE 1.13 Standard implementation of the binary search tree in Figure 1.12b.

In Figure 1.12b, some numbers are assigned to vertices of the binary tree in Figure 1.12a. Note that a number assigned to each parental vertex is larger than all the numbers in its left subtree and smaller than all the numbers in its right subtree. Such trees are called ***binary search trees***. Binary trees and binary search trees have a wide variety of applications in computer science; you will encounter some of them throughout the book. In particular, binary search trees can be generalized to more general types of search trees called ***multiway search trees***, which are indispensable for efficient access to very large data sets.

As you will see later in the book, the efficiency of most important algorithms for binary search trees and their extensions depends on the tree's height. Therefore, the following inequalities for the height h of a binary tree with n nodes are especially important for analysis of such algorithms:

$$\lfloor \log_2 n \rfloor \leq h \leq n - 1.$$

A binary tree is usually implemented for computing purposes by a collection of nodes corresponding to vertices of the tree. Each node contains some information associated with the vertex (its name or some value assigned to it) and two pointers to the nodes representing the left child and right child of the vertex, respectively. Figure 1.13 illustrates such an implementation for the binary search tree in Figure 1.12b.

A computer representation of an arbitrary ordered tree can be done by simply providing a parental vertex with the number of pointers equal to the number of its children. This representation may prove to be inconvenient if the number of children varies widely among the nodes. We can avoid this inconvenience by using nodes with just two pointers, as we did for binary trees. Here, however, the left pointer will point to the first child of the vertex, and the right pointer will point to its next sibling. Accordingly, this representation is called the ***first child–next sibling representation***. Thus, all the siblings of a vertex are linked via the nodes' right pointers in a singly linked list, with the first element of the list pointed to by the left pointer of their parent. Figure 1.14a illustrates this representation for the tree in Figure 1.11b. It is not difficult to see that this representation effectively transforms an ordered tree into a binary tree said to be associated with the ordered tree. We get this representation by "rotating" the pointers about 45 degrees clockwise (see Figure 1.14b).

Sets and Dictionaries

The notion of a set plays a central role in mathematics. A ***set*** can be described as an unordered collection (possibly empty) of distinct items called ***elements*** of the

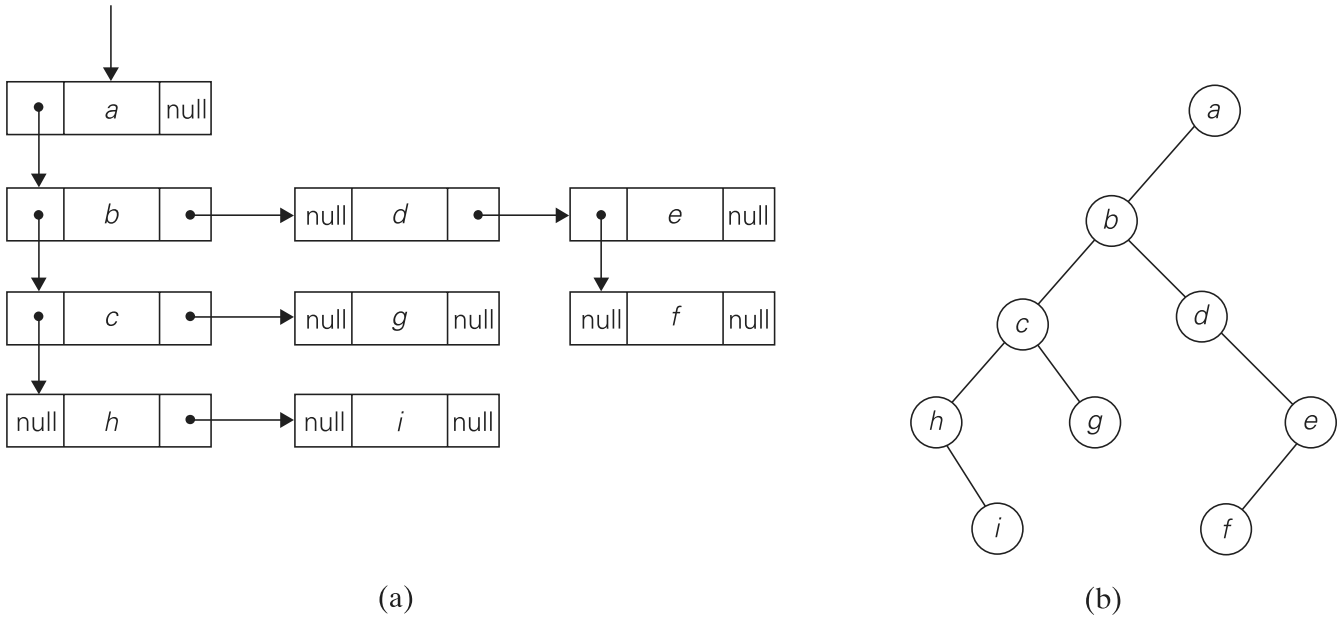


FIGURE 1.14 (a) First child–next sibling representation of the tree in Figure 1.11b. (b) Its binary tree representation.

set. A specific set is defined either by an explicit listing of its elements (e.g., $S = \{2, 3, 5, 7\}$) or by specifying a property that all the set's elements and only they must satisfy (e.g., $S = \{n: n \text{ is a prime number smaller than } 10\}$). The most important set operations are: checking membership of a given item in a given set; finding the union of two sets, which comprises all the elements in either or both of them; and finding the intersection of two sets, which comprises all the common elements in the sets.

Sets can be implemented in computer applications in two ways. The first considers only sets that are subsets of some large set U, called the ***universal set***. If set U has n elements, then any subset S of U can be represented by a bit string of size n, called a ***bit vector***, in which the ith element is 1 if and only if the ith element of U is included in set S. Thus, to continue with our example, if $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $S = \{2, 3, 5, 7\}$ is represented by the bit string 011010100. This way of representing sets makes it possible to implement the standard set operations very fast, but at the expense of potentially using a large amount of storage.

The second and more common way to represent a set for computing purposes is to use the list structure to indicate the set's elements. Of course, this option, too, is feasible only for finite sets; fortunately, unlike mathematics, this is the kind of sets most computer applications need. Note, however, the two principal points of distinction between sets and lists. First, a set cannot contain identical elements; a list can. This requirement for uniqueness is sometimes circumvented by the introduction of a ***multiset***, or ***bag***, an unordered collection of items that are not necessarily distinct. Second, a set is an unordered collection of items; therefore, changing the order of its elements does not change the set. A list, defined as an ordered collection of items, is exactly the opposite. This is an important theoretical distinction, but fortunately it is not important for many applications. It is also worth mentioning that if a set is represented by a list, depending on the application at hand, it might be worth maintaining the list in a sorted order.

In computing, the operations we need to perform for a set or a multiset most often are searching for a given item, adding a new item, and deleting an item from the collection. A data structure that implements these three operations is called the ***dictionary***. Note the relationship between this data structure and the problem of searching mentioned in Section 1.3; obviously, we are dealing here with searching in a dynamic context. Consequently, an efficient implementation of a dictionary has to strike a compromise between the efficiency of searching and the efficiencies of the other two operations. There are quite a few ways a dictionary can be implemented. They range from an unsophisticated use of arrays (sorted or not) to much more sophisticated techniques such as hashing and balanced search trees, which we discuss later in the book.

A number of applications in computing require a dynamic partition of some n-element set into a collection of disjoint subsets. After being initialized as a collection of n one-element subsets, the collection is subjected to a sequence of intermixed union and search operations. This problem is called the ***set union problem***. We discuss efficient algorithmic solutions to this problem in Section 9.2, in conjunction with one of its important applications.

You may have noticed that in our review of basic data structures we almost always mentioned specific operations that are typically performed for the structure in question. This intimate relationship between the data and operations has been recognized by computer scientists for a long time. It has led them in particular to the idea of an ***abstract data type*** (***ADT***): a set of abstract objects representing data items with a collection of operations that can be performed on them. As illustrations of this notion, reread, say, our definitions of the priority queue and dictionary. Although abstract data types could be implemented in older procedural languages such as Pascal (see, e.g., [Aho83]), it is much more convenient to do this in object-oriented languages such as C++ and Java, which support abstract data types by means of ***classes***.

Exercises 1.4

1. Describe how one can implement each of the following operations on an array so that the time it takes does not depend on the array's size n.
   a. Delete the ith element of an array ($1 \le i \le n$).
   b. Delete the ith element of a sorted array (the remaining array has to stay sorted, of course).
2. If you have to solve the searching problem for a list of n numbers, how can you take advantage of the fact that the list is known to be sorted? Give separate answers for
   a. lists represented as arrays.
   b. lists represented as linked lists.
3. a. Show the stack after each operation of the following sequence that starts with the empty stack:

   push(a), push(b), pop, push(c), push(d), pop

   b. Show the queue after each operation of the following sequence that starts with the empty queue:

   enqueue(a), enqueue(b), dequeue, enqueue(c), enqueue(d), dequeue

4. a. Let A be the adjacency matrix of an undirected graph. Explain what property of the matrix indicates that
      i. the graph is complete.
      ii. the graph has a loop, i.e., an edge connecting a vertex to itself.
      iii. the graph has an isolated vertex, i.e., a vertex with no edges incident to it.
   b. Answer the same questions for the adjacency list representation.
5. Give a detailed description of an algorithm for transforming a free tree into a tree rooted at a given vertex of the free tree.