CS6643 Computer Vision

Assignment 1 - Geometric camera models and calibration

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Theoretical Problem Report

Problem 1: Pinhole Camera

1) **"A straight line in the world space is projected onto a straight line at the image plane". Prove by geometric consideration (qualitative explanation via reasoning). Assume perspective projection.**

ANSWER: Firstly, I assume this straight line doesn't go through the focal point **O**. Then, I assume **A** and **B** are the points on the straight line in the world space. After that we know **A**, **B**, **O** can form a plane because they are not on the same line. Next, line **AO** will intersect with image plane at point **A'**, line **BO** will intersect with image plane at point B'. Then, on the image plane, **A'** and **B'** can form a line which is also the projection of the straight line **AB** in the world space. Since **A**, **B** arbitrary point pair on the straight line, so the whole line will be projected onto a straight line (line **A'B'**) at the image plane. (See details in the **Figure 1**)

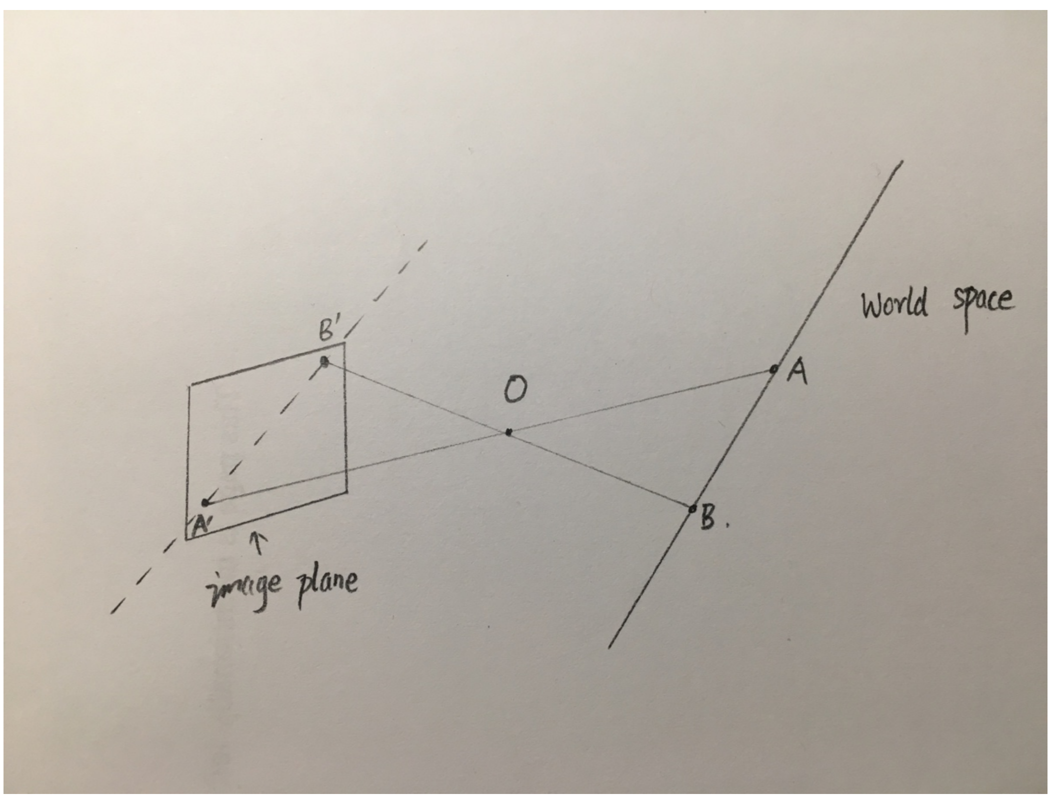

Figure 1, straight line projection

2) Show that, in the pinhole camera model, three collinear points in 3-D space are imaged into three collinear points on the image plane (show via a formal solution).

ANSWER: I assume that the coordinates of that three collinear points are $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$. Because they are collinear in 3-D space, so they have following relations:

i. $\frac{x_1}{x_3} = \frac{z_1}{z_3}, \frac{y_1}{y_3} = \frac{z_1}{z_3}$

ii. $\frac{x_2}{x_3} = \frac{z_2}{z_3}, \frac{y_2}{y_3} = \frac{z_2}{z_3}$

Then according to the pinhole camera model, the corresponding points in the image plane of P_1, P_2 and P_3 are P_1', P_2' and P_3'. After that we can also get the coordinates of P_1', P_2' and P_3' easily by using perspective projection equation $(x, y, z) \Rightarrow (f\frac{x}{z}, f\frac{y}{z})$:

$$P_1' : (\frac{f}{z_1}x_1, \frac{f}{z_1}y_1)$$

$$P_2' : (\frac{f}{z_2}x_1, \frac{f}{z_2}y_2)$$

$$P_3' : (\frac{f}{z_3}x_1, \frac{f}{z_3}y_3)$$

Next we know P_1' and P_2' will form a line on the image plane, then I assume the function of this line is $y = mx + n$. For this function m and n are unknown, in order to solve it, we plug coordinates of P_1' and P_2' into $y = mx + n$ and we can get m and n:

$$\frac{f}{z_1}y_1 = m\cdot\frac{f}{z_1}x_1 + n$$

$$\frac{f}{z_2}y_2 = m\cdot\frac{f}{z_2}x_2 + n$$

$$\Downarrow$$

$$m = \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}$$

$$n = \frac{f}{z_1}(y_1 - \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}x_1)$$

Then we can check if P_3' on this line by plug coordinates of P_3' into $y = mx + n$:

$$mx + n \Rightarrow \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}\cdot\frac{f}{z_3}x_3 + \frac{f}{z_1}(y_1 - \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}x_1)$$

$$\Rightarrow \frac{f}{z_1}y_1 + \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}(\frac{f}{z_3}x_3 - \frac{f}{z_1}x_1)$$

$$\Rightarrow \frac{f}{z_1}\cdot\frac{z_1}{z_3}y_3 + \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}(\frac{f}{z_3}x_3 - \frac{f}{z_1}\cdot\frac{z_1}{z_3}x_3) \text{ (because of } \frac{x_1}{x_3} = \frac{z_1}{z_3} \textit{ and } \frac{y_1}{y_3} = \frac{z_1}{z_3})$$

$$\Rightarrow \frac{f}{z_3}y_3 + \frac{y_1z_2 - y_2z_1}{x_1z_2 - x_2z_1}(\frac{f}{z_3}x_3 - \frac{f}{z_3}x_3) = \frac{f}{z_3}y_3$$

In conclusion, we can see P_3' satisfy the equation: $y = mx + n$, so P_1', P_2' and P_3' are collinear in the image plane.

Problem 2: Perspective Projection

See Fig. 1.4 from textbook on page 6 (pdf handouts) for reference.

a. **Prove geometrically that the projections of two parallel lines lying in some plane Q appear to converge on a horizon line H formed by the intersection of the image plane with the plane parallel to Q and passing through the pinhole.**

ANSWER: In this problem I arbitrary choose two points $\mathbf{P_1}$, $\mathbf{P_2}$ which have the same z value, and their projections $\mathbf{P_1'}$, $\mathbf{P_2'}$ also form a line parallel to plane **Q** lying on the image plane. Then $\mathbf{P_1}$, $\mathbf{P_2}$ and pinhole can form a plane $\mathbf{\Phi}$ at an angle $\boldsymbol{\alpha}$ to the plane **Q**. When $\mathbf{P_1}$, $\mathbf{P_2}$ are infinitive far, which means the **z** value of $\mathbf{P_1}$, $\mathbf{P_2}$ are infinitive large, and the distance between $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_1'}$, $\mathbf{P_2'}$ are infinitive small, and angle $\boldsymbol{\alpha}$ will be infinitive small too(angle α is infinitivally close to 0 degree). In the infinitive situation, angle $\boldsymbol{\alpha}$ is 0, so plane $\mathbf{\Phi}$ is parallel to plane **Q**. Then the distance between $\mathbf{P_1'}$, $\mathbf{P_2'}$ are 0, which means they are intersected in the image plane. What's more because $\mathbf{P_1'}$, $\mathbf{P_2'}$ are on the plane $\boldsymbol{\alpha}$ and image plane, so they converge on the intersection of plane $\boldsymbol{\alpha}$ and image plane, which is the horizon line H.

In the conclusion, because I choose $\mathbf{P_1}$, $\mathbf{P_2}$ randomly, so that the projections of two parallel lines lying in some plane Q appear to converge on a horizon line H formed by the intersection of the image plane with the plane parallel to Q and passing through the pinhole.

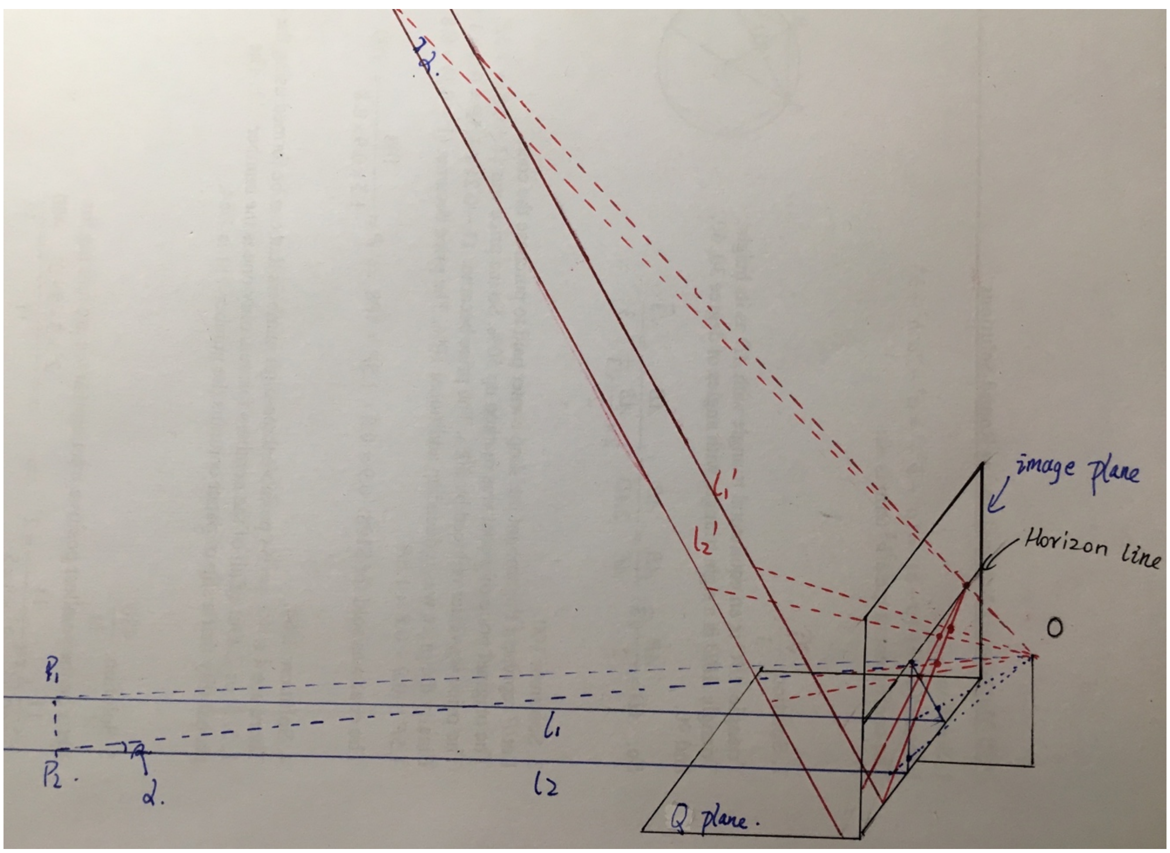

Figure 2, horizon line and vanishing point example (blue and red are two examples)

b. Prove the same result algebraically using the perspective projection equation. You can assume for simplicity that the plane Q is orthogonal to the image plane (as you might see in an image of railway tracks, e.g.).

ANSWER: I choose two points $\mathbf{P_1}$**,** $\mathbf{P_2}$ from parallel line, and the coordinates are $\mathbf{P_1}(x_1, y_1, z_1)$, $\mathbf{P_2}(x_2, y_2, z_2)$. According to the perspective projection equation, the coordinates of the corresponding points on the image plane are $\mathbf{P_1}'(x_1', y_1')$, $\mathbf{P_2}(x_2', y_2')$:

$$x_1' = \frac{f'}{z_1} x_1$$

$$y_1' = \frac{f'}{z_1} y_1$$

$$x_2' = \frac{f'}{z_2} x_2$$

$$y_2' = \frac{f'}{z_2} y_2$$

Then we can assume when $\mathbf{P_1}$**,** $\mathbf{P_2}$ are infinitive far, z_1 and z_2 will be infinitive large. So in the infinitive situation, y_1' and y_2' will be 0, because z_1 and z_2 are infinitive large. What's more the distance between P_1' and P_2' are:

$$x_1' - x_2' = \frac{f'}{z_1} x_1 - \frac{f'}{z_2} x_2 = f'(\frac{z_2 x_1 - z_1 x_2}{z_1 z_2})$$

when z_1 and z_2 are infinitive large, $x_1' - x_2'$ will be 0, so in this situation P_1' and P_2' are intersected on the line y = 0 which is the horizon line on the image plane. Because I choose P_1 and P_2 randomly, so that the projections of two parallel lines lying in some plane Q appear to converge on a horizon line H formed by the intersection of the image plane with the plane parallel to Q and passing through the pinhole.

Problem 3: Coordinates of Optical Center

Let O denote the homogeneous coordinate vector of the optical center of a camera in some reference frame, and let M denote the corresponding perspective projection matrix. Show that MO = 0. (Hint: Think about the coordinates of the optical center in the world coordinate system, use the notion of transformations between world and camera, and plug this into the projection equation.)

ANSWER: According to the text book, we know that **M** = **k(R t)**, then because vector **t** is the translation vector from world coordinate to camera coordinate, but the first three elements of **O** are form a vector of translation from camera coordinate to world frame, so $\mathbf{O} = \begin{bmatrix} -\boldsymbol{t} \\ \mathbf{1} \end{bmatrix}$. Then **MO** = **k(R t)O**, because the forth element of vector O is 1, so **MO** = **k(R t)O** = $\mathbf{k(R\ t)}\begin{bmatrix} -\boldsymbol{t} \\ \mathbf{1} \end{bmatrix}$ = **k(R·-t + t)**.

Then because **O** is already the optical center of camera, so we have no need to rotate the camera.

After that R is an identity matrix (because when $\boldsymbol{\theta}$ is 0 degree, $\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} =$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = identity matrix). Then we can get:

$$\mathbf{MO} = \mathbf{k(R\ t)O} = \mathbf{k(R\ t)}\begin{bmatrix} -\boldsymbol{t} \\ \mathbf{1} \end{bmatrix} = \mathbf{k(Id\ \ t)}\begin{bmatrix} -\boldsymbol{t} \\ \mathbf{1} \end{bmatrix} = \mathbf{k(-t + t)} = \mathbf{0}$$

In conclusion, **MO** = **0**