JOSEPHSON EFFECT AND IT'S APPLICATION IN QUBIT

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Superconductivity is the phenomena in which the electrical resistance of a specific metal or alloy drops to zero and they behave as a perfect diamagnetic substance when they are cooled up to a certain temperature called critical temperature T_c.In the superconducting state, the magnetic susceptibility of the substance is $\chi = -1$.That implies, the superconductor behaves as ideal diamagnet. Also as we cool the substance to its critical state the electrons become more ordered and hence the entropy decreases sharply. This orderliness of electrons also result in decrease of thermal conductivity of the material as the electrons do not transfer much energy.The superconductors are classified mainly in type-1 and type-2 superconductors. The superconductors in which the magnetic field is totally excluded from the interior of superconductors below a certain magnetising field H_c and at H_c the material loses superconductivity and the magnetic field penetrates fully are termed as type-1 or soft superconductors and the superconductors in which the material loses magnetisation gradually rather than suddenly are termed as type-2 or hard superconductors.

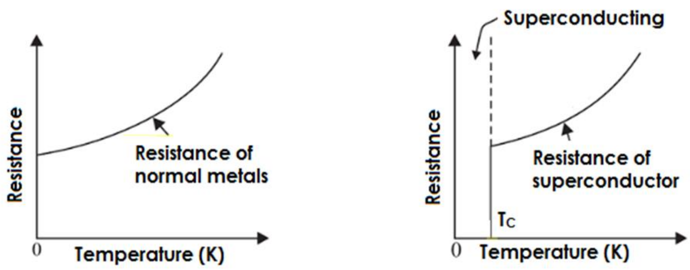

Fig. 1.1: Temperature vs Resistance

1.1 BCS Theory

For explaining the properties of superconductors Bardeen, Cooper, and Schrieffer in 1957 put forward a theory of superconductivity which is based on electron-phonon- electron interaction. It is one of the accurate quantum descriptions of superconductivity. BCS theory explains superconductivity on the basis of electron-phonon-electron interaction and formation of cooper pairs inside the material. When a material is cooled below it's critical temperature $T_C$and applied an electro motive force(voltage source) on the material then electrons start moving and due to the columbic attraction between electrons and ions the ions get displaced from their mean position creating a positive vicinity. That attracts the next electron and when the electron passes away, they get in their previous position. In this way the ions vibrate about their equilibrium position. Though ions are causing electrons to flow but the electron behaves as they are bound to each other. In this way the electrons interact attractively via the lattice distortion. The interaction of electrons with lattice can be quantized in the form of phonon. Let an electron of momentum K is scattered with a momentum $K - p_p$ after emitting a virtual phonon of momentump_p. Another electron of momentum K' interacts with the distorted lattice and absorbs emitted virtual phonon of momentum p_p and then get emitted with momentum $K' + p_p$. In this way electrons interact with each other by exchanging phonons and the attractive force produced between the electrons overcomes the repulsive columbic force.

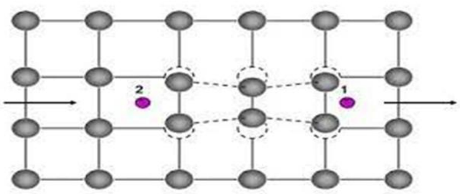

Fig. 1.2: Cooper Pairs

1.2 Cooper Pairs

The pair of electrons formed by phonon interaction is called cooper pair named after an American physicist Leon Cooper who described it first. The Cooper pair has two electrons that interact attractively in phonon field and it has a total spin either 0 or 1 depending on the pairing electrons spins. Hence the Cooper pairs act like bosons since they have integral spin.Energy of the paired electrons is slightly less than the energy of unpaired electrons and the difference between these energies is known as binding energy of a Cooper pair. This binding energy is of the order of $10^{-3}eV$which is very small that's why the superconductivity is a very low temperature event. According to the BCS theory, in between the temperature range 0 to T_c there are many ground states as well as excited states. In these states the Cooper pairs are supposed to be in condensed state with definite phase coherence. At critical temperature this coherence disappears and the Cooper pairs are broken resulting the transition from superconducting state to normal state. In order to understand further let us consider a simple model of two electrons added to a Fermi sea at T=0, with the assumption that the extra electrons interact with each other but not with those in the sea, except via the exclusion principle. We take a two-particle wave function. For two electrons must have equal and opposite momentum, we expect the lowest energy state to have total momentum zero. Hence, an orbital wavefunction can be written

$$\psi_0(\boldsymbol{r_1}, \boldsymbol{r_2}) = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}.\boldsymbol{r_1}} e^{-i\boldsymbol{k}.\boldsymbol{r_2}} \tag{1.1}$$

This eqation can be rewritten either as with the antisymmetric singlet spin function $(\alpha_1\beta_2 - \alpha_2\beta_1)$, where α- "up" spin state of particle and β - "down" spin state of the particle. Because of an attractive interaction, the singlet coupling is expected to have lower energy. This is mainly due to the cosinusoidal dependence of its orbital wavefunction on $(\boldsymbol{r_1} - \boldsymbol{r_2})$ gives a larger probability amplitude for electrons to be near each other, this causes singlet coupling to have lower energy. Thus, a two-electron singlet wave function.

$$\psi_0(\boldsymbol{r_1} - \boldsymbol{r_2}) = \left[\sum_{\boldsymbol{k} > \boldsymbol{k_f}} g_{\boldsymbol{k}} \cos \boldsymbol{k}.(\boldsymbol{r_1} - \boldsymbol{r_2}) \right] (\alpha_1\beta_2 - \beta_1\alpha_2) \tag{1.2}$$

Putting the value of Eq:1.2, in to Schrodinger equation. And we can find weighing coefficient $g_{\boldsymbol{k}}$ and the energy eigenvalue E by:

$$(E - 2\epsilon_{\boldsymbol{k}})\, g_{\boldsymbol{k}} = \sum_{\boldsymbol{k'} > \boldsymbol{k_f}} V_{\boldsymbol{kk'}} g_{\boldsymbol{k'}} \tag{1.3}$$

Where ϵ_k, unperturbed plane wave energies and $V_{kk'}$ are matrix element of the interaction potential.

$$V_{kk'} = \Omega^{-1} \int V(\boldsymbol{r}) e^{i(\boldsymbol{k'} - \boldsymbol{k}).\boldsymbol{r}} d\boldsymbol{r} \tag{1.4}$$

Where r is the distance between two electrons, Ω normalization volume.If $E < 2E_F$ and g_k satisfy Eq:1.3 then bound pair exists. Since general $V_{\boldsymbol{kk'}}$ is not very easy to analyze, Cooper introduced the approximation that all $V_{\boldsymbol{kk'}} = -V$ for $k$states out to a cutoff energy $\hbar\omega_c$ away from E_F and $V_{\boldsymbol{k}k'}$ beyond E_F. Now from Eq:1.3

$$g_{\boldsymbol{k}} = \frac{V \Sigma g_{k'}}{2\epsilon_k - E} \tag{1.5}$$

After simplification and converting summation to integration

$$\frac{1}{V} = N(0) \int_{E_F}^{E_F + \hbar\omega_c} \frac{d\epsilon}{2\epsilon - E} = \frac{1}{2} N(0) ln \frac{2E_F - E + 2\hbar\omega_c}{2E_F - E} \tag{1.6}$$

For most classic superconductor $N(0)V << 0.3$. This allows use of the weak coupling approximation which is valid for $N(0)V << 1$. Hence solution of preceding equation

$$E \approx 2E_F - 2\hbar\omega_c e^{-2/N(0)V} \tag{1.7}$$

Since $k > k_F$ i.e., Kinetic energy is in excess of E_F. Hence there is a bounded state with negative energy with respect to the Fermi surface.

Thus, there is a bound state with negative energy concerning the Fermi surface made up completely of electrons with $k > k_F$,i.e., with kinetic energy over E_F. The grant to the energy of the attractive potential out-weighs this excess kinetic energy, leading to binding regardless of how small V is. Note that the form of the binding energy is not analytic at V= 0 (i.e., it is can not be expanded in the power of V). Hence it can not be achieved by perturbation theory, a fact that greatly delayed the advent of the theory. Furthermore, we get that wave function is proportional to $\sum\limits_{k>k_F} \frac{cos\boldsymbol{k.r}}{2\xi_{\boldsymbol{k}}+E'}$.

Here if we compare this equation with initial wave function Eq.1.2, we detected that g_k depends on $(2\xi_{\boldsymbol{k}} + E')^{-1}$. Its value decrease at a high value of $\xi_{\boldsymbol{k}}$ and get maximum value of $1/E'$ at $\xi_{\boldsymbol{k}} = 0$. Since $g_{\boldsymbol{k}}$ depends only on $\xi_{\boldsymbol{k}}$, this outcome has spherical symmetry; hence, it is an S state as well as a singlet spin state. Thus, the electron state within a range of energy $\sim E$' above E_F are those most strongly involved in forming the bound state. Since $E' << \hbar\omega_c$ for $N(0)V << 1$, this demonstrates that the detailed behavior of $V_{\boldsymbol{kk'}}$ out around $\hbar\omega_c$ will not have any great effect on the outcome.This fact gives us some justification for making such a raw approximation to $V_{\boldsymbol{kk'}}$. The second outcome of this small range of energy states that the uncertainty pricniple, implies that the size of the bound pair is not less than $\sim \frac{\hbar\nu_F}{E'}$. since kT_c turns out to be of the order of E', this infers that the size of the Cooper pair state is $\sim \xi_0 = \frac{a\hbar\nu_F}{kT_c}$ much larger than the interparticle distance Thus, the pairs are highly overlapping.

1.3 Origin of Attractive Interaction Between Electrons

Now the question arises that is $V_{kk'}$ negative for superconductivity? To examine this, we consider bare Coulomb interaction, $V(r) = \frac{e^2}{r}$.

$$V(q) = V(k - k') = V_{kk'} = \Omega^{-1} \int V(r)e^{iq.r}dr \tag{1.8}$$

We get

$$V(q) = \frac{4\pi e^2}{\Omega q^2} \tag{1.9}$$

Since V(q)>0 (always) .If we consider the dielectric function $\epsilon(q,\omega)$ of the medium, $V(q)$ is reduced by $\epsilon^{-1}(\boldsymbol{q},\boldsymbol{\omega})$ times.

$$\epsilon = 1 + \frac{k_s^2}{q^2} \quad (From\, Fermi - Thomas\, Approximation) \tag{1.10}$$

$$V(q) = \frac{4\pi e^2}{q^2 + k_s^2} \tag{1.11}$$

$V_{kk'}$ is positive even if $q = 0$. Hence still superconductivity is not observed. For $V_{kk'}$ to be negative, we consider the motion of the ion cores. In 1950, Frohlich suggested that the electron-lattice interaction for explaining superconductivity. By conservation of momentum, the electron is scattered from k to$k$'. the corresponding phonon must carry the momentum $q = k - k'$ and characteristic frequency must be the phonon frequency ω_q. If $\omega < \omega_q$; then $V_{kk'}$ would be positive. Hence no superconductivity would result. To show $V_{kk'}$ is negative we use the Jellium model. It gives

$$V(q) = \frac{4\pi e^2}{q^2 + k_s^2} + \frac{4\pi e^2}{q^2 + k_s^2}\frac{\omega_{\boldsymbol{q}}^2}{\omega^2 - \omega_{\boldsymbol{q}}^2} \tag{1.12}$$

Here, 1st and 2nd terms represent –Coulomb repulsion and phonon mediated interaction (attractive for $\omega < \omega_{\boldsymbol{q}}$) respectively. Thus, we observe that phonon-mediated attraction is the basis for superconductivity.

1.4 The BCS Ground State

Here we will analyze the ground state of cooper pairs using BCS theory. When net interaction is attractive, Fermi sea is unstable against the formation of bound Cooper pair. Thus, we can predict that pairs condense until an equilibrium point is reached. As the binding energy for a pair has gone to zero, the state of the system changed from the Fermi sea. To handle this complicated state, we introduce the mathematical form that comes from BCS wavefunction. If we write wave function for more than two electrons, the wave function does not follow the antisymmetric in the same way as the single Cooper pair. So, to make it convenient we replace it with a scheme$N \times N$,slatter determinant which prescribes N-electron Anti symmetrized product function. Now define $c^*_{k\uparrow}$ "creation operator" which prescribes occupied state. $c^*_{-k\downarrow}$ "annihilation operator" which prescribes an unoccupied state. Now, the singlet wave function can be written as

$$|\psi_0\rangle = \sum_{k>k_F} g_k c^*_{k\uparrow} c^*_{-k\downarrow} |F\rangle \tag{1.13}$$

where represent Fermi sea with all state occupied up to k_F

$$[c_{k\sigma}, c_{k'\sigma'}] = c_{k\sigma} c^*_{k'\sigma'} + c^*_{k'\sigma'} c^*_{k\sigma} = \delta_{kk'} \delta_{\sigma\sigma'} \Rrightarrow [c_{k\sigma}, c_{k'\sigma'}]_+ = [c^*_{k\sigma}, c^*_{k'\sigma'}]_+ = 0 \tag{1.14}$$

The particle number operator is defined as

$$n_{k\sigma} = c^*_{k\sigma} c_{k\sigma} \tag{1.15}$$

If we operate particle number operator on the occupied and unoccupied state it gives eigenvalue of 1 and 0 respectively. For electron wavefunction in term of momentum eigenfunction we get as following:

$$|\psi_N\rangle = \Sigma g(k_i, ...k_l) c^*_{k_i\uparrow} c^*_{-k_i\downarrow} .....c^*_{k_l\uparrow} c^*_{-k_l\downarrow} |\phi_0\rangle \tag{1.16}$$

Where $|\phi_0\rangle$ is a vacuum state. The number of ways of choosing the$N/2$state for pair occupancy is given by combinatorics relation:

$$\frac{M!}{[M-(N/2)]!} * (N/2) \tag{1.17}$$

The trio BCS put an argument that when there are so many particles involved it would be a good approximation to use mean-field approach, in which each state's occupancy depends solely on the average occupancy of other states. This give relaxation to the constraint on the total number of particles fixed as N since we treat occupancies statistically. But since N is very large we can work in a system in which N is fixed. Here we use a grand canonical ensemble. The BCS ground state:

$$|\psi_G\rangle = \prod_{k=k_1,k_2,...k_M} (u_k + v_k c^*_{k\uparrow} c^*_{-k\downarrow}) |\phi_0\rangle \qquad where\ |u_k|^2 + |v_k|^2 = 1 \tag{1.18}$$

For pair $(k\uparrow, -k\downarrow,)$ probability for occupancy $= |v_k|^2$ probability for unoccupancy is $|1 - |v_{k|}|^2$

$$|\psi_G\rangle = \sum_N \lambda_N |\psi_N\rangle \tag{1.19}$$

$$\bar{N} = \Sigma\, 2|v_k|^2 \tag{1.20}$$

Thus we can represent ground state wave function in terms of N particle electron wave function. Here each term of expansion contains $N/2$ pairs. The modulus of eignen value i.e $|\lambda_N|^2$ are sharply peaked at average value of Ni.e at $\bar{N}$

2 Josephson Effect

When two superconductors are separated by a very thin insulator layer at temperature lower than critical temperature, current flows from one superconductor to another superconductor without applying potential difference between them. This current depends on the characteristic of the insulator. This effect was first discovered by Brain Josephson in 1962 and hence named after him. Josephson effect is the consequence of tunneling of Cooper pairs from one superconductor to another through the thin insulating layer. The wavefunctions of quantum waves of Cooper pairs on each side of the insulator penetrate the insulating layer with exponentially decreasing amplitude. If the insulating layer is thin enough the wave functions overlap sufficiently to form stationary waves and Cooper pairs can then pass through the insulating layer. The current across the junction is given by:

$$I_s = I_c \sin(\Delta\phi) \tag{2.1}$$

The following image represents Josephson junction schematically:

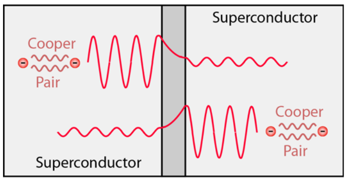

Fig. 2.1: Josephson Junction

where $\Delta\phi$is the phase difference between 'Ginzburg-Landau wave-functions' in the two electrodes and $I_c$represents the critical current.He had also precited that if a phase difference $\Delta\phi$ is maintained across the junction the difference would evolve according to the equation:

$$\frac{d(\Delta\phi)}{dt} = \frac{2eV}{\hbar} \tag{2.2}$$

which give rise to an alternating current of amplitude Ic and frequency $\nu = \frac{2eV}{h}$. The energy $h\nu$ denotes energy change of a Cooper pair transferred across the junction. Accordingly, there are two effects; the AC Josephson junction and the DC Josephson junction effect. Initially, the effect was restricted to quantum mechanical tunneling of electrons through the barrier layer, but now any two superconducting electrodes that are connected by a weak link can have this effect, hence it is much more generalized. The weak link can be insulating material or a normal metal layer made up of weakly superconducting metal with a *proximity effect.* In this proximity effect, Cooper pairs from a superconductor diffuse into normal metal when their proximity is very close. The third type is simply a short, narrow constriction in a continuous superconducting material. Accordingly they are abbreviated as S-N-S, S-I-S and S-c-S linkes where S stands for superconductor, N for normal metal, I for insulator and c for constriction.

With the relation Eq:3.1 and Eq:3.2 we can derive the coupling free energy stored in the junction. This can be done by integrating electrical work$\int I_s V dt = \int I_s(\frac{\hbar}{2e})d(\Delta\phi)$ done by a current source in changing the phase by $\Delta\phi$. Thus we find

$$F = const - E_J cos(\Delta\phi) \quad where\, E_J = \frac{\hbar I_c}{2e}$$

The minimum energy occurs when $\Delta\phi = 0$ i.e equal phases; which corresponds to energy minimum in the absence of phase gradients in a bulk superconductor. Critical current is a parameter that gives an idea about how strongly the phases of the superconductor are coupled through the weak link. This depends on dimensions, properties of material at the junction. Typically I_C lies in the range of $\mu A$to mA.

2.1 The Josephson Critical Current

We will derive the Josephson current for the case of short one-dimensional Metallic weak links. We make simple consideration of special cases where two massive superconducting electrodes are separated by a short, one-dimensional link of length $L << \xi$ all of the same superconductor. We have the one-dimensional Ginzburg-Landau equation describing the superconductor bridge as

$$\xi^2 \frac{d^2 f}{dx^2} + f - f^3 = 0 \tag{2.3}$$

where $f = \frac{\psi}{\psi_\infty}$. We assume that electrodes are in equilibrium, so that $|f| = 1$in both of them. And since relative phase only matters we consider one of phase as 0 and the other one as $\Delta\phi$ without any loss of generality. The necessary boundary conditions are $f = 1\, at\, x = 0$ and $f = e^{i\Delta\phi}\, at\, x = L$. Under the assumptions we made first term in Eq.2.3 dominates and this reduces to Laplace Equation in one dimension i.e $\frac{d^2 f}{dx^2} = 0$; which has the most general solution a s$f = a + bx$. Applying boundary condition we get

$$f = (1 - \frac{x}{L}) + (\frac{x}{L})e^{i\Delta\phi} \tag{2.4}$$

Substitung in Eq.2.3 we get the final form

$$I_s = I_c \sin(\Delta\phi) \tag{2.5}$$

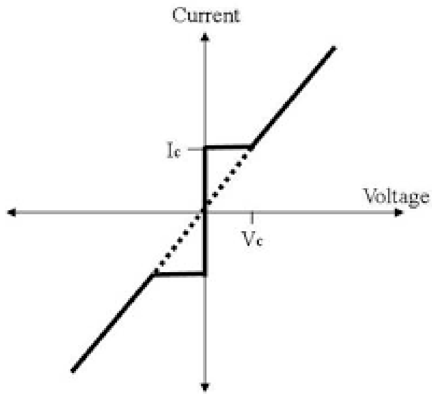

Fig. 2.2: $I_c\ vs\ V$

where $I_c$is given by

$$I_c = (2e\hbar\psi_\infty^2/m^*)(\mathcal{A}/L) \tag{2.6}$$

where $\mathcal{A}$ is the cross-sectional area of the bridge. The theoretical value of critical current differ by a factor $(3^{3/2}/2)(\xi/L)$. Since the approximation demanded $(\xi/L) >> 1$, the bridge critical current will be always greater than that of a long wire of the same cross section. On the physical grounds this can be attributed to that the strongly superconducting banks help to support ψ in the bridge allowing it to sustain a higher phase gradient $(\sim 1/L)$than the limiting value $(\sim 1/\xi)$ in the long wire.

2.2 DC Josephson Effect

When two superconducting materials are separated by a thin insulating barrier, the wavefunctions of Cooper pair overlap with each other and there is a current flow in the thin barrier. If Ψ_1 denotes the wavefunction of electron pair inside the superconductor on one side and Ψ_2 is the wavefunction of electron pair on the other side then these two are bound by a transfer interaction and they satisfy the equations.

$$i\hbar\frac{\partial\Psi_1}{\partial t} = \hbar T\Psi_2 \quad and \quad i\hbar\frac{\partial\Psi_2}{\partial t} = \hbar T\Psi_1 \tag{2.7}$$

where T is the transfer interaction. The wavefunctions Ψ_1 and Ψ_2 can be written as:

$$\Psi_1 = \sqrt{n_1}e^{(i\theta_1)} \qquad and \qquad \Psi_2 = \sqrt{n_2}e^{(i\theta_2)} \tag{2.8}$$

Substituting these in the above equations

$$\frac{1}{2\sqrt{n_2}}e^{i\theta_2}\frac{\partial n_2}{\partial t} + i\Psi_2\frac{\partial\theta_2}{\partial t} = -iT\Psi_1 \tag{2.9}$$

$$\frac{1}{2\sqrt{n_1}}e^{i\theta_1}\frac{\partial n_1}{\partial t} + i\Psi_1\frac{\partial\theta_1}{\partial t} = -iT\Psi_2 \tag{2.10}$$

Solving these equations gives us,

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial\theta_1}{\partial t} = -iT\sqrt{n_1 n_2}e^{i\delta} \tag{2.11}$$

Where δ is the difference between the phases of the wavefunctions Ψ_1 and Ψ_2. Comparing real and imaginary parts of the above equation, we get

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2}sin\delta \tag{2.12}$$

$$\frac{\partial\theta_2}{\partial t} = -T\sqrt{\frac{n_1}{n_2}}cos\delta \tag{2.13}$$

For two identical superconductors, the probability amplitudes are nearly equal $n_1 \approx n_2$.
Then from above equations we get

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \quad and \quad \frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} \tag{2.14}$$

Integrating the above equation over leads to $\theta_2 - \theta_1 = \delta = Constant$ The current flowing depends on $\frac{\partial n_1}{\partial t}$ *and* $\frac{\partial n_2}{\partial t}$ which is proportional to $sin\delta$
Hence $J = J_0\ sin\delta \quad = \quad J_0 \sin(\theta_2 - \theta_1) = -2T\sqrt{n_1 n_2}\sin(\theta_2 - \theta_1)$
Hence even in the absence of applied voltage, there is a current flow that depends on the difference between phases of wavefunctions on the two sides. This current is constant over time because the phase difference δ *is a constant*. Hence this phenomenon is called DC Josephson effect.

2.3 AC Josephson Effect

Now if we apply a DC voltage V across the junction, the potential experienced by electron pair is qV where $q = 2e$.We can take this potential as eV on one side and–eV on the other side, because the potential experienced by electron pair is $eV - (-eV) = 2eV$,remains unchanged.

$$i\hbar\frac{\partial \Psi_1}{\partial t} = \hbar T\Psi_2 - eV\Psi_1 \tag{2.15}$$
$$i\hbar\frac{\partial \Psi_2}{\partial t} = \hbar T\Psi_1 + eV\Psi_2 \tag{2.16}$$

Putting the values of Ψ_1 and Ψ_2 from Eq:2.1;

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial \theta_1}{\partial t} = \frac{ieV}{\hbar}n_1 - iT\sqrt{n_1 n_2}e^{i\delta}$$

and

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + in_2\frac{\partial \theta_2}{\partial t} = -\frac{ieV}{\hbar}n_2 - iT\sqrt{n_1 n_2}e^{-i\delta}$$

Again comparing real and imaginary parts we get; For real part:

$$\frac{\partial n_1}{\partial t} = 2T\sqrt{n_1 n_2}sin\delta \tag{2.17}$$
$$\frac{\partial n_2}{\partial t} = -2T\sqrt{n_1 n_2}sin\delta \tag{2.18}$$

Similarly for imaginary part:

$$\frac{\partial \theta_1}{\partial t} = \frac{eV}{\hbar} - T\sqrt{\frac{n_2}{n_1}}\cos\delta \tag{2.19}$$
$$\frac{\partial \theta_2}{\partial t} = -\frac{eV}{\hbar} - T\sqrt{\frac{n_1}{n_2}}\cos\delta \tag{2.20}$$

Due to the applied voltage across the superconductors, the phase difference varies with time.As a result, n_1 $n_2 \longrightarrow \frac{\partial(\theta_2-\theta_1)}{\partial t} = -2\frac{eV}{\hbar}$.
Integrating the above equation with respect to time we get

$$\theta_2 - \theta_1 = \delta = \delta_0 - 2\frac{eV}{\hbar}t \tag{2.21}$$

Hence the current $J = J_0\ sin\delta = J_0 sin\,(\theta_2 - \theta_1) = J_0 sin(\delta_0 - 2\frac{eV}{\hbar}t) = J_0 sin(\delta_0 - \omega t)$
Hence an AC current flows through the insulating barrier which oscillates with the frequency $\omega = 2\frac{eV}{\hbar}$.This flow of alternating current after applying some voltage to the superconductor is known as the AC Josephson effect

3 Introduction to Quantum Computing and Qubit Circuits

Quantum computing is one of the fastest-growing fields which is an application of quantum mechanics and information science. The ability to solve complex problems using many efficient algorithms and by using relatively fewer bits makes them superior to present-day computing. In contrast to the binary bits 0 and 1 which are the most basic form of information in present-day computers; whereas quantum computers have Quantum Bits (qubits) as the counterpart of classical bits. Since quantum mechanics allow phenomena like superposition and entanglement behavior of qubits are different from classical bits as expected. Qubit is a two-level system where the ground state is represented as $|0\rangle$ and excited state as $|1\rangle$. There are infinitely many states in contrast to classical counterpart with all the equations satisfying the following relation :

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{3.1}$$

where α and β are complex number which satisifes condition $|\alpha|^2 + |\beta|^2 = 1$ The superposition states can be easily visualized using the Bloch sphere representation. It consists of a unit sphere where the north and south pole are represented by $|0\rangle$,$|1\rangle$ respectively any vector can be represented as a radius vector in this sphere. Generally, the state vector is written as follows. The Bloch sphere is represented as shown in figure no:

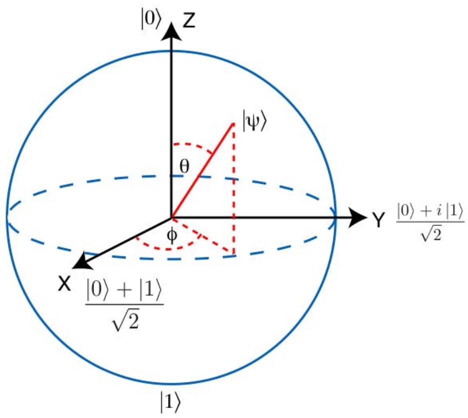

Fig. 3.1: Bloch Sphere

If sym contains N qubits, the total state of that system can be a superposition of 2^N different states $|000...00|\rangle$, $|100...00|\rangle$,$|010...00|\rangle$,.....,$|111...10|\rangle$, $|111...11\rangle$. This shows that there should be 2^Nclassical qubits to represent this system. Researchers have been able to develop algorithms that can speed up factorization database search, the solving of systems of linear equations, and several other computations. These superior properties are not due to a quantum computer exploring many of the states in a superposition at the same time, but due algorithms which interfere between the complex probability amplitudes of these states in an efficient way.

3.1 Josephson Circuits and Qubits

For the realization of quantum computers, we have to bring qubits into reality, we need certain circuits. For classical computing, this was accomplished by transistors which can be switched between on and off states very fast by appropriate bias voltages. Whereas qubits are two-state systems that need to be excited between $|0\rangle$ and $|1\rangle$ states. Earlier it was done using single atoms or ions by their controlled flow, but since parameters related to particles are fixed they cannot be manipulated as per our requirement. These are realized by some special circuits or naively called *artificial atoms* circuits whose parameters can be tuned and altered as per our requirement.

The superposition state is susceptible to losses; it is difficult to maintain coherence for a long time. The average time of present day qubits is around some miutes . With the help of superconducting material and operating below critical temperature T_c we can reduce the resistive losses thus increasing coherence.

The superconducting circuit requires a nonlinear element. This can be understood as precedes. Consider an LC resonating circuit having resonance frequency $\omega_r = \dfrac{1}{\sqrt{LC}}$. At low temperatures i.e at $\hbar\omega_r >> k_B T$, the thermal noise does not significantly affect quantum coherence in the system and the circuit can be treated as a quantum harmonic oscillator. But for the realization, we need two-qubit levels and must be well decoupled from other states else the energy required to excite energy from the lower level will be the same as that needed for excitation between higher levels since their energy spacing is the same for all consecutive levels. This can be overcome by introducing nonlinear elements to the circuit. Thus we can manipulate the lower level only when energy is supplied. So nonlinear element becomes necessary to achieve this. With the help of Josephson junction circuits, we can introduce nonlinear elements in the circuit. This can be understood as shown in the following figure:

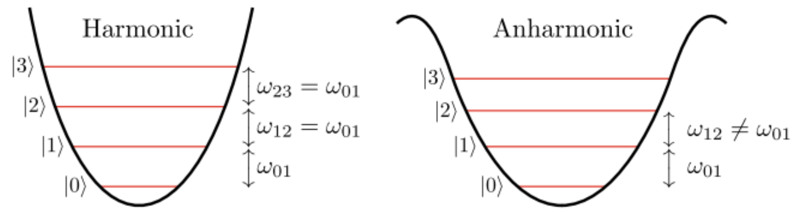

Fig. 3.2: Harmonic and Anharmonic Systems

Josephson junctions play an important in devices that are needed to read out and control superconducting qubits, e.g., switches, mixers, beam-splitters, amplifiers, etc. The transition frequencies of the Josephson junction are of the range 1-10GHz in general. This rangle is well below the plasma frequency of the Josephson junctions, and also matches with frequency ranges for industrial application in different sectors like electronics. The Josephson-junction qubits are operated at temperatures on the order of $10mK$,which ensure that $T << T_c$ and $\hbar\omega_{01} >> k_B T$, these temperature are attainable with present-day technology.

3.2 Cirucit Theory

To quantize the circuit we start with the classical Lagrangian of the circuit and using appropriate generalized coordinate and generalized momenta. Using this we will arrive at the Hamiltonian. From here we can build canonical commutation relations. We treat the simple case of a single Josephson junction circuit. An electrical circuit can be characterized by the number of nodes connected through circuit elements. We use node fluxes as generalized coordinates where V_n is node voltage at nth node. The generalized coordinates given as

$$\Phi_n(t) = \int_{-\infty}^{t} V_n(t')dt' \tag{3.2}$$

The correspondign genralized momenta are generally given by node charges

$$Q_n(t) = \int_{-\infty}^{t} I_n(t')dt' \tag{3.3}$$

Laws like Kirchoff's law can reduce the number of degrees of freedom and acts like constraint. This can be understood as follows; consider a loop l in the circuit, the voltage drop around that loop should be zero, which implies

$$\sum_{b\, around\, l} \Phi_b = \Phi_{ext} \tag{3.4}$$

where $\Phi_b$and Φ_{ext} are branch and external magnetic flux through the loop l. The external magnetic flux is subjected to constraint of quantization of flux conditon which follows $\Phi_{ext} = m\Phi_0$where $m \in \mathbb{Z}$ and $\Phi_0 = h/2e$.

Now by Legendre trasnformation we can find Hamiltonian of the system

$$H = \sum_n \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_n}\dot{\Phi} - \mathcal{L} \tag{3.5}$$

So far we have used only classical relations; now we will plug in key commutator relation which takes the system to quantum limit i.e commutator relatn between conjugate pairs

$$\left[\Phi_{n,}, \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_m}\right] = i\hbar\delta_{nm} \tag{3.6}$$

Our basic superconducting cirucit have three elemtns: capacitors, indcutors, and Josephson junction. We model the Josephson junction as a capacitor $C_J$in parallel to a part characterized by energy E_J. Now we obtain Lagrangian for capacitor C and inductor L. Since Lagrangian $\mathcal{L} = K.E - P.E$ for conservative system we can write directly.

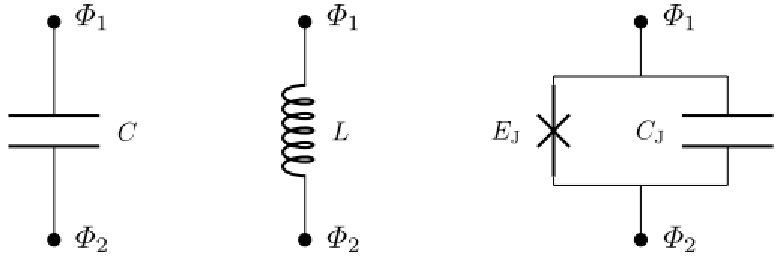

Fig. 3.3: Elements of Qubit cirucit

For capacitor,

$$\frac{CV^2}{2} = \frac{C(\dot{\Phi}_1 - \dot{\Phi}_2)^2}{2} \Rightarrow \mathcal{L}_{\mathcal{C}} = \frac{C(\dot{\Phi}_1 - \dot{\Phi}_2)^2}{2} \tag{3.7}$$

Similarly for inductor,

$$\frac{LI^2}{2} = \left\{V = L\dot{I}\right\} = \frac{(\Phi_1 - \Phi_2)^2}{2L} \Rightarrow \mathcal{L}_{\mathcal{C}} = -\frac{(\Phi_1 - \Phi_2)^2}{2L} \tag{3.8}$$

Now we will find parameters of Josephson junction. From previous discussion we know that

$$I_J = I_c \sin(\phi)$$

$$\dot{\phi} = \frac{2e}{\hbar}V(t)$$

where notations have same meaning as discussed in earlier section. $V(t)$is voltage across the junction, and $\phi = \frac{2e(\Phi_1 - \Phi_2)}{\hbar} = \frac{2\pi(\Phi_1 - \Phi_2)}{\Phi_0}$ is the phase difference across the junction. Now we can calculate energy in the following way

$$\int_{-\infty}^{t} I(\tau)V(\tau)d\tau = E_J(1 - \cos\phi) \tag{3.9}$$

where Josephson energy $E_J$is given by $E_J = \frac{\hbar I_c}{2e}$. Now by knowing value of C_J we can write the Lagrangian for a Josephson junction eqaution as follows

$$\mathcal{L}_{\mathcal{JJ}} = \frac{C_J(\dot{\Phi}_1 - \dot{\Phi}_2)^2}{2} - E_J(1 - \cos\phi) \tag{3.10}$$

We can see that Lagrangian have a cosine term which induces nonlinearity in the circuit. If the Lagrangian had quadriatic term in Φ it will resemble that of harmonic oscillator. The transition frequency ω_{01} and and other properties of the qubit are controlled by Josephson-junction part of a superconducting qubit . For single junction the Josephson energy $E_J$is fixed at the stage of fabrication. However, by using two Josephson junctions in a SQUID configuration, a tunable Josephson energy can be achieved, which means that various qubit parameters can be tuned during an experiment. It works as a single junction with an effective Josephson energy which depends on the external magnetic flx through the SQUID loop. Due to our limited knowledge we were unable to derive the following result for two Junction SQUID mode. We directly state the result from the reference:

$$E_{J,eff} = (E_{J,1} + E_{J,2})\cos((\pi\Phi_{ext})\sqrt{1 + d^2 tan^2\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)}$$

$$d = \frac{E_{J,2} - E_{J,1}}{E_{J,2} + E_{J,1}}$$

this parameter is measure of junction asymmetry. The Lagrangian and Hamiltonian we derived are usefull in analysis of different types of qubit ciruit designs which are devloped using above components.

3.3 Basic Types of Josephson-Junction Qubits

There are three basic Josephson-Junction qubits, charge qubit ,flux qubit and phase qubit.The charge qubit is like a box for charge which controlled by an external voltage V_g ; the flux qubit acts as a loop which controlled by an external magnetic flux Φ_{ext} and the phase qubit is a Josephson-junction biased by current I_b.The main difference between the three is the ratio of Josephson energy vs Charging energy (the required energy for one cooper pair to charge the total capacitance in the circuit).Circuits of these qubits are shown in Fig3.4- (a)Charge qubit ,(b)Flux qubit ,(c)Phase qubit

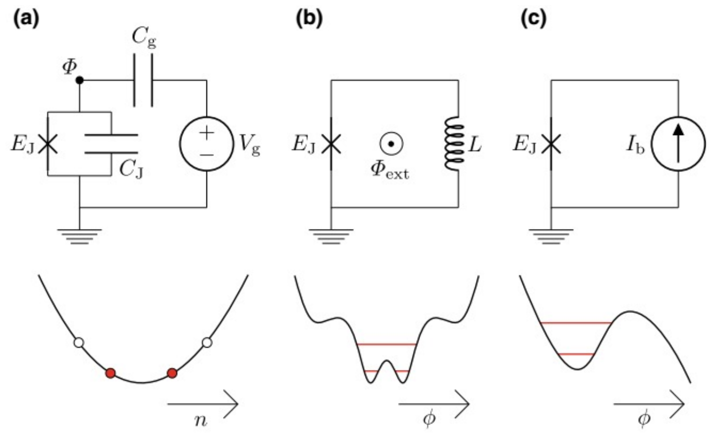

Fig. 3.4: Circuit Theory

These qubit circuits can be interpreted by generalised coordinates and conjugate generalised momenta. If we consider ϕ be the phase difference between the Josephson junction and n (the number of cooper pairs on each of superconducting island of the junction) be the conjugate variable. Then commutation relation of these variable

$$\left[e^{\iota\phi}, n\right] = e^{\iota\phi} \tag{3.11}$$

To understanding the working of Josephson junction qubits , we take the ratio of Josephson energy(E_J) and charging energy $E_c = \frac{e^2}{(2C)}$ (where C is suitable capacitance in the circuit) as a important parameter .For charge qubit $\frac{E_J}{E_c}$<<1 ,for this case number of charge(Coope pairs) n is well defined while phase has large quantum fluctuation . The opposite holds in case $\frac{E_J}{E_c}$>>1,this is case for both the flux qubit and the phase qubit.

3.3.1 Charge Qubit

The charge qubit is also referred as the cooper- pair box (CPB). It was the first superconducting qubit which invented .The small superconducting island (black dot with node flux Φ),which is attached to a superconducting reservoir through the Josephson junction.Through the junction cooper pair electron can tunnel on and off the island. The superconducting island is attached to the voltage source V_g through gate capacitance C_g. Due to electromagnetic enviroment, the background charge $n_g = \frac{C_g V_g}{(2e)}$(measures in per units of cooper pair) is induced on the superconducting island.

The Langrangian of the cooper-pair box circuit is writen as

$$\mathcal{L}_{CPB} = \frac{C_g(\dot{\Phi} - \mathrm{Vg})^2}{2} + \frac{C_J\dot{\Phi}^2}{2} - E_J\left[1 - cos(\frac{2\pi\Phi}{\Phi_0})\right] \tag{3.12}$$

The conjugate momentum is nothing but charge on the superconducting island ,which is calculated as $Q=(C_J+C_g)\dot{\Phi}-C_gV_g$,apply Legendre transformation from Eq.3.5 and ignore the constant term because they do not give any contribution in dynamics.We get the Hamiltonian of CPB

$$H_{CPB} = 4E_c(n-n_g)^2 - E_J cos\phi \tag{3.13}$$

where $n = -Q/2e$, the number of cooper pair on island ; $\phi = 2e\Phi/\hbar$ and $E_c = C_J + C_g$ is total capacitance.

To study the circuit quantum mechanically we expand the Φ and Q to operator using commutation relation in Eq3.11. If $\mid n\rangle$ be a system state ,which written in charge basis counting the number of cooper pair. The commutation relation gives $e^{\pm\iota\phi}|n\rangle = |n \mp 1\rangle$,we get the hamiltonian of CPB

$$H_{CPB} = \sum \left[4E_c(n-n_g)^2|n\rangle\langle n| - \frac{1}{2}E_J(|n+1\rangle\langle n| + |n-1\rangle\langle n|)\right] \tag{3.14}$$

This is tight binding hamiltonian with $E_c$and $n_g$calculating the on site energy and E_J setting the tunneling matrix element between the neighbouring charge state. We can tune the energy level of the system by an external voltage during an experiment .Using SQUID in place of Josephson junction one can further tuned the energy level.

The half-integral values of the background charge,$n_g = m + \frac{1}{2}, m \in Z$,are special due to some reason:

- The eigenstates of the system have well defined parities ,for these values of n_g.
- At these points, for two charge state $|m\rangle$ and $|m+1\rangle$,the effctive charging energies $4E_c(n-n_g)^2$ are degenerated.
- At these points , the two energy levels of the system are well differentiated from the other energy levels in the system,this bring for a good qubit.The transition frequency for the qubit is set by Josephson junction energy E_J because there is degeneracy between the two energy levels.
- The qubit is less sensitive to charge noise at these points.These points are sometimes knows as sweet spots for charge qubit.

3.3.2 Flux Qubit

Flux qubit is also known as a persistent current qubit. The flux qubit consist of a superconducting loop which is interrupted by one Josephson junction. For flux qubit working as qubit there must be at least two state in the local minimum of the potential energy. To satisfy the above conditions require a large self inductance .For the large self inductance loop should be large in size .But large loop will be more sensitive to flactuations in the external magnetic flux which is unwanted for a circuit operating as a qubit.

To overcome the problem of inductance and loop size we use three Josephson junction in place of one. Above these three junction two are similar with Josephson energy $E_J$while the last is lesser in Josephson energy αE_J.The value of α is between the range 0.6-0.7, it determine the potential energy landscape of the circuit and make the circuit less sensitive for charge noise.

The Hamiltonian of flux qubit with three Josephson junction can be written as

$$H_{flux} = \frac{P_p^2}{2M_p} + \frac{P_m^2}{2M_m} + 2E_J(1 - cos\phi_p cos\phi_m) + \alpha E_J\left[1 - cos(2\pi\frac{\Phi_{ext}}{\Phi_0} + 2\phi_m)\right] \tag{3.15}$$

here $M_p = 2C_J(\Phi_0/2\pi)^2$,$M_m = M_p(1+2\alpha), P_p = -\iota\hbar\frac{\partial}{\partial\phi_p}$,and $P_m = -\iota\hbar\frac{\partial}{\partial\phi_m}$. $\phi_1$and ϕ_2 be the phase difference between two large junction ,they form new variable $\phi_p = \phi_1 + \phi_2$and $\phi_m = \phi_1 - \phi_2$.

The above Hamiltonian treated as describing a system having anisotropic mass and moving under a two dimentional periodic potential. As like an external voltage tuned the energy levels in charge qubit ,in flux qubit potential energy term by manage the external flux tuned the energy level. Replacing one of the junction by SQUID one can further tuned the energy level.

At the point when the ratio $\frac{\Phi_{ext}}{\Phi_0} = 0.5$ is important when taking the flux qubit:

- Potential energy term becomes symmetric at this point. Due to symmetric potential, the eigenstates of the system having well defined parities.When we go away from this point, the system loses it's symmetricity in potential and eigenstates no longer have well defined parities.

- At the value of Φ_{ext} such that $\Phi_{ext}/\Phi_0 \approx 0.5$,the two lowest energy levels are well seperated from other energy levels in the system. It makes the circuit good for qubit.The Hamiltonian for these two levels at this points can be written as
$$H = \frac{\varepsilon\sigma_z + \delta\sigma_x}{2} \tag{3.16}$$
where $\varepsilon = I_p(2\Phi_{ext} - \Phi_0)$,$\sigma_z$and $\sigma_x$are Pauli operator which are defined as $\sigma_z = |\circlearrowleft\rangle\langle\circlearrowright| - |\circlearrowright\rangle\langle\circlearrowleft|$ and $\sigma_x = |\circlearrowleft\rangle\langle\circlearrowright| + |\circlearrowright\rangle\langle\circlearrowleft|$.Here the basis states are $|\circlearrowright\rangle$ and $|\circlearrowleft\rangle$ known as circulating clockwise and anticlockwise supercurrent $I_p$respectively in the loop.Each circulating state is related to one potential well; each potential wells are attached by tunneling matrix element δ.

3.3.3 Phase Qubit

A phase qubit is a current biased Josephson junction which is operate in zero voltage state with non zero current bias. In phase qubit there is a large Josephson junction ($E_J/E_c \approx 10^6$)which is controlled by a applied bias current I_b. A " tilted washboard " potential for the circuit is set by the bias currentI_b . It tuned close to critical current I_c .

The Hamiltonian for phase qubit circuit is

$$H_{phase} = \frac{2\pi}{\Phi_0}\frac{p^2}{2C_j} - \frac{\Phi_0}{2\pi}I_b\phi - E_J cos\phi \tag{3.17}$$

where momentum p is related to the charge $Q = 2ep/\hbar$ on the capacitance of the Josephson junction.For phase qubit the resulting eigenenergies have small anharmonicity . But we know that a qubit can be defined by considering only two lowest energy levels.The ratio of Josephson junction energy and charging energy is large ,it makes the phase qubit insensitive to charge noise. Due to the symmetric point, the flux qubit and the charge qubit are insensitive to noise sources ,but in phase qubit there is no any symmetry point ,hence at any symmetric point the phase qubit is specially protected from noise sources.

3.4 Transmon Qubit

The transmission-line shunted plasma oscillation qubit is abbreviated as Transmon qubit; is one of the buzzing qubits in industry. As we already discussed qubits are sensitive particularly to charge noise and this implies true for the Cooper Pair Box, as a result, it was the least favorite of researchers. The solution to the charge noise problem of the Cooper Pair Box hedged on designing a qubit with higher-order energy levels. The transmon qubit consists of a Cooper pair box(charge qubit) shunted by a large external capacitance $C_{ext} >> C$. Now adding this extra capacitance will reduce the charging energy E_c in the circuit. This modification changes $\frac{E_J}{E_C}$ ratio from 10^{-1} to 10^2 approximately. The charge qubit goes from having well-defined n to having a well-defined ϕ. Thus the resulting energy levels are prone to fluctuation in n_g. But this comes at the cost of reduced anharmonicity of the circuit. As charge drifts across the junction are screened by capacitance, which prevents change of transition frequency between the ground and first excited state of the device.When $E_J >> E_C$ using perturbation theory in the small variable $\frac{E_C}{E_J}$ gives the energy level E_M of the circuit approximated by the following relation.
$$E_m = -E_J + \sqrt{8E_JE_C}(m + \frac{1}{2}) - \frac{E_C}{12}(6m^2 + 6m + 3) \tag{3.18}$$
We can obtain the qubit transition frequency as:

$$\omega_{01} = \frac{(\sqrt{8E_JE_C} - E_C)}{\hbar} \tag{3.19}$$

and the expression for anharmonicity as :

$$\omega_{12} - \omega_{01} = \frac{-E_C}{\hbar} \tag{3.20}$$

Where m is mth energy level Further study shows the trade off between decrease in sensitivity to noise and anharmoticity is favourable one. Because decrease in sensitivity to charge noise varies as $exp(\sqrt{E_J/E_C})$ while anharmonicty only decreases linearly in $\sqrt{E_J/E_C})$ when scaled by ω_{01}.

Now we will write a simple python code to compare anharmonicty and harmonic nature of transon and normal harmonic oscillator. We consider upto five enrgy levels. We consider the Josephson junction with energy E_J =20e9, the frequency of oscillator is 5e9. Where phase is divided from $-\pi$ to π in an interval of 100 steps.

```
Python code:
# Finding Potential energies of Quantum Harmonic Oscillator and Transmon
import matplotlib.pyplot as plt
E_J = 20e9
w = 5e9
anharm = -300e6
N_phis = 101
phis = np.linspace(-np.pi,np.pi,N_phis)
mid_idx = int((N_phis+1)/2)

U_QHO = 0.5*E_J*phis**2
U_QHO = U_QHO/w
U_transmon = (E_J-E_J*np.cos(phis))
U_transmon = U_transmon/w

# Solving for Energies of Oscillators
from qutip import destroy
N = 35
N_energies = 5
c = destroy(N)
H_QHO = w*c.dag()*c
E_QHO = H_QHO.eigenenergies()[0:N_energies]
H_transmon = w*c.dag()*c + (anharm2)*(c.dag()*c)*(c.dag()*c - 1)
E_transmon = H_transmon.eigenenergies()[0:2*N_energies]

# Printing Energies
print(E_QHO[:4])
print(E_transmon[:8])

# Plotting and Visualization

fig, axes = plt.subplots(1, 1, figsize=(6,6))
axes.plot(phis, U_transmon, '-', color='red', linewidth=3.0)
axes.plot(phis, U_QHO, '--', color='blue', linewidth=3.0)

for eidx in range(1,N_energies):
        delta_E_QHO = (E_QHO[eidx]-E_QHO[0])/w
        delta_E_transmon = (E_transmon[2*eidx]-E_transmon[0])/w
        QHO_lim_idx = min(np.where(U_QHO[int((N_phis+1)/2):N_phis] > delta_E_QHO)[0])
        trans_lim_idx = min(np.where(U_transmon[int((N_phis+1)/2):N_phis] > delta_E_transm
        trans_label, = axes.plot([phis[mid_idx-trans_lim_idx-1], phis[mid_idx+trans_lim_id
        qho_label, = axes.plot([phis[mid_idx-QHO_lim_idx-1], phis[mid_idx+QHO_lim_idx-1]],
        [delta_E_QHO, delta_E_QHO], '--', color='blue', linewidth=3.0)

axes.set_xlabel('Phase $\phi$', fontsize=24)
axes.set_ylabel('Energy Levels / $\hbar\omega$', fontsize=24)
axes.set_ylim(-0.2,5)

qho_label.set_label('QHO Energies')
trans_label.set_label('Transmon Energies')
axes.legend(loc=2, fontsize=14
```

The result has been shown in this graph. We can clearly see how energy levels are different in both cases and how Anharmonicty plays an important role. Note: One need to use qutip package in order to simulate.

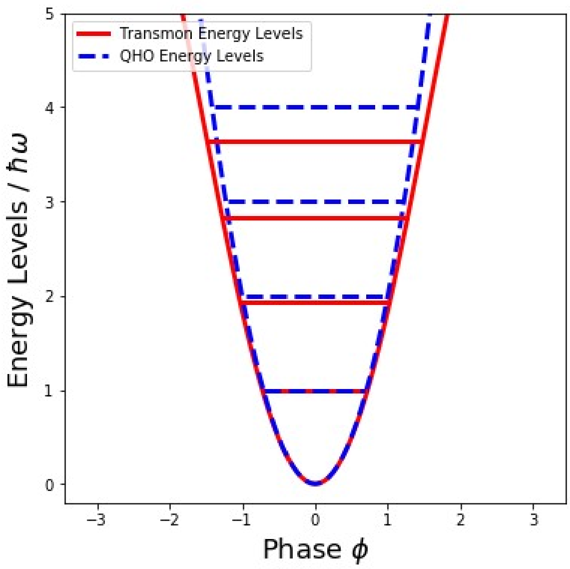

Fig. 3.5: Energy levels of Harmonic and Anharmonic Oscillators

3.5 The DiVincenzo Criteria.

Finally, we will look at the criteria which judges about the efficiency of qubits. This condition is given by a criterion known as DiVincenzo criteria. These are set of five conditions that need to be fulfilled by Josephson-junction qubits, trapped ions, or any other quantum-computing architecture. They are:

1. ***(DV1) Qubits*** : Multitple qubits should be able to fabricated on a single physical cirucit. There should not be any interference between them.
2. ***(DV2) Initialization***: it must be possible to initialize these qubits to a simple, known state, e.g., $|000...00\rangle$: Qubits are intitialized to state $|0\rangle$by convention. This is accomplished by projecting the system to ground state or by flipping $|1\rangle$to $|0\rangle$.
3. ***DV3) Gates***: it must be possible to perform both single- and two-qubit gates on the qubits with high fidelity. Single-qubit gates can be represented by rotations on the Bloch sphere; while two-qubit gates are quantum analogues of classical two-bit gates like XOR. The set of available gates must be universal, i.e., they must together enable any conceivable program to be implemented on the quantum computer.
4. ***(DV4) Readout***: It must be possible to measure the states of the qubits.
5. ***(DV5) Coherence***: The coherence times of the qubits must be long enough to allow a large number of gates to be performed in sequence before a significant loss of quantum coherence occurs.

4 Conclusion

We have studied the theoretical aspects of superconductivity, Josephson junctions, and how the Josephson junction is useful in the application of Qubit circuits. We showed how anharmonicity plays an important role in Qubit circuits by modifying the energy spacing of qubits. In the end, we studied how computational efficiency can be compared using the DiVincenzo criterion.

5 Acknowledgement

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