ODE and MVC

(ordinary Differential Equations and multi variable calculus)

1. First order ordinary Differential Equations (14M)
2. ordinary Differential Equations of higher order. (14M)
3. Multi variable calculus (Integration) (14M)
4. Vector Differentiation (14M)
5. vector Integration (14M)

UNIT - I

First order ODE — Ind, Dep

ODE → $\frac{dy}{dx}$ → 1 - D, 1 - I.

PDE → $\frac{\partial y}{\partial x \partial z}$ → 1 - D, x, z → I.

$\frac{df}{dx}$ 1 - D, ∞ - I.

Differential Eq^n :- (or).

2 types

1. ODE
2. PDE

ODE :- Eg :- $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$

$$\frac{d^3y}{dx^3} + 7\frac{dy}{dx} = \cos x$$

∴ an ODE

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0.$$

PDE :- two (or) more independent

Eg :- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ $\qquad$ $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial x}{\partial y} = 0.$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

order of Differential Eq^n :- highest derivative

Degree " " " :- It eg

which occurs

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = 3.$$

$\boxed{\text{order} = 2}$

which occurs
radicals and fractions.

Eg:-
1) $\left(\frac{d^2y}{dx^2}\right)^3 - 5x\left(\frac{d^1y}{dx}\right)^5 + 6xy = 0.$

order = 2
degree = 3

2). $\left(\frac{d^1y}{dx}\right)^2 = \tan x$ — order = 1, degree = 2.

3) $\left(\frac{d^3y}{dx^3}\right)^{4/5} \quad \left(x + \frac{d^2y}{dx^2}\right)$

order = 3
degree = 4.

$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^{4} = \left(x + \frac{d^2y}{dx^2}\right)^{5}$

4). $\left(\frac{d^2y}{dx^2} + x\right)^{3/4} = \left(\frac{d^3y}{dx^3}\right)^{1}$

order = 3
degree = 1.

5) $\left(\frac{d^3y}{dx^3}\right)^{3/4} + x = 2.$

$\left(\frac{d^3y}{dx^3}\right)^{3/4} = 2 - x$

order = 3.
degree = 3.

$(\quad) + (\quad) + (\quad) + (\quad) =$

6). $\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right) = \left(\frac{d^3y}{dx^3}\right)^{1/4}$

7). $\left(\frac{d^2y}{dx^2}\right)^{1} + y\cos x + y = \tan x$

order = 2
degree = 1

$\boxed{\text{order} = 2}$
degree = 3.

$\left(\frac{d^2y}{dx^2}\right)^{1} + \left(\frac{d^1y}{dx}\right)^3 = 2.$

$\frac{d^3y}{dx^3}$

Eg:- $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

sol:-
$y - x\frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Sq on b.s.

$\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$

$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} \quad \left(\frac{dy}{dx}\right)^2 = 0$

$$y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} - 1 - \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(x^2-1) - 2xy\frac{dy}{dx} + (y^2-1) = 0$$

order = 1 ; degree = 2.

Formation of a differential Eqⁿˢ :-

$$f(x, y, c_1, c_2, \dots, c_n) = 0 \quad \text{——— (1)}$$

$c_1, c_2, \dots, c_n$ by (1) 'n' times.

Here x is independent variable

y is dependent "

$$f\left(x, y, \frac{dy}{dx}, c_1, c_2, \dots, c_n\right) = 0 \quad \text{——— (2)}$$

Again (2) 2nd time w.r.to 'x', we get

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, c_1, c_2, \dots, c_n\right) = 0 \quad \text{——— (3)}$$

$$f(x, y, y', y'', y''', y^{iv}, \dots, y^n, c_1, c_2, c_3, \dots, c_n) = 0$$

$$f(y^n, y^{(n-1)}, \dots, y', y, x) = 0$$

① $y = ae^x + be^{-x}$ ——— (1)

Solⁿ:- Here a, b are arbitrary constants

Diff.. (1) w.r.to 'x'

$$\frac{dy}{dx} = ae^x + be^{-x}(-1)$$

$$\frac{dy}{dx} = ae^x - be^{-x} \quad \text{——— (2)}$$

Again diff.. Eqⁿ(2) w.r.to 'x'

$$\frac{d^2y}{dx^2} = ae^x - be^{-x}(-1)$$

$$\frac{d^2y}{dx^2} = \underbrace{ae^x + be^{-x}}_{\text{from (1)}} \quad \text{——— (3)}$$

$$\frac{d^2y}{dx^2} = y \qquad [\because y = ae^x + be^{-x}]$$

(or)

$$y'' = y$$

$$\boxed{y'' - y = 0}$$

② $xy = ae^x + be^{-x} + x^2$

Solⁿ:- $xy - x^2 = ae^x + be^{-x}$ ——— (1)

Here a, b are arbitrary constants

Diff.. (1) w.r.to 'x'

$$x\frac{dy}{dx} + y(1) - 2x = ae^x + be^{-x}(-1)$$

$$x\frac{dy}{dx} + y - 2x = ae^x - be^{-x} \quad \text{——— (2)}$$

Diff.. (2) w.r.to 'x'

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) + \frac{dy}{dx} - 2(1) = ae^x - be^{-x}(-1)$$

$$\Rightarrow xy'' + y' + y' - 2 = \underbrace{ae^x + be^{-x}}$$

$$\Rightarrow xy'' + 2y' - 2 = xy - x^2 \quad [\because (1)]$$

$$\Rightarrow xy'' + 2y' - xy + x^2 - 2 = 0.$$

③ $\log\left(\frac{y}{x}\right) = cx$

Solⁿ:- $\log y - \log x = cx$ ——— (1)

where 'c' is an arbitrary constant.

Diff.. (1) w.r.to 'x'

$\Rightarrow \frac{1}{y}\frac{dy}{dx} - \frac{1}{x} = c$

$\Rightarrow \frac{1}{y}(y') - \frac{1}{x} = \left(\frac{\log y - \log x}{x}\right) \quad [\because (1)]$

$\Rightarrow \left(\frac{y'}{y} - \frac{1}{x}\right)x = \log y - \log x$

$\Rightarrow \left(\frac{y'}{y} - \frac{1}{x}\right)x = \log(y/x)$

(4). $\sin^{-1}x + \sin^{-1}y = c.$

sol:- $y' = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

(5). $y = Ae^{-2x} + Be^{5x}$ —— (1)

sol:- 'A' & 'B' are arbitrary constants

Diff.. (1) w.r.to 'x'

$\frac{dy}{dx} = Ae^{-2x}(-2) + Be^{5x}(5)$

$y' = -2Ae^{-2x} + \underline{5B}e^{5x}$

$y' = \underbrace{-2Ae^{-2x} - 2Be^{5x}} + 7Be^{5x}$

$y' = -2\left[\underbrace{Ae^{-2x} + Be^{5x}}\right] + 7Be^{5x}$

$y' = -2y + 7Be^{5x}$ —— (2) $[\because (1)] \Rightarrow y' + 2y = 7Be^{5x}$

Again diff.. (2) w.r.to 'x'

$y'' = -2y' + [7Be^{5x}](5)$

$y'' = -2y' + 5(y' + 2y) \quad [\because (2)]$

$y'' + 2y' - 5y' - 10y = 0$

$\boxed{y'' - 3y' - 10y = 0}$

(or).

Given $y = Ae^{-2x} + Be^{5x}$ —— (1).

Diff.. w.r.to 'x'

$y' = -2Ae^{-2x} + 5Be^{5x}$ —— (2).

Again Diff.. w.r.to 'x'

$y'' = 4Ae^{-2x} + 25Be^{5x}$ —— (3)

$$\Rightarrow \begin{vmatrix} e^{-2x} & e^{5x} & -y \\ -2e^{-2x} & 5e^{5x} & -y' \\ 4e^{-2x} & 25e^{5x} & -y'' \end{vmatrix} = 0$$

+ − +

$$\Rightarrow \begin{vmatrix} -2e^{-2x} & 5e^{5x} & -y' \\ 4e^{-2x} & 25e^{5x} & -y'' \end{vmatrix}$$

$$\Rightarrow e^{-2x} e^{5x} \begin{vmatrix} 1 & 1 & -y \\ -2 & 5 & -y' \\ 4 & 25 & -y'' \end{vmatrix} = 0$$

$$\Rightarrow (-5y'' + 25y') - 1(2y'' + 4y') - y(-50 - 20) = 0$$

$$\Rightarrow -5y'' + 25y' - 2y'' - 4y' - \quad = 0$$

$$\Rightarrow \boxed{y'' - 3y' - 10y = 0}$$

(6). $y^2 = (x - c)^3$

Non Homogenous D.E (N.H.D.E) :- N.H.D.E of 1st order & 1st degree in 'x' and 'y' if a_1, b_1, c_1 and a_2, b_2, c_2 are constants and atleast one of 'c_1' and 'c_2' is not zero, then $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ is N.H.D.E.

Procedure for solving N.H.D.E :-

consider $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ —— (1)

case (1) :- if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ —

put $\left.\begin{array}{l} x = X + h \Rightarrow X = x - h \\ y = Y + K \Rightarrow Y = y - k \end{array}\right\}$ —— (2)

then $\frac{dy}{dx} = \frac{dY}{dX}$ —— (3)

(2) & (3) in (1)

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+K) + c_1}{a_2(X+h) + b_2(Y+K) + c_2}$$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Let us Assume $a_1h + b_1k + c_1 = 0$

$a_2h + b_2k + c_2 = 0$

then solve 'h & k' values $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

$\therefore$ H.D.E $X = x - h$; $Y = y - k$,

case (2) :- If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then we take

$$\boxed{\frac{a_2}{a_1} = \frac{b_2}{b_1} = k.}$$ where 'k' is a constant

$\Rightarrow a_2 = ka_1 ; b_2 = kb_1$

sub. a_2, b_2 values in (1), we get

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} = \frac{a_1x+b_1y+c_1}{ka_1x+kb_1y+c_2} = \frac{a_1x+b_1y+c_1}{k\underbrace{(a_1x+b_1y)}+c_2}$$

$z = a_1x+b_1y$

case (3) :- $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$ $\Rightarrow$ $\boxed{\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \frac{1}{m}}$

$$\frac{dy}{dx} = m.$$

$\Rightarrow dy = m\,dx$ $\Rightarrow \int dy = \int m\,dx$ $\Rightarrow \boxed{y = mx + c}$ //

I. on. b

① $(x+y-1)\frac{dy}{dx} = x-y+2$

$$\frac{kx-ky+2}{kx+ky-1}.$$

$$\frac{\cancel{k}(x-y+2/k)}{\cancel{k}(x+y-1/k)}$$

sol:- $\frac{dy}{dx} = \frac{x-y+2}{x+y-1}$ —— (1).

$\therefore$ N.H.D.E. $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $a_1 = 1 ; b_1 = -1 ; c_1 = 2$

$a_2 = 1 ; b_2 = 1 ; c_2 = -1$

$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ $\left(\Rightarrow \frac{1}{1} \neq \frac{-1}{1}\right)$

put $x = X+h$ —(2) ; $y = Y+k$ —(3) & $\frac{dy}{dx} = \frac{dY}{dX}$ —(4).

$\Rightarrow \boxed{X = x-h}$ $\boxed{Y = y-k}$

(2), (3) & (4) in (1).

$$\frac{dY}{dX} = \frac{(X+h)-(Y+k)+2}{(X+h)+(Y+k)-1} \Rightarrow \frac{dY}{dX} = \frac{X-Y+(h-k+2)}{X+Y+(h+k-1)} \quad —(5)$$

$h-k+2=0$

$h+k-1=0.$

solving above two Eq $\boxed{h = -1/2}$; $\boxed{k = 3/2}$

Sub. 'h' & 'k' values in (5).

$$\frac{dY}{dX} = \frac{X-Y+0}{X+Y+0} \Rightarrow \boxed{\frac{dY}{dX} = \frac{X-Y}{X+Y}} \quad —— (6)$$

H.D.E.

Since $f(X,Y) = \frac{X-Y}{X+Y}$

$$f(kX, kY) = \frac{kX-kY}{kX+kY}$$

$$f(kX,kY) = k^0 f(X,Y)$$

$$f(kx, ky) = k^0 f(x,y)$$

put $\boxed{y = v\cancel{x}}$ — (7) $\Rightarrow$ $\boxed{v = y/\cancel{x}}$

Diff. (7)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{— (8)}$$

sub. (8) ~~&~~ (7). in (6).

$$(6) \Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y} \Rightarrow v + x\frac{dv}{dx} = \frac{x - vx}{x + vx}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v-v-v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \left(\frac{1-2v-v^2}{1+v}\right) \Rightarrow \cancel{x}\frac{dv}{dx} = -\frac{(v^2+2v-1)}{1+v}$$

$$\Rightarrow \left(\frac{1+v}{v^2+2v-1}\right)dv = -\frac{1}{x}dx$$

$\Rightarrow$ Integrating on b.sides

$$\Rightarrow \int \frac{1+v}{v^2+2v-1}\,dv = -\int \frac{1}{\cancel{x}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{2+2v}{v^2+2v-1}\,dv = -\log|\cancel{x}| + \log|c|$$

$$\Rightarrow \frac{1}{2}\log|v^2+2v-1| = \log\left(\frac{c}{\cancel{x}}\right)$$

$$\Rightarrow \log(v^2+2v-1) = 2\log(c/\cancel{x})$$

$$\Rightarrow \log(v^2+2v-1) = \log(c/\cancel{x})^2$$

$$\Rightarrow v^2+2v-1 = \frac{c^2}{\cancel{x}^2}$$

$$\Rightarrow \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{c^2}{x^2} \qquad \left(\because v = \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y-3/2)^2}{(x+1/2)^2} + \frac{2(y-3/2)}{(x+1/2)} - 1 = \frac{c^2}{(x+1/2)^2} \qquad \left(\begin{array}{l}\because X = x-h = x+1/2 \\ \because Y = y-k = y-3/2\end{array}\right)$$

(or)

$$\Rightarrow (y-3/2)^2 + 2(y-3/2)(x+1/2) - (x+1/2)^2 = c^2 \quad //$$

② $\dfrac{x+2y+1}{2x+4y+3} = \dfrac{dy}{dx}.$

1 ~~st~~ $a_1 = 1$; $a_2 = 2$
$b_1 = 2$; $b_2 = 4$
$c_1 = 1$; $c_2 = 3$

$$\therefore \boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}} \qquad \left(\frac{1}{2} = 2/4\right)$$

$\Rightarrow \dfrac{dy}{}$ $x+2y+1$

$$\frac{\quad}{dx} = \frac{\quad}{2x+4y+3}$$

$$\frac{dy}{dx} = \frac{x+2y+1}{2(x+2y)+3} \quad \text{---} (1)$$

let $x+2y = z$ ——— (2)

Diff. on b.s. w.r.to 'x'

$$1 + 2\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{dz}{dx}-1}{2} \quad \text{---} (3).$$

Now sub. (2) & (3) in (1)

$$\Rightarrow \frac{\frac{dz}{dx}-1}{2} = \frac{z+1}{2z+3} \Rightarrow \frac{dz}{dx} - 1 = \frac{2z+2}{2z+3}.$$

$$\Rightarrow \frac{dz}{dx} = \frac{2z+2}{2z+3} + 1 \Rightarrow \frac{dz}{dx} = \frac{2z+2+2z+3}{2z+3}$$

$$\Rightarrow \frac{dz}{dx} = \frac{4z+5}{2z+3}.$$

$$\Rightarrow \left(\frac{2z+3}{4z+5}\right) dz = dx$$

$$\Rightarrow \int \frac{2z+3}{4z+5}\, dz = \int dx$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{4z+6}{4z+5}\right) dz = x + c$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{(4z+5)+1}{4z+5}\right) dz = x+c$$

$$\Rightarrow \frac{1}{2}\int \frac{\cancel{4z+5}}{\cancel{4z+5}}\, dz + \frac{1}{2}\int \frac{1}{4z+5}\, dz = x+c$$

$$\Rightarrow \frac{1}{2}z + \frac{1}{8}\int \frac{4}{4z+5}\, dz = x+c$$

$$\Rightarrow \frac{1}{2}z + \frac{1}{8}\log|4z+5| = x+c$$

$$\Rightarrow \frac{1}{2}(x+2y) + \frac{1}{8}\log(4x+8y+5) = x+c$$

(or)

$$\Rightarrow 4(x+2y) + \log(4x+8y+5) = 8x+8c \quad //$$

Exact D.E :- Let $M(x,y)\,dx + N(x,y)\,dy = 0$ (or) $M dx + N dy = 0$ be M, N are real valued functions for some x, y. $M dx + N dy = 0$ is said to an Exact D.E (E.D.E). If $\exists$ a f^n 'f' such that $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

Eg:- D.E $\underset{M}{2xy}\,dx + \underset{N}{x^2}\,dy = 0$ is an Exact D.E

Since $\exists$ a f^n $f(x,y) = x^2 y$ such that

$$\frac{\partial f}{\quad} = 2xy \; ; \; \frac{\partial f}{\quad} = x^2(1)$$

$$\frac{dy}{dx} = \frac{(x^2+y)}{(y^2+x)}$$

Since $\exists$ a f^n $f(x,y) = x^2y$ such that

$$\frac{\partial f}{\partial x} = 2xy \; ; \; \frac{\partial f}{\partial y} = x^2(1) = x^2$$

$\downarrow$ M $\qquad$ $\downarrow$ N

$\frac{dy}{dx} = \frac{(x+y)}{(y+x)}$

$\Rightarrow \int x\,dx - \int y\,dy$

② $\underset{N}{x}dy + \underset{M}{y}dx = 0$ $\qquad f(x,y) = xy$. since $\frac{\partial f}{\partial x} = y = M$; $\frac{\partial f}{\partial y} = x = N$

(or)

$\int M dx + \int N dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\downarrow$

non-Exact

If $M(x,y)$ and $N(x,y)$ are. $M dx + N dy = 0$.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Soln of an E.D.E :- $\int M dx + \int N dy = C$.

y constant $\qquad$ donot take x-terms

① $\frac{dy}{dx} = -\left(\frac{x+2y-1}{2x+y-2}\right)$

Sol:- Given $(2x+y-2)\,dy = -(x+2y-1)\,dx$

$\Rightarrow (x+2y-1)\,dx + (2x+y-2)\,dy = 0$.

$M \quad dx + N \quad dy = 0$.

Here $M = x+2y-1$; $N = 2x+y-2$.

$$\frac{\partial M}{\partial y} = 0+2(1)-0 = 2 \quad ; \quad \frac{\partial N}{\partial x} = 2+0-0 = 2$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\therefore$ Given D.E is called an E.D.E

G.S of given D.E is

$\Rightarrow \int M dx + \int N dy = C$

y constant $\qquad$ donot take x-terms

$\Rightarrow \int (x+2y-1)\,dx + \int (2x+y-2)\,dy = C$.

$\Rightarrow \int x\,dx + 2y\int 1\,dx - \int 1\,dx + \int y\,dy - 2\int dy = C$.

$\Rightarrow \frac{x^2}{2} + 2y(x) - x + \frac{y^2}{2} - 2(y) = C$

$\Rightarrow \frac{x^2}{2} + 2xy - x + \frac{y^2}{2} - 2y = C$

(or)

$\Rightarrow x^2 + 4xy - 2x + y^2 - 4y = 2C$ //

③. $\left(x e^{xy} + 2y\right)\frac{dy}{dx} + y e^{xy} = 0$.

Sol:-

$$\Rightarrow \frac{(x e^{xy}+2y)\,dy + y e^{xy}\,dx}{dx} = 0$$

$$\Rightarrow (x e^{xy}+2y)\,dy + y e^{xy}\,dx = 0$$

$$\Rightarrow (y e^{xy})\,dx + (x e^{xy}+2y)\,dy = 0$$

$$M\,dx + N\,dy = 0.$$

Here $M = y e^{xy}$

$N = x e^{xy} + 2y$

$$\frac{\partial M}{\partial y} = y e^{xy}(x) + e^{xy}(1)$$

$$= e^{xy}(xy+1)$$

$$\frac{\partial N}{\partial x} = x e^{(xy)}(y) + e^{xy}(1) + 0$$

$$= e^{xy}(xy+1)$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

G.S is $\boxed{\int M\,dx + \int N\,dy = C}$ 'y' constant; donot take 'x' term

$$\Rightarrow \int (y e^{xy})\,dx + \int (x e^{xy} + 2y)\,dy = C.$$

$$\Rightarrow y \int e^{xy}\,dx + \int 2y\,dy = C$$

$$\Rightarrow \cancel{y}\frac{e^{xy}}{\cancel{y}} + \cancel{2}\frac{y^2}{\cancel{2}} = C$$

$$\Rightarrow \boxed{e^{xy} + y^2 = C}$$

③. $$\frac{dy}{dx} = -\left(\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}\right)$$

Sol:-

$M = y\cos x + \sin y + y$; $N = \sin x + x\cos y + x.$

$$\frac{\partial M}{\partial y} = \cos x\,(1) + \cos y + 1 \quad ; \quad \frac{\partial N}{\partial x} = \cos x + \cos y\,(1) + 1$$

$$= \cos x + \cos y + 1 \qquad\qquad = \cos x + \cos y + 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

G.S is $\int M\,dx + \int N\,dy = C \Rightarrow \int (y\cos x + \sin y + y)\,dx + \int (\sin x + x\cos y + x)\,dy = C$

$$\Rightarrow y\int \cos x\,dx + \sin y \int 1\,dx + y\int 1\,dx + 0 = C.$$

$$\Rightarrow y\sin x + \sin y\,(x) + y\,(x) = C$$

$$\Rightarrow y\sin x + x\sin y + xy = C.$$

④. $$x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x.$$

Sol:- $x^3\sec^2 y\,dy + 3x^2\tan y\,dx = \cos x\,dx$

$$\Rightarrow x^3\sec^2 y\,dy + 3x^2\tan y\,dx - \cos x\,dx = 0$$

$$\Rightarrow \underbrace{(3x^2\tan y - \cos x)}_{M}\,dx + \underbrace{(x^3\sec^2 y)}_{N}\,dy = 0$$

Here $M = 3x^2\tan y - \cos x$; $N = x^3\sec^2 y$

$\underline{\text{che}}$ $M = 3x^2\tan y - \cos x$; $N = x^3\sec^2 y$

$\frac{\partial M}{\partial y} = 3x^2\sec^2 y - 0$; $\frac{\partial N}{\partial x} = 3x^2\sec^2 y$

$= 3x^2\sec^2 y$ $\qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

G.S is $\int M dx + \int N dy = C \Rightarrow \int (3x^2\tan y - \cos x) dx + \int (x^3\sec^2 y)\, dy = C$

$\Rightarrow 3\tan y \int x^2 dx - \int \cos x\, dx + 0 = C$

$\Rightarrow \cancel{3}\tan y \frac{x^3}{\cancel{3}} - \sin x = C$

$\Rightarrow x^3\tan y - \sin x = C$

(3). Solve $\underbrace{(e^y+1)\cos x}_{M}\, dx + \underbrace{e^y\sin x}_{N}\, dy = 0$

$\underline{\text{s/n}}$ $M = (e^y+1)\cos x$; $N = e^y\sin x$

$\frac{\partial M}{\partial y} = \cos x\,(e^y)$; $\frac{\partial N}{\partial x} = e^y\cos x.$ $\qquad \therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\therefore$ E.D.E.

G.S is $\int M dx + \int N dy = C \Rightarrow \int ((e^y+1)\cos x)\, dx + \int (e^y\sin x)\, dy = C$

$\Rightarrow \int e^y\cos x\, dx + \int \cos x\, dx + \int \underline{e^y\sin x}\, dy = C$

$\Rightarrow e^y\int \cos x\, dx + \int \cos x\, dx + 0 = C$

$\Rightarrow e^y\sin x + \sin x = C$ (or)

$\Rightarrow \sin x\,(e^y+1) = C.$

H.W: ① $\underbrace{2xy}_{N}\, dy - \underbrace{(x^2-y^2+1)}_{M}\, dx = 0$

② $\underbrace{(x^2-y^2)}_{M}\, dx = \underline{2xy}\, dy.$

$\frac{x^3}{3} - y^2x + x = C.$

$x^3 - 3xy^2 = 3C.$

Non-Exact D.E :- $\boxed{Mdx + Ndy = 0}$ $\qquad$ E.D.E

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Integrating Factor (I.F) :- Let $Mdx + Ndy = 0$ be not on E.D.E,

$u(x,y) \neq 0$, $u(x,y)$ of $Mdx + Ndy = 0.$

Method 1 :- $Mdx + Ndy = 0.$:-

①. $d(xy) = x\,dy + y\,dx$

②. $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

③. $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{}$

⑤. $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

⑥. $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

$$\frac{\quad}{y^2} \qquad\qquad \left[\;\left(y\right)\right] \quad \frac{\quad}{xy}$$

3. $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

4. $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$

9. $d\left[\log(xy)\right] = \frac{x\,dy + y\,dx}{xy}$

10. $d\left[\log(x^2+y^2)\right] = \frac{2(x\,dx + y\,dy)}{x^2+y^2}$

11. $d\left(\frac{e^x}{y}\right) = \frac{y e^x\,dx - e^x\,dy}{y^2}.$

7. $d\left[\tan^{-1}(y/x)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

8. $d\left[\tan^{-1}(x/y)\right] = \frac{y\,dx - x\,dy}{x^2+y^2}$

1. $x\,dx + y\,dy = \frac{x\,dy - y\,dx}{x^2+y^2}$

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Sol:- $d\left(\frac{x^2+y^2}{2}\right) = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$

Integrating on b.s.

$\int d\left(\frac{x^2+y^2}{2}\right) = \int d\left[\tan^{-1}(y/x)\right]$

$\Rightarrow \frac{x^2+y^2}{2} = \tan^{-1}(y/x) + c \Rightarrow \frac{x^2+y^2}{2} - \tan^{-1}(y/x) = c.$

2. $x\,dx + y\,dy = a^2\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right)$

Sol:- $d\left(\frac{x^2+y^2}{2}\right) = a^2\, d\left[\tan^{-1}(y/x)\right]$

Integrating on b.s.

$\int d\left(\frac{x^2+y^2}{2}\right) = a^2 \int d\left[\tan^{-1}(y/x)\right]$

$\frac{x^2+y^2}{2} = a^2 \tan^{-1}(y/x) + c$ //

3. $(y - x^2)\,dx + (x^2 \cot y - x)\,dy = 0$

Sol:- $y\,dx - x^2\,dx + x^2 \cot y\,dy - x\,dy = 0$

$\Rightarrow (y\,dx - x\,dy) = x^2\,dx - x^2 \cot y\,dy$

$\Rightarrow$ Dividing 'x^2' on b.s.

$\Rightarrow \frac{y\,dx - x\,dy}{x^2} = dx - \cot y\,dy$

$\Rightarrow -\frac{(x\,dy - y\,dx)}{x^2} = dx - \cot y\,dy$

$\Rightarrow -d\left(\frac{y}{x}\right) = dx - \cot y\,dy$

Integrating on b.s

$\Rightarrow -\int d(y/x) = \int dx - \int \cot y\,dy$

$\Rightarrow -\left(\frac{y}{x}\right) = x - \log|\sin y| + c$

(or).

$\Rightarrow x + \frac{y}{x} - \log(\sin y) = c$

4. $x\,dy - y\,dx = xy^2\,dx.$

Sol:- Dividing "y^2" on b.s.

$\frac{x\,dy - y\,dx}{y^2} = \frac{x\cancel{y^2}}{\cancel{y^2}}\,dx$

$\Rightarrow -\frac{(y\,dx - x\,dy)}{y^2} = x\,dx$

$\Rightarrow -\int d\left(\frac{x}{y}\right) = \int x\,dx$

$\Rightarrow -\frac{x}{y} = \frac{x^2}{2} + C$

$\rightarrow \frac{x^2}{2} + \frac{x}{y} = C$ //

⑤. $y\left(2x^2y + e^x\right)dx = \left(e^x + y^3\right)dy$

sln:- $2x^2y^2\,dx + ye^x\,dx = e^x\,dy + y^3\,dy$

$\Rightarrow 2x^2y^2\,dx + ye^x\,dx - e^x\,dy - y^3\,dy = 0$

Dividing with 'y^2' on b.s.

$\Rightarrow 2x^2\,dx + \left(\frac{ye^x\,dx - e^x\,dy}{y^2}\right) - y\,dy = 0$

$\Rightarrow 2x^2\,dx + d\left(\frac{e^x}{y}\right) - y\,dy = 0$

Applying integrating on b.s.

$\Rightarrow \int 2x^2\,dx + \int d\left(\frac{e^x}{y}\right) - \int y\,dy = 0$

$\Rightarrow 2\frac{x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = C$ //

HW.

⑥. $\frac{y(xy + e^x)\,dx - e^x\,dy}{y^2} = 0. \Longrightarrow \frac{x^2}{2} + \frac{e^x}{y} = C.$

⑦. $y\,dx + x\,dy + xy\left(y\,dx - x\,dy\right) = 0.$

sln:- $\left(y\,dx + x\,dy\right) + xy^2\,dx - x^2y\,dy = 0$

$xy\left(y\,dx - x\,dy\right)$

$\int xy\,d\left[\log(yx)\right]$

$\Rightarrow$ Dividing with $\frac{1}{x^2y^2}$

$\Rightarrow \frac{y\,dx + x\,dy}{x^2y^2} + \frac{\cancel{x}\cancel{y^2}}{x^{\cancel{2}}\cancel{y^2}}\,dx - \frac{\cancel{x^2}y}{\cancel{x^2}y^{2}}\,dy = 0$

$d\left(\frac{-1}{xy}\right) + \frac{1}{x}\,dx - \frac{1}{y}\,dy = 0$

$= -d\left(\frac{1}{xy}\right)$

$= -\left[\frac{xy(0) - (x\,dy + y\,dx)}{(xy)^2}\right]$

$= +\left(\frac{x\,dy + y\,dx}{(xy)^2}\right)$

Integrating on b.s.

$-\int d\left(\frac{1}{xy}\right) + \int \frac{1}{x}\,dx - \int \frac{1}{y}\,dy = 0$

$-\frac{1}{xy} + \log x - \log y = 0$ // (or) $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = 0$ //

Method 2 :- $M\,dx + N\,dy = 0$.

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a Homogeneous D.E &

$Mx + Ny \neq 0$ and $M\,dx + N\,dy = 0$ is not ~~an E.D.E~~, then

$Mx+Ny \neq 0$ and $Mdx+Ndy=0$ is not ~~an E.D.E~~, then

$\boxed{\dfrac{1}{Mx+Ny}}$ is an integrating factor (I.F) of $Mdx+Ndy=0$.

1). $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

2). $f(kx,ky) = k^0 f(x,y)$.

3). $Mx+Ny \neq 0$. (4). $\dfrac{1}{Mx+Ny} = $ I.F (5). multiply. (1).

(1) Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$. —— (1) $\Rightarrow \dfrac{dy}{dx} = \dfrac{x^2y}{x^3+y^3}$

Sol: $M\,dx + N\,dy = 0$.

where $M = x^2y$; $N = -(x^3+y^3)$

$\dfrac{\partial M}{\partial y} = x^2(1) = x^2$; $\dfrac{\partial N}{\partial x} = -3x^2 - 0 = -3x^2$

$\therefore \boxed{\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}}$.

$\therefore$ Given D.E is not a E.D.E.

$\Rightarrow$ Let $f(x,y) = \dfrac{dy}{dx} = \dfrac{x^2y}{x^3+y^3}$ ✓

$f(kx,ky) = \dfrac{(k^2x^2)(ky)}{k^3x^3+k^3y^3} = \dfrac{\cancel{k^3}\,x^2y}{\cancel{k^3}(x^3+y^3)} = k^0 f(x,y)$.

$\therefore f(kx,ky) = k^0 f(x,y)$.

$\therefore$ eqn (1) is a H.D.E.

$\Rightarrow$ check $Mx+Ny = \cancel{x^3y} - \cancel{x^3y} - y^4$

$= -y^4 \neq 0$.

$I.F = \dfrac{1}{Mx+Ny} = \dfrac{1}{-y^4} = \dfrac{-1}{y^4}$.

Now multiplying eqn (1) with '$\dfrac{-1}{y^4}$', we get—

$$-\frac{x^2y}{y^4}\,dx + \frac{(x^3+y^3)}{y^4}\,dy = 0.$$

$$\Rightarrow \left(\frac{-x^2}{y^3}\right) dx + \left(\frac{x^3}{y^4}+\frac{1}{y}\right) dy = 0. \quad \text{———} \quad (2)$$

which is in the form of

$M_1\, dx + N_1\, dy = 0.$ Here $M_1 = \frac{-x^2}{y^3}$; $N_1 = \frac{x^3}{y^4}+\frac{1}{y}$

$= -x^2 y^{-3}$

$$\frac{\partial M_1}{\partial y} = -x^2(-3)y^{-4} \quad ; \quad \frac{\partial N_1}{\partial x} = \frac{1}{y^4}(3x^2)+0$$

$$= \frac{3x^2}{y^4} \qquad\qquad = \frac{3x^2}{y^4}$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ Eqn (2) is an E.D.E.

$\therefore$ G.S is $\int M_1\, dx + \int N_1\, dy = C.$

y constant. do not take x-terms

$$\int\left(\frac{-x^2}{y^3}\right) dx + \int\left(\frac{x^3}{y^4}+\frac{1}{y}\right) dy = C$$

$$\Rightarrow -\frac{1}{y^3}\int x^2\, dx + 0 + \int\frac{1}{y}\, dy = C$$

$$\Rightarrow -\frac{1}{y^3}\frac{x^3}{3} + \log y = C \quad \text{//.}$$

(or)

$$\log y - \frac{x^3}{3y^3} = C \quad \text{//}$$

2. $xy\,dx - (x^2+2y^2)\,dy = 0.$ —— (1) $\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+2y^2}$

Sol:

$$\frac{\partial M}{\partial y} = x \quad ; \quad \frac{\partial N}{\partial x} = -2x.$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\therefore$ (1) is not an E.D.E.

$\Rightarrow$ Let $f(x,y) = \frac{dy}{dx} = \frac{xy}{x^2+2y^2}$

$$f(kx,ky) = k^0 f(x,y)$$

$\therefore$ Eqn (1) is a H.D.E.

$\Rightarrow Mx+Ny = -2y^3 \neq 0.$

$$I.F = \frac{1}{Mx+Ny} = \frac{-1}{2y^3}$$

Now multiplying Eqn (1) with $\frac{-1}{2y^3}$ on b.s., we get -

Now multiplying Eqn (1) with $\frac{-1}{2y^3}$ on b.s., we get -

$$-\left(\frac{xy}{2y^3}\right)dx + \left(\frac{x^2+2y^2}{2y^3}\right)dy = 0$$

$$\Rightarrow \left(\frac{-x}{2y^2}\right)dx + \left(\frac{x^2}{2y^3} + \frac{\cancel{2}y^{\cancel{2}}}{\cancel{2}y^{\cancel{3}}}\right)dy = 0$$

$$\Rightarrow \left(\frac{-x}{2y^2}\right)dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0.$$

$$M_1 \quad dx + \quad N_1 \quad dy = 0.$$

Ans - $\log y - \dfrac{x^2}{4y^2} = C.$

Method : III :- (To find an I.F of $Mdx + Ndy = 0$)

$Mdx + Ndy = 0$ is (2) N.H.D.E , (1) N.E.D.E.

(3) $y\, f(x,y)dx + x\, g(x,y)dy = 0$

$$I.F = \frac{1}{Mx - Ny}, \text{ where } Mx - Ny \neq 0.$$

Rough

(1).

(2). $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ (1) & Homogeneous (2)

$$I.F = \frac{1}{Mx + Ny}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y(x^2y^2+2)}{x(2-2x^2y^2)}$$

(1). $y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$ ——— (1)

Sol:- $\quad M \quad dx + \quad N \quad dy = 0.$

where $M = y(x^2y^2+2)$ $\quad ; \quad N = x(2-2x^2y^2)$

$= y^3x^2 + 2y \quad\quad = 2x - 2x^3y^2$

$$\frac{\partial M}{\partial y} = x^2(3y^2) + 2(1) \quad ; \quad \frac{\partial N}{\partial x} = 2 - 2y^2(3x^2)$$

$$= 3x^2y^2 + 2 \quad\quad = 2 - 6x^2y^2$$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$\therefore$ Given D.E is non-Exact D.E.

Let $f(x,y) = \dfrac{dy}{dx} = \dfrac{-y(x^2y^2+2)}{x(2-2x^2y^2)}$

$$f(kx,ky) = -\frac{(\cancel{k}y)(k^2x^2k^2y^2+2)}{\cancel{k}x(2-2k^2x^2k^2y^2)}$$

$$= \frac{-y(k^4x^2y^2+2)}{x(2-2k^4x^2y^2)}$$

$$f(kx,ky) \neq k^0 f(x,y)$$

$\therefore$ Given D.E is n-H-D.E.

(1), is in the form of $y\, f(x,y)\,dx + x\, g(x,y)\,dy = 0$.

Now check $Mx - Ny = (y^3x^2 + 2y)x - (2x - 2x^3y^2)y$

$= x^3y^3 + \cancel{2yx} - \cancel{2xy} + 2x^3y^3$

$= 3x^3y^3 \neq 0$

$\therefore$ I.F $= \dfrac{1}{Mx - Ny} = \dfrac{1}{3x^3y^3}$

multiplying I.F with (1).

$$(1) \Rightarrow \frac{1}{3x^3\cancel{y^3}}\left(\cancel{y}(x^2y^2+2)\right)dx + \frac{1}{3\cancel{x^3}y^3}\left(\cancel{x}(2-2x^2y^2)\right)dy = 0$$

$$\Rightarrow \underbrace{\left(\frac{x^2y^2+2}{3x^3y^2}\right)}_{M_1} dx + \underbrace{\left(\frac{2-2x^2y^2}{3x^2y^3}\right)}_{N_1} dy = 0. \qquad (2)$$

which is in the form of

$M_1 dx + N_1 dy = 0$

Here $M_1 = \dfrac{x^2y^2+2}{3x^3y^2}$; $N_1 = \dfrac{2-2x^2y^2}{3x^2y^3}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{3x^3}\frac{\partial}{\partial y}\left(\frac{x^2y^2+2}{y^2}\right)$$

$$= \frac{1}{3x^3}\left[\frac{y^2(2x^2y+0) - (x^2y^2+2)(2y)}{y^4}\right] = \frac{1}{3x^3y^4}\left[\cancel{2x^2y^3} - \cancel{2y^3x^2} - 4y\right]$$

$$= \frac{-4y}{3x^3y^{\cancel{4}}} = \frac{-4}{3x^3y^3}$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{3y^3}\frac{\partial}{\partial x}\left[\frac{2-2x^2y^2}{x^2}\right]$$

$$= \frac{1}{3y^3}\left[\frac{x^2(0-4y^2x) - (2-2x^2y^2)(2x)}{x^4}\right]$$

$$= \frac{1}{3x^4y^3}\left[\frac{\cancel{4x^3y^2} - 4x + \cancel{4y^2x^3}}{1}\right] = \frac{-4}{3x^3y^3}.$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ (2) is an E.D.E.

G.S is $\int\limits_{y\text{-constant}} M_1\,dx + \int\limits_{\text{don't take } x\text{-term}} N_1\,dy = C.$

$$\Rightarrow \int\limits_{y\text{-constant}} \left(\frac{x^2y^2+2}{3y^2x^3}\right)dx + \int\limits_{D-x} \left(\frac{2-2x^2y^2}{3x^2y^3}\right)dy = C.$$

$$\Rightarrow \frac{1}{3y^2}\int\limits_{y-C}\left(\frac{x^2y^2+2}{x^3}\right)dx + \int\limits_{D-x}\frac{2}{3x^2y^3}\,dy - \int\limits_{D-x}\frac{2\cancel{x^2}\cancel{y^2}}{3\cancel{x^2}y^{\cancel{3}}}\,dy = C.$$

$$\Rightarrow \frac{1}{3y^2}\int\limits_{y-C}\left(\frac{\cancel{x^2}y^2}{x^{\cancel{3}}}+\frac{2}{x^3}\right)dx + 0 - \frac{2}{3}\int\frac{1}{y}\,dy = C$$

$$\Rightarrow \frac{1}{3y^2}\int\limits_{y-C}\frac{y^2}{x}\,dx + \frac{2}{3y^2}\int\frac{1}{x^3}\,dx - \frac{2}{3}\log y = C$$

$$\Rightarrow \frac{\cancel{y^2}}{3\cancel{y^2}}\int\frac{1}{x}\,dx + \frac{2}{3y^2}\left(\frac{x^{-3+1}}{-3+1}\right) - \frac{2}{3}\log y = C$$

$$\Rightarrow \frac{1}{3}\log x + \frac{\cancel{2}}{3y^2x^2(-\cancel{2})} - \frac{2}{3}\log y = C$$

$$\Rightarrow \frac{1}{3}\log x - \frac{1}{3}\,\frac{1}{y^2x^2} - \frac{2}{3}\log y = C. \quad //$$

(2). $\underbrace{y(1+xy)}_{M}dx + \underbrace{x(1-xy)}_{N.}dy = 0$ ——(1) $\Rightarrow \frac{dy}{dx} = -\frac{y(1+xy)}{x(1-xy)}.$

Sol:-

$\frac{\partial M}{\partial y} = 1+2xy \qquad \frac{\partial N}{\partial x} = 1-2xy \qquad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ (1) is N-E-D-E.

Let $f(x,y) = \frac{dy}{dx} = \frac{-y(1+xy)}{x(1-xy)}$

$f(kx,ky) \neq k^0 f(x,y)$

(1) is N-H-D-E.

(1) is in the form of $y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0.$

check $Mx - Ny = 2x^2y^2 \neq 0.$

$I.F = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}.$

Now multiplying Eq (1) with $I.F = \frac{1}{2x^2y^2}.$

$$(1) \Rightarrow \frac{\cancel{y}(1+xy)}{2x^2y^{\cancel{2}}}dx + \frac{\cancel{x}(1-xy)}{2x^{\cancel{2}}y^2}dy = 0.$$

$$\Rightarrow \left(\frac{1+xy}{2x^2y}\right)dx + \left(\frac{1-xy}{2xy^2}\right)dy = 0. \qquad (2)$$

$$M_1\, dx + N_1\, dy = 0$$

where $M_1 = \dfrac{1+xy}{2x^2y}$, $N_1 = \dfrac{1-xy}{2xy^2}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2x^2}\frac{\partial}{\partial y}\left(\frac{1+xy}{y}\right) = \frac{-1}{2x^2y^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{2y^2}\frac{\partial}{\partial x}\left(\frac{1-xy}{x}\right) = \frac{-1}{2x^2y^2}$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}} \quad \therefore (2) \text{ is an E.D.E.}$$

G.S is $\underset{y\text{-constant}}{\int M_1\, dx} + \underset{\text{donot take } x\text{-terms}}{\int N_1\, dy} = C.$

$$\Rightarrow \int\left(\frac{1+xy}{2x^2y}\right)dx + \int\left(\frac{1-xy}{2xy^2}\right)dy = C$$

$$\Rightarrow \frac{1}{2y}\int\left(\frac{1+xy}{x^2}\right)dx + \int\left(\cancel{\frac{1}{2xy^2}} - \frac{\cancel{x}y}{2\cancel{x}y\,y}\right)dy = C$$

$$\Rightarrow \frac{1}{2y}\int\frac{1}{x^2}\,dx + \frac{1}{2y}\int\frac{\cancel{x}y}{\cancel{x}^2}\,dx + 0 - \frac{1}{2}\int\frac{1}{y}\,dy = C$$

$$\Rightarrow \frac{1}{2y}\left(\frac{x^{-2+1}}{-2+1}\right) + \frac{\cancel{y}}{2\cancel{y}}\int\frac{1}{x}\,dx - \frac{1}{2}\log y = C$$

$$\Rightarrow \frac{-1}{2yx^1} + \frac{1}{2}\log x - \frac{1}{2}\log y = C \quad //$$

$$\Rightarrow \frac{-1}{xy} + \log x - \log y = 2C$$

$$\Rightarrow \frac{-1}{xy} + \log\left(\frac{x}{y}\right) = 2C \quad //$$

Method - IV :- $Mdx + Ndy = 0.$

$Mdx+Ndy=0$ is Non-~~Exact~~, Non ~~H.~~-D.E & $y\, f(x,y)\,dx + x\, g(x,y)\,dy = 0$

$f(x)$ suchthat $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x).$ $\quad e^{\int f(x)\,dx}$ is I.F

(1). $(x^3 - 2y^2)\,dx + 2xy\,dy = 0 \qquad f(x)$

N

1. $\underbrace{(x^3 - 2y^2)}_{M}\,dx + \underbrace{2xy}_{N}\,dy = 0$. ——— $f(x)$. ——— (1) $\Rightarrow \dfrac{dy}{dx} = -\dfrac{(x^3-2y^2)}{2xy}$

Sol:

$M\,dx + N\,dy = 0$.

where $M = x^3 - 2y^2$; $N = 2xy$.

$\dfrac{\partial M}{\partial y} = -4y$; $\dfrac{\partial N}{\partial x} = 2y$. $\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

$\therefore$ (1) is not an E.D.E.

let $f(x,y) = \dfrac{dy}{dx} = -\dfrac{(x^3-2y^2)}{2xy}$

$f(kx, ky) \neq k^0 f(x,y)$

(1) is Not a H.D.E.

Now check $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{-4y - 2y}{2xy} = \dfrac{-6y}{2xy} = -\dfrac{3}{x} = f(x)$

$\therefore$ I.F $= e^{\int f(x)\,dx} = e^{\int(-3/x)\,dx} = e^{-3\int 1/x\,dx} = e^{-3\log x} = e^{\log_e x^{-3}}$

$= x^{-3} = \dfrac{1}{x^3}$

$\therefore \boxed{\text{I.F} = \dfrac{1}{x^3}}$

Multiplying (1) with I.F $= \dfrac{1}{x^3}$ on b.s.

(1) $\Rightarrow \dfrac{(x^3 - 2y^2)}{x^3}\,dx + \dfrac{(2\cancel{x}y)}{x^{\cancel{3}}}\,dy = 0$.

$\Rightarrow \left(\dfrac{x^3 - 2y^2}{x^3}\right)dx + \left(\dfrac{2y}{x^2}\right)dy = 0$.

$\Rightarrow \underbrace{\left(1 - \dfrac{2y^2}{x^3}\right)}_{M_1}dx + \underbrace{\left(\dfrac{2y}{x^2}\right)}_{N_1}dy = 0$ ——— (2)

$\therefore$ It is in the form of $M_1\,dx + N_1\,dy = 0$.

where $M_1 = 1 - \dfrac{2y^2}{x^3}$; $N_1 = \dfrac{2y}{x^2}$

$\dfrac{\partial M_1}{\partial y} = -\dfrac{4y}{x^3}$; $\dfrac{\partial N_1}{\partial x} = -\dfrac{4y}{x^3}$ $\therefore \boxed{\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}}$

$\therefore$ (2) is an E.D.E.

G.S is $\int M_1\,dx + \int N_1\,dy = C$

$\int\left(1 - \dfrac{2y^2}{x^3}\right)dx + \int \dfrac{2y}{x^2}\,dy = C$

G.S is $\int M_1 dx + \int N_1 dy = C$

$$\Rightarrow \int \left(1 - \frac{2y^2}{x^3}\right) dx + \int \frac{2y}{x^2} dy = C$$

y - constant ; don't take x - terms

$y(\ ) + x(\ ) = 0$

$$\Rightarrow x - 2y^2 \int \frac{1}{x^3} dx + 0 = C$$

$$\Rightarrow x - 2y^2\left(\frac{x^{-3+1}}{-3+1}\right) = C \Rightarrow x + \frac{\cancel{2}y^2}{\cancel{2}x^2} = C$$

$$\Rightarrow \boxed{\frac{x^3+y^2}{x^2} = C}$$

H.W. ① $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$. — II - method.

② $y - x\frac{dy}{dx} = x + y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}(y+x) = y - x$

$dy/dx = \frac{y-x}{y+x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$$

$$\underbrace{(y-x)}_{M}\,dx - \underbrace{(y+x)}_{N}\,dy = 0$$

A :- $\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C$

A :- $\log(x^2+y^2)^{1/2} + \tan^{-1}(y/x) = C$.

Method (5) :- (To find an I.F of $Mdx + Ndy = 0$)

① $Mdx + Ndy = 0$ — (1)

② $M =\ \ ;\ N =$

③ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ — N.E.D.E.

④ $f(kx, ky) \neq k^n f(x,y)$ — N.H.D.E.

⑤ $y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0$ (III)

⑥ $\boxed{I.F = e^{\int g(y)\,dy}}$

check $\left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) = g(y)$

z

$M_1 dx + N_1 dy = 0$

$M(x,y)\,dx + N(x,y)\,dy = 0$ is not ~~Exact~~ and not ~~H.D.E~~

~~Method III & IV~~ $\exists$ a continuous and $g(y)$ $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ then

$e^{\int g(y)\,dy}$ is an I.F of $Mdx + Ndy = 0$.

① Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$. (1) $\Rightarrow \frac{dy}{dx} = \frac{-(xy^3+y)}{2x^2y^2+2x+2y^4}$

Sol: $\underbrace{(xy^3+y)}_{M}\,dx + \underbrace{2(x^2y^2+x+y^4)}_{N}\,dy = 0$.

Here $M = xy^3 + y$; $N = 2(x^2y^2 + x + y^4)$

$= 2x^2y^2 + 2x + 2y^4$

$\frac{\partial M}{\partial y} = 3xy^2 + 1$; $\frac{\partial N}{\partial x}$ [illegible]

$$= 2x^2y^2 + 2x + 2y^4$$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \quad ; \quad \frac{\partial N}{\partial x} = 4xy^2 + 2 + 0$$
$$= 4xy^2 + 2.$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is a Non Exact D.E.

Ck $f(x,y) = \dfrac{dy}{dx} = \dfrac{-(xy^3+y)}{2x^2y^2+2x+2y^4}$

$$f(kx,ky) = -\left(\frac{kx\,k^3y^3 + ky}{2k^2x^2k^2y^2 + 2kx + 2k^4y^4}\right)$$

$$= \left(\frac{k^4xy^3 + ky}{2k^4x^2y^2 + 2kx + 2k^4y^4}\right)$$

$$f(kx,ky) \neq k^0 f(x,y)$$

$\therefore$ (1) is non-H-D-E.

(1) is not in the form of $y f(x,y)\,dx + x\,g(x,y)\,dy = 0$.

check $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = \dfrac{1}{xy^3+y}\left(4xy^2+2 - 3xy^2 - 1\right)$

$$= \frac{1}{xy^3+y}\left(xy^2+1\right)$$

$$= \frac{1}{y(\cancel{xy^2+1})}\left(\cancel{xy^2+1}\right)$$

$$= 1/y$$

$$= g(y)$$

$$\therefore I.F = e^{\int g(y)\,dy} = e^{\int 1/y\,dy} = e^{\log_e y} = y.$$

$$\therefore \boxed{I.F = y}$$

Now multiply Eq^n (1) with I.F = y. on b.s

$$y(xy^3+y)\,dx + 2y(x^2y^2+x+y^4)\,dy = 0$$

$$\Rightarrow \underbrace{(y^4x + y^2)}_{M_1}dx + \underbrace{(2x^2y^3 + 2xy + 2y^5)}_{N_1}dy = 0 \qquad (2).$$

which is in the form of $M_1dx + N_1dy = 0$.

where $M_1 = xy^4 + y^2$; $N_1 = 2x^2y^3 + 2xy + 2y^5$.

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \quad ; \quad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y + 0$$

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \quad ; \quad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y + 0$$
$$= 4xy^3 + 2y.$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ (2) is an E.D.E.

G.S is $\int M_1\,dx + \int N_1\,dy = C.$

y-constant — don't take x-terms

$$\Rightarrow \int (xy^4 + y^2)\,dx + \int (2x^2y^3 + 2xy + 2y^5)\,dy = C$$

$$\Rightarrow y^4 \int x\,dx + y^2 \int 1\,dx + 2\int y^5\,dy = C.$$

$$\Rightarrow y^4 \frac{x^2}{2} + y^2(x) + 2\,\frac{y^6}{6} = C$$

(or).

$$\Rightarrow 3x^2y^4 + 6xy^2 + 2y^6 = 6C$$

H.W. ① $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$ $\left(xy + \frac{2x}{y^2} + y^2 = C\right)$

H.W. ② $y\,(xy + e^x)\,dx - e^x dy = 0.$ $\left(\frac{x^2}{2} + \frac{e^x}{y} = C\right)$

H.W. ③ $y\,(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$ $\left(\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C\right)$

H.W. ④ $(x^2 + y^2 + x)\,dx + xy\,dy = 0.$ $(3x^4 + 6y^2x^2 + 4x^3 = C)$

Linear Differential Eqn :- $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$

P & Q are (or) 'x' only L.D.E. of 'y'.

Procedure :- ① $\boxed{I.F = e^{\int P(x)\,dx}}$

② Soln is $y \times (I.F) = \int Q(x) \times (I.F)\,dx + C.$

Note :- ① 11y L.D.E 'x' is of the form $\frac{dx}{dy} + P(y)\,x = Q(y).$

$I.F = e^{\int P(y)\,dy}$ & G.S is $x\,(I.F) = \int Q(y)\,I.F\,dy + C$

① Solve. $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$ $\quad M\,dx + N\,dy = 0$

Sol :-
$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{4x^2}{1+x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right) y = \frac{4x^2}{1+x^2}$$

which is in the form of $\frac{dy}{dx} + P(x)\,y = \Phi(x)$

where $P(x) = \frac{2x}{1+x^2}$; $\Phi(x) = \frac{4x^2}{1+x^2}$.

$$I.F = e^{\int P(x)\,dx} = e^{\int \left(\frac{2x}{1+x^2}\right) dx} = e^{\log_e(1+x^2)} = 1+x^2 \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

G.S $\boxed{y\,(I.F) = \int \Phi(x)\,(I.F)\,dx + c}$

$$\Rightarrow \quad y\,(1+x^2) = \int \left(\frac{4x^2}{\cancel{1+x^2}}\right) \cancel{(1+x^2)}\,dx + c$$

$$\Rightarrow \quad y\,(1+x^2) = \int 4x^2\,dx + c$$

$$\Rightarrow \quad y\,(1+x^2) = 4\frac{x^3}{3} + c. \quad //$$

(2). Solve. $\cos^2 x \frac{dy}{dx} + y = \tan x$

Sol:
$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}.$$

$$\frac{dy}{dx} + (\sec^2 x)\,y = \tan x \sec^2 x.$$

which is in the form of $\frac{dy}{dx} + P(x)\,y = \Phi(x)$

where $P(x) = \sec^2 x$; $\Phi(x) = \tan x \sec^2 x$.

$$\therefore I.F = e^{\int P(x)\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}.$$

G.S is $\boxed{y\,(I.F) = \int \Phi(x)\,(I.F)\,dx + c.}$

$$y\,e^{\tan x} = \int (\tan x \sec^2 x)\, e^{\tan x}\,dx + c. \quad \text{——} \ (1)$$

put $\tan x = t$ —— (2)

$\sec^2 x\,dx = dt$ —— (3)

sub. (2) & (3) in (1)

$$y\,e^{\tan x} = \int t\, e^{t}\,dt + c.$$

$$y\,e^{\tan x} = (t-1)\,e^{t} + c. \qquad \left(\because \int t e^{t}\,dt = (t-1)e^{t} + c\right)$$

$$\because \int t e^{-t}\,dt = -(t+1)e^{-t} +$$

$$y e^{\tan x} = (t-1)e^{t} + c.$$

$$\because \int t e^{-t} dt = -(t+1)e^{-t} + c$$

$$y e^{\tan x} = (\tan x - 1) e^{\tan x} + c. \;//$$

③. $$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right) y = \frac{1}{(1+x^2)^2}$$

$$y(1+x^2) = \tan^{-1}x + c.$$

④. $$\frac{dy}{dx} + y = e^{e^x} \text{ ——— (1)}$$

Ans:- $$\frac{dy}{dx} + P(x)y = \Phi(x)$$

Here $P(x) = 1$; $\Phi(x) = e^{e^x}$.

$$\text{I.F} = e^{\int P(x)dx} = e^{\int 1 dx} = e^{x}.$$

G.S is $$y \times (\text{I.F}) = \int \Phi(x)(\text{I.F})\,dx + c.$$

$$y \times e^x = \int e^{e^x}\left(e^x\right) dx + c$$

$$\left(\begin{array}{l} \text{let } e^x = t \\ e^x dx = dt \end{array}\right)$$

$$y e^x = \int e^t \, dt + c$$

$$y e^x = e^t + c$$

$$\boxed{y e^x = e^{e^x} + c}.$$

⑤. $$(x+y+1)\frac{dy}{dx} = 1.$$

Sol:- $$\frac{dy}{dx} = \frac{1}{x+y+1}$$

$$\frac{dx}{dy} = x+y+1$$

$$\left[\because \frac{dx}{dy} + P(y)x = \Phi(y)\right]$$

$$\frac{dx}{dy} - x = y+1.$$

which is in the form of $\frac{dx}{dy} + P(y)x = \Phi(y)$

Here $P(y) = -1$; $\Phi(y) = y+1$.

$$\text{I.F} = e^{\int P(y).dy} = e^{\int -1\, dy} = e^{-y}$$

G.S is $$x \times (\text{I.F}) = \int \Phi(y)(\text{I.F})\,dy + c.$$

$$\Rightarrow x \times e^{-y} = \int (y+1) e^{-y} dy + c.$$

$$\Rightarrow x e^{-y} = \int \underset{u}{(y+1)} \underset{v}{(e^{-y})} dy + c.$$

ILATE

Applying integration by parts

Applying integrating by parts.

$$\left[\because \int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx\right]$$

$$\Rightarrow x e^{-y} = (y+1)\int e^{-y}dy - \int (1+0)\frac{e^{-y}}{(-1)}\,dy$$

$$= (y+1)\frac{e^{-y}}{(-1)} + \int e^{-y}dy$$

$$= -(y+1)e^{-y} + \frac{e^{-y}}{(-1)} + c$$

$$= -(y+1)e^{-y} - e^{-y} + c \;//$$

$$= e^{-y}(-y-1-1) + c$$

$$\Rightarrow x e^{-y} = e^{-y}(-y-2) + c. \;//$$

⑥. $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0. \Rightarrow dr = -(2r\cot\theta + \sin 2\theta)\,d\theta.$

Sol:-

$$\Rightarrow \frac{dr}{d\theta} = -2r\cot\theta - \sin 2\theta.$$

$$\Rightarrow \frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} + (2\cot\theta)\,r = -\sin 2\theta$$

$$\left[\because \frac{dy}{dx} + \boxed{P(x)}y = Q(x)\right]$$

$$I.F = e^{\int P(x)dx}$$

$$y(I.F) = \int Q(x)\, I.F\, dx + c.$$

which is in the form of $\left[\frac{dr}{d\theta} + P(\theta)r = Q(\theta)\right]$

Here $P(\theta) = 2\cot\theta$; $Q(\theta) = -\sin 2\theta.$

$$I.F = e^{\int P(\theta)d\theta} = e^{\int 2\cot\theta\, d\theta} = e^{2\int\cot\theta\, d\theta}$$

$$= e^{2\log|\sin\theta|} \qquad \left(\because \int\cot x = \log|\sin x| + c\right)$$

$$= e^{\log_e(\sin\theta)^2}$$

$$\boxed{I.F = \sin^2\theta.}$$

G.S is $\boxed{r(I.F) = \int Q(\theta)(I.F)\,d\theta + c.}$

$$\Rightarrow r\sin^2\theta = \int -\sin 2\theta \cdot \sin^2\theta\, d\theta + c.$$

$$= \int -2\sin\theta\cos\theta\sin^2\theta\, d\theta + c$$

$$= -2\int \underbrace{\sin^3\theta\cdot\cos\theta}_{3=n}\, d\theta + c$$

$$= -2\int (\underbrace{\sin\theta}_{f(x)})^3 \underbrace{\cos\theta}_{f'(x)}\,d\theta + c \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= -2\,\frac{(\sin\theta)^{3+1}}{3+1} + c$$

$$= -\frac{2}{4}\sin^4\theta + c$$

$$\Rightarrow r\sin^2\theta = -\frac{1}{2}\sin^4\theta + \frac{c}{1}$$

(or)

$$\Rightarrow 2r\sin^2\theta + \sin^4\theta = 2c.$$

(or)

$$\Rightarrow \sin^2\theta\left(2r + \sin^2\theta\right) = 2c$$

//

H.W. $(x + 2y^3)\dfrac{dy}{dx} = y$.

$$\frac{dy}{dx} = \frac{y}{x+2y^3}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{1x}{y} = 2y^2$$

$$\frac{dx}{dy} + P(y)x = \phi(y)$$

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{1}{(1+x^2)^2},\quad \text{Given that } y(0) = 1.$$

Sol: $\left[\because \dfrac{dy}{dx} + P^{(x)}y = \phi(x)\right]$

where $P(x) = \dfrac{2x}{1+x^2}$; $\phi(x) = \dfrac{1}{(1+x^2)^2}$

$$I.F = e^{\int P(x)dx} = e^{\int 2x/1+x^2\,dx} = e^{\log_e(1+x^2)} = 1+x^2.$$

G.S is $y\times(I.F) = \int \phi(x)(I.F)\,dx + c.$

$$y\times 1+x^2 = \int \frac{1}{(1+x^2)^{\cancel{2}}}(1+\cancel{x^2})\,dx + c$$

$$y(1+x^2) = \int \frac{1}{1+x^2}\,dx + c$$

$$\boxed{y(1+x^2) = \tan^{-1}x + c} \quad \text{——— (1)}$$

Given $y(0) = 1$.

i.e, $x=0,\ y=1$.

sub, $x=0$ & $y=1$ in (1)

$$1 = 0 + c$$

$1 = 0 + C$

$\boxed{C = 1}$

Sub, $\boxed{C=1}$ in (1)

$\therefore \boxed{y(1+x^2) = \tan^{-1}x + 1}$

HW ① $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$ [A:- $\frac{y}{x+1} - \frac{e^{3x}}{3} = C$.]

② $(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$. [A:- $xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + C$.

non-linear (or) Bernoulli's D.E :-

$$\frac{dy}{dx} + P(x)y = \Phi(x)\, y^{n} \quad \text{—— (1)}$$ is called.

where 'P' & 'Q' are functions of 'x' & 'n' is a real constant.

case (1):- If $n=1$ (1) can be written of

$$\frac{dy}{dx} + P(x)y = \Phi(x)y$$

$$\frac{dy}{dx} = \Phi(x)y - P(x)y$$

$$\frac{dy}{dx} = y\left[\Phi(x) - P(x)\right]$$

by using variable-separable method

$$\frac{dy}{y} = \left[\Phi(x) - P(x)\right]dx.$$

Integrating on b.s.

$$\int \frac{dy}{y} = \int\left[\Phi(x) - P(x)\right]dx + C.$$

(or)

$$\int \frac{dy}{y} - \int\left[\Phi(x) - P(x)\right]dx = C$$

case (2):- If $\boxed{n \neq 1}$

$$\frac{dy}{dx} + P(x)y = \Phi(x)y^n \quad \text{—— (1)}$$

multiply Eqn (1) with y^{-n}.

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1}\, y^{-n} = \Phi(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)\left(y^{1-n}\right) = \Phi(x). \quad \text{—— (2)}$$

Now $y^{1-n} = z$ —— (3)

Diff w.r.to 'x' on b.s.

$$(1-n)\, y^{\cancel{1}-n-\cancel{1}} \frac{dy}{dx} = \frac{dz}{dx}$$

$$(1-n)\, y^{-n} \frac{dy}{dx} = \frac{dz}{dx}.$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dz}{dx} \quad \text{—— (4)}$$

Eqⁿ(4) sub. in Eqⁿ(2)

$$(2) \Rightarrow \frac{1}{1-n} \frac{dz}{dx} + \frac{P(x)\, z}{1} = Q(x)$$

$$\frac{dz}{dx} + \underline{(1-n)}P(x) \cdot \underline{z} = \underline{Q(x)\,(1-n)}$$

This is in the form of L.D.E and can be solved by L.D.E procedure.

$\boxed{\text{Note}}$: 'x' is $\frac{dx}{dy} + P(y) \cdot x = Q(y)\, x^n$.

①. $x \frac{dy}{dx} + y = x^2 y^6$.

Sol:- Divide with 'x'.

$$\frac{dy}{dx} + \frac{y}{x} = x y^6.$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x y^6 \quad \text{—— (1)}$$

$$\frac{dy}{dx} + P(x) y = Q(x) y^n \quad (\text{Bernoulli's D.E})$$

Multiply Eqⁿ(1) with y^{-6}.

$$y^{-6} \frac{dy}{dx} + \left(\frac{1}{x}\right) y^{-5} = x. \quad \text{—— (2)}$$

Put $y^{-5} = z$ —— (3)

Diff. w.r.t 'x'.

$$(-5)\, y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-6} \frac{dy}{dx} = \frac{-1}{5} \frac{dz}{dx} \quad \text{—— (4)}$$

sub. (3) & (4) in (2)

$$-\frac{1}{5} \frac{dz}{dx} + \frac{1}{x}(z) = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{5}{x}(z) = -5x \quad \text{—— (5)}$$

$\therefore$ Eq in the form of L.D.E is $\frac{dy}{dx} + P(x)y = Q(x)$.

$P(x) = \frac{-5}{x}$; $Q(x) = -5x$.

$I.F = e^{\int P(x)dx} = e^{\int -5/x \, dx} = e^{-5\log x} = \frac{1}{x^5}$ (or) x^{-5}.

$G.S = \left[y(I.F) = \int Q(x)(I.F)\, dx + c \right]$

$\Rightarrow Z(I.F) = \int Q(x).(I.F)\, dx + c$

$\Rightarrow Z x^{-5} = \int (-5x)(x^{-5})\, dx + c$

$\Rightarrow Z x^{-5} = -5\int x^{-4} dx + c$

$\Rightarrow Z x^{-5} = -5\left(\frac{x^{-3}}{-3}\right) + c$

$\Rightarrow Z x^{-5} = \frac{5}{3} x^{-3} + c.$

$\Rightarrow y^{-5} x^{-5} = \frac{5}{3x^3} + c$ $\quad (\because Z = y^{-5})$.

(or)

$\Rightarrow \frac{1}{(xy)^5} = \frac{5}{3x^3} + c$ //

(2). $\frac{dy}{dx}\left(x^2y^3 + xy \right) = 1$

Sol:- $\frac{dy}{dx} = \frac{1}{x^2y^3 + xy}$.

$\frac{dx}{dy} = x^2y^3 + xy$

$\Rightarrow \frac{dx}{dy} - xy = x^2 y^3$ —— (1)

$\left(\because \frac{dx}{dy} + P(y)x = Q(y).x^n\right)$

multiply on b.s with " x^{-2} "

$x^{-2}\frac{dx}{dy} - x^{-1}y = y^3$ —— (2)

put $x^{-1} = z$ —— (3)

Diff. w.r.to 'y' on b.s.

$(-1)x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$.

$x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$ —— (4).

$$\frac{dx}{dy} = -\frac{dz}{dy} \text{ ——— (4).}$$

sub. Eqn (3) & (4) in (2).

$$-\frac{dz}{dy} - zy = y^3$$

$$\frac{dz}{dy} + yz = -y^3.$$

which is in the form of L.D.E

i.e., $\frac{dx}{dy} + P(y)x = Q(y)$.

where $P(y) = y$; $Q(y) = -y^3$.

$$I.F = e^{\int P(y)\,dy} = e^{\int y\,dy} = e^{\left[\frac{y^2}{2}\right]} = e^{y^2/2}.$$

$$G.S = \left[\, x(I.F) = \int Q(y)(I.F)\,dy + c.\right.$$

$$z(I.F) = \int Q(y)\,(I.F)\,dy + c.$$

$$z\,e^{y^2/2} = \int(-y^3)\,e^{y^2/2}\,dy + c$$

$$\Rightarrow z e^{y^2/2} = -\int \underset{y^2 \cdot y}{y^3} \cdot e^{y^2/2}\,dy + c \Rightarrow z e^{y^2/2} = -\int y^2 \cdot y\, e^{y^2/2}\,dy + c$$

put $\frac{y^2}{2} = t$. $\Rightarrow y^2 = 2t$

$\frac{2y\,dy}{2} = dt$. (or) $y\,dy = dt$.

$$= -\int 2t\, e^{t}\,dt + c$$

$$z e^{y^2/2} = -2\int t e^{t}\,dt + c$$

$$z e^{y^2/2} = -2\; e^{t}(t-1) + c$$

$$x^{-1} e^{y^2/2} = -2 e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c \qquad \left(\because\ t = y^2/2,\ z = x^{-1}\right)$$

(or).

$$\frac{1}{x} e^{y^2/2} = -2\,e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c \ //$$

$y(I.F) = \int Q(x)\, I.F\,dx + c.$

$I.F = e^{\int P(x)\,dx}$

$\frac{dy}{dx} + P(x)y = Q(x)$

$\frac{dx}{dy} + P(y)x = Q(y)$.

$I.F = e^{\int P(y)\,dy}$

$x(I.F) = \int Q(y)\,I.F\,dy + c.$

③ $\frac{dy}{dx} = e^{x-y}\left(e^{x} - e^{y}\right)$ ——— (1)

sol:

$$\frac{dy}{dx} = \frac{e^{x}}{e^{y}}\left(e^{x} - e^{y}\right)$$

$$\Rightarrow e^{y}\frac{dy}{dx} = e^{x}\left(e^{x} - e^{y}\right)$$

$$\Rightarrow e^{y}\frac{dy}{dx} = e^{2x} - e^{x} \cdot e^{y}$$

$$\Rightarrow e^{y}\frac{dy}{dx} = e^{2x} - e^{x}\cdot e^{y}$$

$$\Rightarrow e^{y}\frac{dy}{dx} + e^{x}e^{y} = e^{2x} \quad \text{—— (2)}$$

Put $e^{y} = z$ —— (3)

$$e^{y}\frac{dy}{dx} = \frac{dz}{dx} \quad \text{—— (4)}$$

(3) & (4) in (2)

$$\frac{dz}{dx} + e^{x}z = e^{2x} \quad \text{—— (5)}$$

which is in the form of L.D.E

i.e. $\frac{dz}{dx} + P(x)z = Q(x)$.

where $P(x) = e^{x}$; $Q(x) = e^{2x}$.

$$I.F = e^{\int P(x)\,dx} = e^{\int e^{x}dx} = e^{e^{x}}.$$

$$G.S \Rightarrow z\cdot(I.F) = \int Q(x)(I.F)\,dx + c.$$

$$z(e^{e^{x}}) = \int e^{2x}\cdot e^{e^{x}}\,dx + c.$$

$$\Rightarrow z e^{e^{x}} = \int e^{x}\cdot e^{x}\cdot e^{e^{x}}\,dx + c$$

put $e^{x} = t$

$e^{x}dx = dt$

$$\Rightarrow z e^{t} = \int t\, e^{t}\,dt + c$$

$$\Rightarrow z e^{t} = e^{t}(t-1) + c$$

$$\Rightarrow e^{+y}e^{t} = e^{e^{x}}(e^{x}-1) + c \quad (\because t = e^{x} \;\&\; z = e^{y}).$$

$$\Rightarrow e^{y}e^{e^{x}} = e^{e^{x}}(e^{x}-1) + c$$

(4) $\sec^{2}y\,\frac{dy}{dx} + 2x\tan y = x^{3}$

Sol: put $\tan y = z$.

(5). $\frac{dy}{dx} + x\sin 2y = x^{3}\cos^{2}y$

Sol: $\Rightarrow \frac{dy}{dx} + x(2\sin y\cos y) = x^{3}\cos^{2}y.$

$$\Rightarrow \frac{1}{\cos^{2}y}\frac{dy}{dx} + \frac{x(2\sin y\,\cancel{\cos y})}{\cancel{\cos^{2}y}} = x^{3}.$$

$$\sec^{2}y\,\frac{dy}{dx} + 2x\tan y = x^{3}.$$

$\tan y = z$.

$\cos^{2}y = z$.

$$2\cos y(-\sin y)\frac{dy}{dx} = \frac{dz}{dx}.$$

A:- $(\tan y)e^{x^{2}} = \frac{1}{2}e^{x^{2}}(x^{2}-1) + c.$

(1). $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^{x}\sec y.$

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y(1+x)} = (1+x)e^x$$

$$\Rightarrow \cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x. \quad \text{—— (1)}$$

put $\sin y = z$.

w.r.t 'x'.

$$\cos y\frac{dy}{dx} = \frac{dz}{dx}.$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x.$$

$$\left[\because \frac{dz}{dx} + P(x)z = Q(x)\right]$$

Here $P(x) = -\frac{1}{1+x}$; $Q(x) = (1+x)e^x$.

$$\text{I.F} = e^{\int P(x)\,dx} = e^{\int(-1/1+x)\,dx} = e^{-\log(1+x)} = e^{\log(1+x)^{-1}} = (1+x)^{-1} = 1/1+x$$

G.S is $z(\text{I.F}) = \int Q(x)(\text{I.F})\,dx + C$

$$\Rightarrow z\left(\frac{1}{1+x}\right) = \int \cancel{(1+x)}e^x\left(\frac{1}{\cancel{1+x}}\right)dx + C$$

$$\Rightarrow \frac{z}{1+x} = \int e^x\,dx + C$$

$$\Rightarrow \frac{z}{1+x} = e^x + C$$

$$\Rightarrow \frac{\sin y}{1+x} = e^x + C \qquad (\because z = \sin y)$$

(OR)

$$\sin y = (1+x)(e^x + C)$$

H.W.

② $(1-x^2)\frac{dy}{dx} + xy = y^3\sin^{-1}x$ //

$$\frac{1-x^2}{y^2} = -2\left(x\sin^{-1}x + \sqrt{1-x^2}\right) + C$$

$$\frac{dy}{dx} + \left(\frac{xy}{1-x^2}\right) = \frac{y^3\sin^{-1}x}{1-x^2}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

③ $e^x\frac{dy}{dx} = 2xy^2 + y\cdot e^x.$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

Sol:- $\frac{dy}{dx} = \frac{2xy^2}{e^x} + y$ (Divide with e^x)

$$\frac{dy}{dx} - y = \left(\frac{2x}{e^x}\right)y^2 \qquad \left[\because \frac{dy}{dx} + P(x)y = Q(x)y^n\right]$$

Divide "y^2" on b.s (or) multiply y^{-2} on b.s.

$$y^{-2}\frac{dy}{dx} - y^{-1} = \frac{2x}{e^x} \quad \text{—— (1)}$$

let $y^{-1} = z$.

w.r.t 'x' on b.s.

$$(-1)y^{-2}\frac{dy}{dx}=\frac{dz}{dx}.$$

$$y^{-2}\frac{dy}{dx}=-\frac{dz}{dx}. \quad \text{—— (2)}$$

Now sub. (2) in (1). $\Rightarrow -\frac{dz}{dx}-z=\frac{2x}{e^x}$

$$\Rightarrow \frac{dz}{dx}+z=-\frac{2x}{e^x}.$$

$$\left[\because \frac{dz}{dx}+P(x)z=Q(x)\right]$$

where $P(x)=1$; $Q(x)=\frac{-2x}{e^x}.$

$$I.F=e^{\int P(x)dx}=e^{\int 1dx}=e^x$$

G.S is $z(I.F)=\int Q(x).(I.F)\,dx+c.$

$$z(e^x)=\int\frac{-2x}{\cancel{e^x}}(\cancel{e^x})\,dx+c$$

$$ze^x=-\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)+c$$

$$y^{-1}e^x=-x^2+c. \qquad (\because z=y^{-1})$$

(or).

$$x^2+e^xy^{-1}=c.$$

Applications of on ODE :-

1. Newton's Law of cooling
2. the law of Natural growth or Decay
3. Electrical circuits.
4. orthogonal Trajectories.

Newton's Law of cooling :- 'θ' be the 't'

'θ_0' (air).

i.e, $\boxed{\frac{d\theta}{dt}\propto(\theta-\theta_0)}$ $\qquad \frac{dy}{dx}.$

$$\Rightarrow \frac{d\theta}{dt}=-k(\theta-\theta_0)$$

where 'k' is a +ve constant.

Here. Applying variable separable method

$\Rightarrow$ dθ ... k dt

$$\frac{\quad}{\theta-\theta_0} = -k\, dt.$$

Integrating on b.s.

$$\Rightarrow \int \frac{1}{\theta-\theta_0}\, d\theta = -k\int 1\, dt$$

$$\Rightarrow \log|\theta-\theta_0| = -kt + c \quad \text{------} \quad (1).$$

Let $\theta=\theta_1$ at the time $t=0$.

$$\Rightarrow \log|\theta_1-\theta_0| = 0+c$$

$$\boxed{c=\log|\theta_1-\theta_0|}$$

sub. 'c' value in (1).

$$(1) \Rightarrow \log|\theta-\theta_0| = -kt + \log|\theta_1-\theta_0|$$

(or)

$$kt = \log|\theta_1-\theta_0| - \log|\theta-\theta_0| \;//$$

① . Air at 30°c It cools down from 80°c to 60°c in 12-minutes. the 24 minutes ?

Sol:- let ~~'θ'~~ — Temp. of body at 't'.

~~θ_0~~ — air.

Given that $\theta_0 = 30°c$

Initially. $\theta = 80°c$ at $t=0$

$\theta = 60°c$. at $t=12$.

$\theta = ?$ at 24-minutes $(t=24)$.

By N.L.C.

W.K.T $\dfrac{d\theta}{dt} \propto (\theta-\theta_0)$

$\dfrac{d\theta}{dt} = -k(\theta-\theta_0)$, 'k' is a +ve constant.

$$\Rightarrow \int \frac{d\theta}{\theta-\theta_0} = \int -k\, dt \quad (\because \text{variable separable method})$$

$$\Rightarrow \log|\theta-\theta_0| = -kt + c$$

$$\Rightarrow \log|\theta-30°| = -kt + c \quad \text{---} (1). \quad (\because \theta_0 = 30°c)$$

Given that $\boxed{\theta = 80°c}$ at time $\boxed{t=0}$ sub. in (1).

$$\log|80°-30°| = -k(0) + c.$$

$$\log|50°| = c.$$

$$\boxed{\therefore\ c = \log|50°|} \quad \text{——— (2)}$$

Sub. (2) in (1).

$$\Rightarrow (1) \Rightarrow \log|\theta - 30°| = -kt + \log|50°|$$

$$\Rightarrow kt = \log|50°| - \log|\theta - 30°| \quad \text{——— (3).}$$

Given that $\boxed{\theta = 60°C}$ at time $\boxed{t = 12}$ sub in (3).

$$\Rightarrow 12k = \log|50°| - \log|60 - 30|$$

$$\Rightarrow 12k = \log|50°| - \log|30°| \quad \text{——— (4).}$$

Now solving (3) & (4).

$$(3) \div (4) \Rightarrow \frac{\cancel{k}t}{12\cancel{k}} = \frac{\log|50°| - \log|\theta - 30°|}{\log|50°| - \log|30°|}$$

$$\Rightarrow \frac{t}{12} = \frac{\log|50/\theta - 30°|}{\log|50/30|} \quad \left[\because \log a - \log b = \log(a/b)\right]$$

Given $\boxed{t = 24}$ then $\theta = ?$

sub in above.

$$\Rightarrow \frac{\cancel{24}^{2}}{\cancel{12}} = \frac{\log(50/\theta - 30°)}{\log(50/30)}$$

$$\Rightarrow 2\log\left(\frac{50}{30}\right) = \log\left(\frac{50°}{\theta - 30°}\right)$$

$$\Rightarrow \log\left(\frac{50}{30}\right)^2 = \log\left(\frac{50°}{\theta - 30°}\right)$$

$$\Rightarrow \left(\frac{50}{30}\right)^2 = \left(\frac{50°}{\theta - 30°}\right)$$

$$\Rightarrow \frac{25\cancel{00}}{9\cancel{00}} = \frac{50}{\theta - 30.}$$

$$\Rightarrow \boxed{\theta = 48°}$$

(2). 100°c to 75°c in ten minutes. air is at 20°c temperature. half an hour. 25°c.

Sol:- Let 'θ_0' be the temperature of the air

'θ' be the " " body at time 't'.

$$\theta_0 = 20°C.$$

Initially $\theta = 100°C$ at $t = 0\,min$

$\theta = 75°c$ at $t = 10\,min$

we have to find (i) $\theta = ?$ after $t =$ half an hour $= 30 min$.

(ii). $\theta = 25^\circ c$ (Given), $t = ?$

by Newton's law of cooling.

W.K.T $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$$\frac{d\theta}{dt} = -k\ (\theta - \theta_0)$$

$$\frac{d\theta}{\theta - \theta_0} = -k\, dt \quad \text{(variable separable)}$$

$$\int \frac{1}{\theta - \theta_0} d\theta = -k \int dt \quad \text{(I.b.s)}$$

$$\log|\theta - \theta_0| = -kt + c \quad \text{——— (1)}$$

$$\log|\theta - 20^\circ| = -kt + c$$

Given that $\boxed{\theta = 100^\circ c}$ at time $\boxed{t = 0}$ sub. in (1)

$$\log(100 - 20) = 0 + c \Rightarrow \boxed{c = \log 80^\circ} \quad \text{——— (2)}$$

sub. eq^n (2) in eq^n (1)

$$\log(\theta - 20^\circ| = -kt + \log 80^\circ$$

$$\Rightarrow kt = \log 80^\circ - \log|\theta - 20^\circ| \quad \text{——— (3)}$$

Given that $\boxed{\theta = 75^\circ c}$ at $\boxed{\text{time}(t) = 10 min}$ sub in (3).

$$\Rightarrow 10k = \log 80^\circ - \log|55^\circ| \quad \text{——— (4)}$$

Now solving (3) & (4).

$$(3) \div (4) \Rightarrow \frac{\cancel{k}t}{10\cancel{k}} = \frac{\log 80^\circ - \log|\theta - 20^\circ|}{\log 80^\circ - \log|55^\circ|}$$

$$\Rightarrow \frac{t}{10} = \frac{\log|80/\theta - 20|}{\log|80/55|} \quad \text{——— (5)}$$

(i). $\theta = ?$ at time $\boxed{t = 30 min}$ sub. in (5)

$$\Rightarrow \frac{3\cancel{0}}{1\cancel{0}} = \frac{\log|80/\theta - 20|}{\log|80/55|}$$

$$\Rightarrow 3\log(80/55) = \log\left(\frac{80}{\theta - 20}\right)$$

$$\Rightarrow \log(80/55)^3 = \log(80/\theta - 20)$$

$$\Rightarrow \left(\frac{80}{55}\right)^3 = \frac{80}{\theta - 20} \Rightarrow \boxed{\theta = 45.99 \simeq 46^\circ c}$$

$$\Rightarrow \left(\frac{80}{55}\right)^3 = \frac{80}{\theta-20} \Rightarrow \boxed{\theta = 45.99 \simeq 46^\circ C}$$

(ii) $t = ?$ at $\boxed{\theta = 25^\circ C}$ sub. in (5)

$$\Rightarrow \frac{t}{10} = \frac{\log(80/25-20)}{\log(80/55)}$$

$$\Rightarrow \frac{t}{10} = \frac{\log_e(80/5)}{\log_e(80/55)} \Rightarrow \frac{t}{10} = \frac{2.77}{0.37} \Rightarrow \boxed{t = 74.8}$$

3. A body kept in air 25°c cools from 140°c to 80°c in 20min. 35°C.

Sol:- Let 'θ' be the tem. of the body at time 't'.

θ_0 " " " air.

Given that $\boxed{\theta_0 = 25^\circ C}$

Initially $\theta = 140^\circ C$ at $t = 0$ min.

$\theta = 80^\circ C$ at $t = 20$ min.

We have to find 't' at $\theta = 35^\circ C$.

by N-L-C. W.K.T.

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\Rightarrow \int \frac{1}{\theta - \theta_0} d\theta = -k \int dt$$

$$\Rightarrow \log|\theta - \theta_0| = -kt + c \quad \text{—— (1)}$$

$$\boxed{c = \log|115^\circ|} \quad \text{—— (2)}$$

$$kt \Rightarrow \log|115^\circ| - \log|\theta - 25^\circ| \quad \text{—— (3)}$$

$$20k = \log|115| - \log|55^\circ| \quad \text{—— (4)}$$

$$\Rightarrow \frac{t}{20} = \frac{\log|115/\theta-25|}{\log|115/55|} \quad \text{—— (5)}$$

we have to find $t = ?$ at $\boxed{\theta = 35^\circ C}$ sub. in (5)

$$\frac{t}{20} = \frac{\log|115/10|}{\log|115/55|} \Rightarrow \boxed{t = 66.25}$$

$\therefore$ The temp will be $35^\circ C$ after 66.25 min.

④. $20^\circ C$

$100^\circ C$ to $80^\circ C$ in 10 min

after 20 minutes. $40^\circ C$.

Sol:-

$\theta_0 = 20^\circ C$.

$\theta = 100^\circ C \rightarrow t = 0$ min

$\theta = 80^\circ C \rightarrow t = 10$ min

(i) $t = 20$ min $\rightarrow \theta = ?$ ✓ $65^\circ C$

(ii) $\theta = 40^\circ C \rightarrow t = ?$ 5 min.

⑤. $100^\circ C$

$30^\circ C$ — r-T.

2 min $\rightarrow 90^\circ C$

5 min

Sol:-

$\theta_0 = 30^\circ C$

$\theta = 100^\circ C \rightarrow t = 0$ min.

$\theta = 90^\circ C \rightarrow t = 2$ min

$\theta = ? \rightarrow t = 5$ min

(77.6)

⑥. $75^\circ C$ cools in an atmosphere of constant temperature $25^\circ C$. at "$k\theta$", 'θ'

after 10 min, $65^\circ C$

20 min, $55^\circ C$. $\Rightarrow \frac{d\theta}{dt} \propto \theta$

Sol:- $\Rightarrow \frac{d\theta}{dt} = -k\theta$

$\Rightarrow \int \frac{1}{\theta} d\theta = k \int dt \Rightarrow \log\theta = -kt + c$ —— (1).

Initially $\boxed{t=0} \Rightarrow \boxed{\theta = 75^\circ C - 25^\circ C = 50^\circ C.}$ sub. in (1)

$\Rightarrow \log 50^\circ = -k(0) + c \Rightarrow \boxed{c = \log 50^\circ C}$ —— (2)

sub. (2) in (1)

$\Rightarrow \log\theta = -kt + \log 50^\circ \Rightarrow kt = \log 50^\circ - \log\theta$ —— (3)

Given that $\boxed{t = 10\text{min}} \Rightarrow \boxed{\theta = 65^\circ C - 25^\circ C = 40^\circ C}$ sub. in (3)

$\Rightarrow 10k = \log 50^\circ - \log 40^\circ$ —— (4)

solving (3) & (4).

$\frac{(3)}{(4)} \Rightarrow \frac{\cancel{k}t}{10\cancel{k}} = \frac{\log 50^\circ - \log\theta}{\log 50^\circ - \log 40^\circ} \Rightarrow \frac{t}{10} = \frac{\log(50/\theta)}{\log(50/40)}$ —— (5)

(i). $\boxed{\theta = ?}$ after $\boxed{t = 20\text{min}}$

sub. above in (5)

$\frac{2\cancel{0}}{1\cancel{0}} = \frac{\log(50/\theta)}{\log(5\cancel{0}/4\cancel{0})}$

$\left(\frac{5}{4}\right)^2 = \frac{50}{\theta} \Rightarrow \frac{25}{16} = \frac{50}{\theta} \Rightarrow \boxed{\theta = 32^\circ}$

(ii). $\boxed{t = ?}$ at $\theta = 55^\circ C - 25^\circ C$

$\boxed{\theta = 30^\circ C}$

sub. in (5)

$\frac{t}{10} = \frac{\log(5\cancel{0}/3\cancel{0})}{\log(5\cancel{0}/4\cancel{0})}$

$\boxed{t = 23.18}$

$\left(\frac{5}{4}\right)^2 = \frac{50}{\theta} \Rightarrow \frac{25}{16} = \frac{50}{\theta} \Rightarrow \boxed{\theta = 32}$

$\log\left(\frac{39}{49}\right)$ $\boxed{t = 23.18}$

⑦. $80°c$; $t=0$ | at $30°c$.
$t=3$.
$50°c$.
$t=?$ $\theta=40°c$.

5.26

Sol:- θ = Temperature of the copper ball.
θ_0 = " " " water i.e, $\boxed{\theta_0 = 30°c}$

Initially $t=0$ min and $\theta = 80°c$.
$t = 3$ min & $\theta = 50°c$
we have to find $t=?$ at $\boxed{\theta = 40°c}$

LAW of Natural Growth and Decay :—
'x' be at time 't',
rate of change of amount 'x' of a substance
$\propto$
" of that substance 't'.

$$\boxed{\frac{dx}{dt} \propto x}$$

Natural Growth :— $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx \Rightarrow \int \frac{dx}{x} = k\int dt$
$(k>0) \Rightarrow \boxed{\log x = kt + c}$

Natural Decay :— $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx \ (k>0) \Rightarrow \int \frac{dx}{x} = -k\int dt$
$\boxed{\log x = -kt + c}$

①. 'N' bacteria in a culture grew at a rate 'N'.
'N' was '100' and → "332" in one hour, 'N' after $1\frac{1}{2}$ hours.

Sol:- 'N' be No. of bacterias i.e, $\boxed{N = 100}$ initially $\boxed{t=0}$
$\boxed{N = 332}$ at time $t = 1$ hour
$\boxed{t = 60\text{min}}$.

Now, we have to find the no. of bacteria at a time
$t = 1\frac{1}{2}$ hour $= 90$min i.e, $\boxed{N=?}$ at $\boxed{t = 90\text{min}}$

by Natural Growth, $\frac{dN}{dt} \propto N$. $\left(\because \frac{dx}{dt} \propto x\right)$

$\frac{dN}{dt} = kN \quad (k>0)$

$$\frac{dN}{dt} = kN \quad (k>0).$$

$$\frac{dN}{N} = k\,dt \quad (\text{variable separable method})$$

$$\int \frac{1}{N}dN = k\int dt \quad (\because \int \text{ on by})$$

$$\log N = kt + c \quad \text{———— (1).}$$

~~Given that~~ initially $\boxed{t=0}$ and $\boxed{N=100}$ sub. in (1)

$$\Rightarrow \boxed{\log 100 = c} \quad \text{—— (2).}$$

Now sub. (2) in (1).

$$\log N = kt + \log 100$$

$$\Rightarrow kt = \log N - \log 100 \quad \text{—— (3).}$$

Given that $t = 60\text{min}$ and $\boxed{N = 332}$ sub. in (3)

$$\Rightarrow 60k = \log 332 - \log 100 \quad \text{—— (4).}$$

Now solving (3) & (4).

$$(3)\div(4) \Rightarrow \frac{\cancel{k}t}{60\cancel{k}} = \frac{\log N - \log 100}{\log 332 - \log 100}$$

$$\Rightarrow \frac{t}{60} = \frac{\log(N/100)}{\log(332/100)} \quad \text{———— (5).}$$

we have to find $N = ?$ at $\boxed{t = 90\text{min}}$ sub. in (5)

$$\Rightarrow \frac{\overset{3}{\cancel{90}}}{\underset{2}{\cancel{60}}} = \frac{\log(N/100)}{\log(332/100)}$$

$$\Rightarrow 3\log\left(\frac{332}{100}\right) = 2\log\left(\frac{N}{100}\right)$$

$$\Rightarrow \log\left(\frac{332}{100}\right)^3 = \log\left(\frac{N}{100}\right)^2$$

$$\Rightarrow \left(\frac{332}{100}\right)^3 = \left(\frac{N}{100}\right)^2$$

$$\Rightarrow \boxed{N = 604.9} \simeq 605.$$

②. $\underline{\text{growing}}$ 200 to 500 grams from $\underbrace{6.\text{am to } 9.\text{am}}$.

grams = ? at noon.

$\underline{\text{Sol}}$:- Let 'N' be the no. of bacteria in a culture at any time $t>0$,

then. Given that $\boxed{N = 200 \text{ grams}}$, initially at $\boxed{t=0}$

$\boxed{N = 500 \text{ grams}}$ at $\boxed{t = 3\text{hours}}$ (6.am to 9.am)

$\boxed{N = 500 \text{ grams}}$ at $\boxed{t = 3 \text{ hours}}$ (6.am to 9.am)

we have to find $\boxed{N = ?}$ at $\boxed{t = 6 \text{ hours}}$ (from 6.am to 12noon)

w.k.t Law of Natural Growth $\frac{dN}{dt} \propto N$

$\frac{dN}{dt} = kN.$

(125.gms)

③. bacteria multiply to the instantaneous 'N' number present. If the original number doubles in 2 hrs ? when it will be tripled ?

Sol: Let 'N' be the No. of bacteria.

Let the original number be 'x'.

Given $\boxed{N = x}$ at $\boxed{t = 0}$

Given $\boxed{N = 2x}$ at $\boxed{t = 2 \text{ hrs}}$

we have to find $\boxed{t = ?}$ at $\boxed{N = 3x}$.

w.k.T. by law of Natural Growth $\frac{dN}{dt} \propto N$

$\frac{dN}{dt} = kN \quad (k > 0)$

$$\Rightarrow \frac{dN}{N} = k\,dt$$

$$\Rightarrow \int \frac{dN}{N} = \int k\,dt$$

$$\Rightarrow \log N = kt + C \quad \text{——— (1)}$$

Given that at $\boxed{t = 0}$ & $\boxed{N = x}$ sub. in (1)

$$\Rightarrow \boxed{\log x = C} \quad \text{——— (2).}$$

Sub. (2) in (1)

$$\Rightarrow \log N = kt + \log x$$

$$\Rightarrow kt = \log N - \log x \quad \text{——— (3).}$$

Given that $\boxed{N = 2x}$ at $\boxed{t = 2 \text{ hrs}}$ sub. in (3)

$$\Rightarrow 2k = \log 2x - \log x \quad \text{——— (4).}$$

$$(3) \div (4) \Rightarrow \frac{\cancel{k}t}{2\cancel{k}} = \frac{\log N - \log x}{\log 2x - \log x}$$

$$\Rightarrow \frac{t}{2} = \frac{\log (N/x)}{\log (2x/x)} \quad \text{——— (5)}$$

we have to find $t = ?$ at $\boxed{N = 3x}$

$$\Rightarrow \frac{t}{2} = \frac{\log(3\cancel{x}/\cancel{x})}{\log(2)}$$

$$\Rightarrow t/_2 = \frac{\log 3}{\log 2} \Rightarrow \boxed{t = 3.04\,\text{kg}}.$$

Q) 30% of a radio disappears in 10 days, how long will it take for 90% of it to disappear?

Sol:- Let the no. of radio active substance is $\underline{x = 100}$. at $\underline{t = 0}$ days (initially)

After 10 days the no. of radio active substance is $\underline{x = 70}$ at $t = \underline{10}$ days

Now, we have to find how long it will take to disappear of 90% of radio active substance i.e $x = \underline{10}$ at $t = ?$

By the law of Natural Decay,

w.k.t $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx$, $(k>0)$

$$\Rightarrow \frac{dx}{x} = -k\,dt$$

$$\Rightarrow \int \frac{1}{x} dx = -k \int dt$$

$$\Rightarrow \log x = -kt + c. \quad \text{—— (1)}$$

Intially $\boxed{x = 100}$ and $\boxed{t = 0}$ sub. in (1).

$$\Rightarrow \log 100 = -k(0) + c \Rightarrow \boxed{c = \log 100} \text{ — (2)}$$

sub. (2) in (1) $\Rightarrow \log x = -kt + \log 100$

$$\Rightarrow kt = \log 100 - \log x. \quad \text{—— (3)}$$

Given that $\boxed{x = 70}$ at $\boxed{t = 10}$ sub. in (3).

$$\Rightarrow 10k = \log 100 - \log 70. \quad \text{—— (4)}$$

Now solving (3) & (4).

(3) $\div$ (4) $\Rightarrow \frac{\cancel{k}t}{10\cancel{k}} = \frac{\log 100 - \log x}{\log 100 - \log 70}$

$$\Rightarrow \frac{t}{10} = \frac{\log(100/x)}{\log(10\cancel{0}/7\cancel{0})} \quad \text{—— (5)}$$

$t = ?$ at $\boxed{x = 10}$ sub. in (5).

$$\Rightarrow \frac{t}{10} = \log(10\cancel{0}/1\cancel{0})$$

$x = ?$ at $\boxed{x = 10}$ sub. ( ).

$$\Rightarrow \frac{t}{10} = \frac{\log(100/10)}{\log(10/7)}$$

$$\Rightarrow t = 10 \times \frac{\log 10}{\log(10/7)}$$

$$\boxed{t = 64.5 ..}$$

5% → 50 years.
100 years.
'x' be 100% of radium.
$x = 100 \rightarrow t = 0$.
$x = 95 \rightarrow t = 50$ years.
$x = ?$ ($t = 100$ years)

Differential Eqn's of first order but not first degree :-

Eg :- ① $\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right) + 2y = 0$. (Second degree & 1st order)

Eg :- ② $\left(\frac{dy}{dx}\right)^3 + \sin x \frac{dy}{dx} = x^2$ (1st order & 3rd degree)

① solvable for 'p' ✓

② " " 'y'

③ " 'x'

④ clairaut's type.

Note: i.e. say $p = \frac{dy}{dx}$.

the above Example Eqn, we can write ① $p^2 + px^2 + 2y = 0$. ✓

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)x^2 + 2y = 0$.

② $p^3 + \sin x(p) = x^2$.

① solve the D.E $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0$. —— (1)

Sol :- : D.E, but not 1st degree.

put $p = \frac{dy}{dx}$.

(1) $\Rightarrow x^2p^2 + xy\,p - 6y^2 = 0$ ✓ ($\because ax^2 + bx + c = 0$)

($\because ap^2 + bp + c = 0$).

clear $a = x^2$

$b = xy$

$c = -6y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -xy + \sqrt{x^2y^2 - 4x^2(-6y^2)}$$

$$\overline{\qquad\qquad}$$
$$2a$$

$$= \frac{-xy \pm \sqrt{x^2y^2 - 4x^2(-6y^2)}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$$

$$p = \frac{-xy \pm 5xy}{2x^2}$$

$$p = \frac{-y \pm 5y}{2x}$$

$$p = \frac{-y+5y}{2x}$$

$$p = \frac{\cancel{4}^{2}y}{\cancel{2}x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2}{x}\,dx \quad (\because \text{Variable separable method})$$

$$\Rightarrow \log y = 2\log x + \log c_1$$

$$\Rightarrow \log y = \log x^2 + \log c_1$$

$$\Rightarrow y = x^2 c_1$$

$$\Rightarrow \boxed{y - x^2 c_1 = 0}$$

$$p = \frac{-y-5y}{2x}$$

$$p = \frac{-\cancel{6}^{3}y}{\cancel{2}x}$$

$$\frac{dy}{dx} = \frac{-3y}{x}$$

$$\Rightarrow \int \frac{1}{y}\,dy = -3\int \frac{1}{x}\,dx$$

$$\Rightarrow \log y = -3\log x + \log c_2$$

$$\Rightarrow \log y = \log x^{-3} + \log c_2$$

$$\Rightarrow y = c_2 x^{-3}$$

$$y = \frac{c_2}{x^3}$$

$$\boxed{y - \frac{c_2}{x^3} = 0}$$

$$\therefore \text{G.S is } \left(y - x^2c_1,\ y - \frac{c_2}{x^3}\right) = 0$$

② $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (2y^2 - x^2) = 0$. (or) $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$

Sol:- $x^2p^2 - 2xyp + (2y^2 - x^2) = 0$. —— (I)

Here $a = x^2$ $\quad(\because ax^2 + bx + c = 0)$

$b = -2xy$ $\quad(\because ap^2 + bp + c = 0)$

$c = (2y^2 - x^2)$.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^4}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{-4x^2y^2 + 4x^4}}{2x^2}$$

$$= \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{\qquad}$$

$$P = \frac{y \pm \sqrt{x^2-y^2}}{x}$$

$$P = \frac{y+\sqrt{x^2-y^2}}{x}$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x} \quad \text{--- (1)}$$

$$\left[\because f(x,y) = \frac{y+\sqrt{x^2-y^2}}{x}\right.$$

$$f(kx,ky) = \frac{ky+\sqrt{k^2x^2-k^2y^2}}{kx}$$

$$= k^0 f(x,y).$$

$\therefore$ $f(x,y)$ is a Homogenous f^n]

O.E

Put $\boxed{y=vx}$ --- (2) $\Rightarrow \boxed{v=\frac{y}{x}}$

Diff. (2) w.r.t x.

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{--- (3)}$$

Now sub. (2), (3) in (1)

$$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2-v^2x^2}}{x}$$

$$\Rightarrow v+x\frac{dv}{dx} = \frac{vx+x\sqrt{1-v^2}}{x}$$

$$\Rightarrow \cancel{v}+x\frac{dv}{dx} = \cancel{v}+\sqrt{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1-v^2} \Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

I. on b.s.

$$\Rightarrow \int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \sin^{-1}v = \log x + \log c_1 \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a)+c\right]$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log(xc_1)$$

$$\Rightarrow \boxed{\sin^{-1}\left(\frac{y}{x}\right) - \log(xc_1) = 0}$$

$$P = \frac{y-\sqrt{x^2-y^2}}{x}.$$

$$\frac{dy}{dx} = \frac{y-\sqrt{x^2-y^2}}{x}$$

HW

HW

(3) $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{dy}{dx}\right) + 6 = 0.$ $\qquad (y-3x-c_1,\ y-2x-c_2)=0$

(4). $\underset{a}{yp^2} + \underset{b}{(x-y)p} \underset{c}{-x} = 0.$ $\qquad (y-x-c_1,\ x^2+y^2-c_2)=0$

(5). $p^2 x(x-1)(x-2) = (3x^2-6x+2)^2$

Sol: $p^2 = \frac{(3x^2-6x+2)^2}{x(x-1)(x-2)}$

$p = \pm\sqrt{\frac{(3x^2-6x+2)^2}{(x^2-x)(x-2)}}$

$\frac{dy}{dx} = \pm \frac{(3x^2-6x+2)}{\sqrt{x^3-3x^2+2x}}$ $\quad \because \left(p = \frac{dy}{dx}\right)$

$\int dy = \pm\int \frac{3x^2-6x+2}{\sqrt{x^3-3x^2+2x}}\,dx$ $\quad \left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c\right]$

$y = \pm 2\sqrt{x^3-3x^2+2x} + c$

$y = 2\sqrt{x^3-3x^2+2x} + c_1$; $y = -2\sqrt{x^3-3x^2+2x} + c_2$

$y - 2\sqrt{x^3-3x^2+2x} - c_1 = 0$; $y + 2\sqrt{x^3-3x^2+2x} - c_2 = 0$.

Gs is $\left(y - 2\sqrt{x^3-3x^2+2x} - c_1,\ y + 2\sqrt{x^3-3x^2+2x} - c_2\right) = 0.$

6. $4xp^2 = (3x-a)^2$

Sol: $\Rightarrow p^2 = \frac{(3x-a)^2}{4x}$

$p = \pm\sqrt{\frac{(3x-a)^2}{4x}} \Rightarrow p = \pm\frac{3x-a}{2\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{3x-a}{2\sqrt{x}}$; $\frac{dy}{dx} = -\frac{3x-a}{2\sqrt{x}}$

$\Rightarrow dy = \left[\frac{3x-a}{2\sqrt{x}}\right]dx$

$\Rightarrow dy = \left(\frac{3}{2}x^{1/2} - \frac{a}{2}x^{-1/2}\right)dx.$

$\Rightarrow \int dy = \frac{3}{2}\int x^{1/2}dx - \frac{a}{2}\int x^{-1/2}dx.$

$\Rightarrow y = \frac{3}{2}\left(\frac{x^{3/2}}{3/2}\right) - \frac{a}{2}\left(\frac{x^{1/2}}{1/2}\right) + c_1$

$y = x^{3/2} - ax^{1/2} + c_1$

$\Rightarrow \boxed{y - x^{3/2} + ax^{1/2} - c_1 = 0}$

H.W $xp^2 = (x-a)^2$ $\quad$ [A:- $y = -\frac{2}{3}x^{3/2} + a\sqrt{x} + c_1$, $\frac{2}{3}x^{3/2} - a\sqrt{x} + c_2$

H.W. $p^2 + 2py\cot x = y^2$ $\quad$ by $\frac{c}{y} = 1-\cos x$, $\quad cy = \frac{1-\cos x}{\sin x}$

H.W. $p^2 + 2py\cot x = y^2$ by $\frac{c}{y} = 1-\cos x$, $cy = \frac{1-\cos x}{\sin x}$

H.W. $p^2 - yp - (x^2 - xy) = 0.$

H.W. $(p+y+x)(xp+y+x)(p+2x) = 0$

Sol:

$p+y+x = 0$

$\frac{dy}{dx} + y + x = 0.$

$\frac{dy}{dx} + y = -x.$

which is in the form of Linear D.E

$\frac{dy}{dx} + P(x)y = Q(x)$

$P(x) = 1$

$Q(x) = -x.$

$xp + y + x = 0$

$x\frac{dy}{dx} + y + x = 0.$

$x\frac{dy}{dx} = -y - x.$

$\frac{dy}{dx} = -\frac{y}{x} - 1.$

$\frac{dy}{dx} + \frac{y}{x} = -1.$

$\left[\because \frac{dy}{dx} + P(x).y = Q(x)\right]$

$P(x) = 1/$

$Q(x) = 1$

$p + 2x = 0.$

$\frac{dy}{dx} = -2x.$

$dy = -2x\,dx.$

Solvable for 'y' :-

Singular soln :- A soln D.E does not consists arbitrary c.soln.

General soln :- A soln of D.E having $c_1, c_2, c_3 \ldots$

Procedure :- Let $f(x, y', p) = 0$ — (1) be the (1). factor

If (1) cannot be split up into and (2).
1st degree 'y'. then (1) for 'y'.

$y = f(x, p)$ — (2).

$\left(\frac{dy}{dx} = p\right)$ (x, y, c)

(1). Solve $y' = a\sqrt{1+p^2}$ — (1).

Sol:

Diffn (1) w.r.to 'x'.

$\frac{dy}{dx} = a\frac{1}{\cancel{2}\sqrt{1+p^2}}(\cancel{2}p)\frac{dp}{dx}.$

$\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \frac{dy}{dx} = \frac{ap}{\sqrt{1+p^2}}.\frac{dp}{dx}.$

$\Rightarrow \cancel{p} = \frac{a\cancel{p}}{\sqrt{1+p^2}}\frac{dp}{dx}$

$\Rightarrow 1 = \frac{a}{\sqrt{1+p^2}}\frac{dp}{dx}$

$\Rightarrow \frac{1}{a}\int dx = \int \frac{1}{\sqrt{1+p^2}}\, dp$ ($\because$ Variable separable)

$\Rightarrow \frac{1}{a}x = \sinh^{-1} p + c$. $\quad \because \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(x/a)+c$.

$\Rightarrow \frac{x}{a} - c = \sinh^{-1} p$

$\Rightarrow \boxed{p = \sinh\left(\frac{x}{a}-c\right)}$ — (2)

Now sub, (2) in (1), we get G.S.

(1) $\Rightarrow y = a\sqrt{1+p^2}$

$y = a\sqrt{1+\sinh^2\left(\frac{x}{a}-c\right)}$

$y = a\sqrt{\cosh^2\left(\frac{x}{a}-c\right)}$

$y = a\cosh\left(\frac{x}{a}-c\right)$ ✓

(2). $y - 2px + p^2 = 0$. — (1)

$p^2 - 2px + y = 0$.

$(ap^2+bp+c=0)$

$-b \pm \sqrt{}$

Sol:- D.E 'p'

$y = 2px - p^2$

Diff. w.r.to 'x'

$\frac{dy}{dx} = 2p(1) + x(2)\frac{dp}{dx} - 2p\frac{dp}{dx}$

$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

$p = 2p + 2\frac{dp}{dx}(x-p)$

$\Rightarrow 2(x-p)\frac{dp}{dx} = -p$.

$\Rightarrow \frac{dp}{dx} = \frac{-p}{2(x-p)}$.

$\int \frac{2(x-p)}{-p}\,dp = \int dx$.

$\int \left(\frac{2x}{-p}+2\right)dp = \int x + c$.

$\Rightarrow \frac{dx}{dp} = \frac{2(x-p)}{-p}$.

$\Rightarrow \frac{dx}{dp} = \frac{2x}{-p} + 2$.

$\Rightarrow \frac{dx}{dp} + \left(\frac{2}{p}\right)x = 2$.

Linear D.E (L.D.E)

$\boxed{\frac{dx}{dp} + P(p)\,x = Q(p)}$ $\rightarrow$

$\frac{dy}{dx} + P(x)y = Q(x)$

I.F $= e^{\int P(x)dx}$

G.S $\Rightarrow y(\text{I.F}) = \int Q(x)$

$P(p) = 2 \quad ; \quad Q(p) = 2$.

$$P(p) = \frac{2}{p} \; ; \; Q(p) = 2.$$

$$\text{I.F} = e^{\int P(p)dp} = e^{\int 2/p \, dp} = e^{2\log p} = e^{\log_e p^2} = p^2$$

$$\text{I.F} = e^{\int P(x)dx}$$

$$\text{G.S} \Rightarrow y(\text{I.F}) = \int Q(x) \, \text{I.F} \, dx + c$$

$$\text{G.S is} \quad x \times (\text{I.F}) = \int Q(p)(\text{I.F})\, dp + c$$

$$\Rightarrow x \times p^2 = \int 2 \cdot p^2 \, dp + c$$

$$\Rightarrow xp^2 = \frac{2p^3}{3} + c. \quad \Rightarrow \boxed{3xp^2 - 2p^3 = c.}$$

③. $y + px = p^2x^4.$ ——— (1).

Sol^n: D.E interms of 'p'.

i.e., $p^2x^4 - px - y = 0.$

$(p^2a + pb + c = 0)$ (where $a = x^4$, $b = -x$, $c = -y$)

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{x \pm \sqrt{x^2 + 4x^4y}}{2x^4} = \frac{x \pm x\sqrt{1 + 4x^2y}}{2x^4} = \frac{1 \pm \sqrt{1 + 4x^2y}}{2x^3}$$

by solved for 'y' method.

Diff. (1) w.r.to 'x' on b.s.

$$\frac{dy}{dx} + p(1) + x\frac{dp}{dx} = p^2(4x^3) + x^4(2p)\frac{dp}{dx}$$

$$\Rightarrow \frac{dy}{dx} + p + x\frac{dp}{dx} = 4p^2x^3 + 2px^4\frac{dp}{dx}$$

$$\Rightarrow p + p + x\frac{dp}{dx} - 4p^2x^3 - 2px^4\frac{dp}{dx} = 0.$$

$$\Rightarrow 2p + x\frac{dp}{dx}(1 - 2px^3) - 4p^2x^3 = 0$$

$$\Rightarrow x\frac{dp}{dx}(1 - 2px^3) + 2p(1 - 2px^3) = 0$$

$$\Rightarrow (1 - 2px^3)\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\Rightarrow 1 - 2px^3 = 0$$

$$1 = 2px^3$$

$$\boxed{p = \frac{1}{2x^3}}$$

will give singular sol^n. (Ignore this)

$$x\frac{dp}{dx} + 2p = 0.$$

$$x\frac{dp}{dx} = -2p.$$

$$\Rightarrow \frac{dp}{p} = -2\frac{dx}{x}. \quad \text{(variable separable)}$$

$$\Rightarrow \int\frac{dp}{p} = -2\int\frac{dx}{x}$$

$\Rightarrow \log p = -2\log x + \log c.$

$\Rightarrow \log p = \log x^{-2} + \log c$

$\Rightarrow \log p = \log(x^{-2}c)$

$p = x^{-2}c$

$\boxed{p = \frac{c}{x^2}}$ will give General soln.

Now sub. 'p' value in (1).

$(1) \Rightarrow y = \left(\frac{c}{\cancel{x^2}}\right)\cancel{x} + \frac{c^2}{\cancel{x^4}}(\cancel{x^4})$

$\Rightarrow y = -\frac{c}{x} + c^2$

$\Rightarrow y = c^2 - \frac{c}{x} \Rightarrow \boxed{yx = c^2x - c}$

④. $y' = 3x + \log p.$

Sol:- $y = 3x + \log p$ —— (1)

by solvable for 'y' method

Diff. (1) w.r.to 'x' on b.s.

$\frac{dy}{dx} = 3(1) + \frac{1}{p}\frac{dp}{dx}.$

$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx}$ $\quad \left(\because \frac{dy}{dx} = p\right)$

$\Rightarrow \frac{1}{p}\frac{dp}{dx} = p - 3.$

$\Rightarrow \int \frac{1}{p(p-3)}\,dp = \int dx$ ($\because$ variable separable) —— (2).

$\Rightarrow$ By partial fractions, we have to L.H.S.

L.H.S $= \frac{1}{p'(p-3)'} = \frac{A}{p} + \frac{B}{p-3}$ —— (3).

$\frac{1}{p(p-3)} = \frac{A(p-3) + Bp}{p(\cancel{p-3})}$

$1 = A(p-3) + Bp$ —— (4)

Put $p - 3 = 0$ in (4) $\Rightarrow 1 = B(3) \Rightarrow \boxed{B = \frac{1}{3}}$

$\boxed{p = 3}$

Put $\boxed{p = 0}$ in (4) $\Rightarrow 1 = A(-3) \Rightarrow \boxed{A = -1/3}$

$\therefore$ 'A' & 'B' values in (3)

$\therefore \frac{1}{p(p-3)} = \frac{-1}{3p} + \frac{1}{3(p-3)}$ —— (5)

Sub. (5) in (2).

① $\frac{1}{x'(x-1)'} = \frac{A}{x} + \frac{B}{x-1}$

②. $\frac{1}{x^2(x-1)'} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

③. $\frac{1}{(x^2+1)'(x-1)'} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

④. $\frac{1}{(x^2+1)(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}.$

$$\Rightarrow \int\left[\frac{-1}{3p}+\frac{1}{3(p-3)}\right]dp = \int dx$$

$$\Rightarrow -\frac{1}{3}\int \frac{1}{p}\,dp + \frac{1}{3}\int \frac{1}{p-3}\,dp = x + c$$

$$\Rightarrow -\frac{1}{3}\log p + \frac{1}{3}\log(p-3) = x + c$$

$$\Rightarrow -\log \underset{b}{p} + \log \underset{a}{(p-3)} = 3x + c$$

$$\Rightarrow \boxed{\log\left(\frac{p-3}{p}\right) = 3x + 3c}$$ //

(5). $y = p\tan p + \log\cos p$ (A:- $\tan p = x + c$.)

(6). $x^2 + xp^2 = yp$.

$xp^2 - yp + x^2 = 0$

$ap^2 + bp + c = 0$

$\downarrow$

sol:-

$$y = \frac{x^2 + xp^2}{p}$$

$$y = \frac{x^2}{p} + xp \quad \text{——— (1).}$$

diff. w.r.to 'x'.

$$\frac{dy}{dx} = \frac{p(2x) - x^2\frac{dp}{dx}}{p^2} + x\frac{dp}{dx} + p(1).$$

$$\cancel{p} = \frac{2xp - x^2\frac{dp}{dx}}{p^2} + x\frac{dp}{dx} + \cancel{p}$$

$$0 = \frac{2x}{p} - \underbrace{\frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx}}$$

$$-\frac{2x}{p} = \frac{dp}{dx}\left(x - \frac{x^2}{p^2}\right).$$

$$-\frac{2\cancel{x}}{p} = \cancel{x}\frac{dp}{dx}\left(1 - \frac{x}{p^2}\right)$$

$$\Rightarrow \frac{-2}{p} = \frac{dp}{dx}\left(1 - \frac{x}{p^2}\right)$$

$$\Rightarrow \frac{-2}{\cancel{p}} = \frac{dp}{dx}\left(\frac{p^2 - x}{p^{\cancel{2}}}\right)$$

$$\Rightarrow \frac{dp}{dx} = \frac{-2p}{p^2 - x}.$$

$$\Rightarrow \frac{dx}{dp} = \frac{p^2 - x}{-2p}$$

$$\Rightarrow \frac{dx}{dp} = -\frac{p}{2} + \frac{x}{2p}$$

$$\Rightarrow \frac{dx}{dp} - \frac{x}{2p} = -\frac{p}{2}.$$

$$IF = e^{\int p(x)dx}$$

$$\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$$

$$\left[\frac{dx}{dp} + P(p)x = Q(p)\right]$$

$$\Rightarrow \frac{dx}{dp} - \frac{x}{2p} = -\frac{p}{2}.$$

$$\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$$

which is in the form L.D.E $\boxed{\frac{dx}{dp} + P(p)x = Q(p)}$

where $P(p) = \frac{-1}{2p}$; $Q(p) = -\frac{p}{2}$

$$I.F = e^{\int P(p)dp} = e^{\int -1/2p \, dp}$$

G.S is $x \times I.F = \int Q(p) \cdot I.F \, dp + $

$A = a/2$
$B = -a/2$
$C = -a/2$

⑦. $y = x + a\tan^{-1}p$ —— (1).

(A:- $x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(1+p^2) - \tan^{-1}p\right]$)

Sol:- Diffn w.r.t 'x' on bs.

$$\frac{dy}{dx} = 1 + a\frac{1}{1+p^2}\frac{dp}{dx} \qquad \left(\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right)$$

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}.$$

$$\frac{a}{1+p^2}\frac{dp}{dx} = p-1$$

$$\frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

$$\Rightarrow \frac{dx}{dp} = \frac{a}{(p-1)(1+p^2)} \quad \text{—— (1)}$$

$$\frac{a}{(p-1)(1+p^2)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$$

$$\Rightarrow a = A(p^2+1) + (Bp+C)(p-1)$$

$$\Rightarrow a = Ap^2 + A + Bp^2 - Bp + Cp - C.$$

$$\Rightarrow a = p^2(A+B) + p(C-B) + (A-C).$$

Differential Eqns of first order but not first degree :-

Eg :- $\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right) + 2y = 0$ (order = 1, degree = 2)

$\left(\frac{dy}{dx}\right)^3 + \sin x \frac{dy}{dx} = x^2$ (order = 1, degree = 3.)

① . Solvable for 'p'

② . solvable for 'y'

③ . Solvable for 'x'

④ . clairaut's type.

The methods to solve the above Eqn :-

① solvable for 'p'

② " 'y'

① solvable for 'p'

② " 'y'

③ " 'x'

④. clairaut's type.

solvable for 'p' method :-

Note :- ①. Expressed in terms of 'p', $\boxed{p = \frac{dy}{dx}}$

Eg :- ①. $\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right) + 2y = 0.$

$\Rightarrow p^2 + x^2 p + 2y = 0.$ $\left(\text{put } p = \frac{dy}{dx}\right)$

②. $p^3 + \sin x \, p = x^2$ (or) $\left(\frac{dy}{dx}\right)^3 + \sin x \left(\frac{dy}{dx}\right) = x^2$

①. $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0.$ —— (1)

Sol :- The given D.E is 1st order D.E, But not 1st degree.

put $\frac{dy}{dx} = p.$ —— (2)

sub., (2) in (1).

$x^2 p^2 + xy\,p - 6y^2 = 0.$ $\quad \because ax^2 + bx + c = 0.$

$\left(a p^2 + b p + c = 0\right)$ $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

$a = x^2;\ b = xy;\ c = -6y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-xy \pm \sqrt{x^2y^2 - 4x^2(-6y)^2}}{2x^2}$$

$$p = \frac{-xy \pm \sqrt{1x^2y^2 + 24x^2y^2}}{2x^2}$$

$$p = \frac{-xy \pm \sqrt{25x^2y^2}}{2x^2}$$

$$p = \frac{-xy \pm 5xy}{2x^2}$$

$$p = \frac{\cancel{x}(-y \pm 5y)}{2x^{\cancel{2}}}$$

$p = -y \pm 5y$

$2x$

$p = \frac{-y+5y}{2x}$

$\frac{dy}{dx} = \frac{\cancel{4}^{2}y}{\cancel{2}x}$ $[\because (2)]$

$\frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \frac{dy}{y} = \frac{2}{x}dx$ ($\because$ variable-separable)

Integrating on b.s.

$\int \frac{1}{y}dy = 2\int \frac{1}{x}dx$

$\log y = 2\log x + \log c_1$

$\log y = \log x^2 + \log c_1$ $(\because \log x^m = m\log x)$

$\log y = \log(x^2 c_1)$ $(\because \log a + \log b = \log ab)$

$y = x^2 c_1$

$\boxed{y - x^2 c_1 = 0}$

$p = \frac{-y-5y}{2x}$

$\frac{dy}{dx} = \frac{-\cancel{6}^{3}y}{\cancel{2}x}$ $[\because (2)]$

$\frac{dy}{dx} = \frac{-3y}{x}$

$\frac{dy}{y} = \frac{-3}{x}dx$

$\int$ on b.s.

$\int \frac{1}{y}dy = -3\int \frac{1}{x}dx$

$\log y = -3\log x + \log c_2$

$\log y = \log x^{-3} + \log c_2$

$\log y = \log(x^{-3}c_2)$

$y = x^{-3}c_2$

(or)

$y = \frac{c_2}{x^3}$

$\boxed{y - \frac{c_2}{x^3} = 0}$

$\therefore$ General soln is $\left(y - x^2 c_1,\ y - \frac{c_2}{x^3}\right) = 0$

② $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + (2y^2 - x^2) = 0$ —— (1)

sol:

Put $\frac{dy}{dx} = p$ —— (2)

Sub. (2) in (1)

$x^2p^2 - 2xy\,p + (2y^2 - x^2) = 0$

$a\,p^2 + b\,p + c = 0$

$a = x^2$; $b = -2xy$; $c = 2y^2 - x^2$

$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$

$p = \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^4}}{2x^2}$

$p = \frac{2xy \pm \sqrt{4x^4 - 4x^2y^2}}{2x^2}$ $\Rightarrow p = \frac{2xy \pm \sqrt{4x^2(x^2 - y^2)}}{2x^2}$

$$p = \frac{2xy \pm 2x\sqrt{x^2-y^2}}{2x^2}$$

$$p = \frac{\cancel{2x}\left[y \pm \sqrt{x^2-y^2}\right]}{\cancel{2}x^{\cancel{2}}}$$

$$p = \frac{y \pm \sqrt{x^2-y^2}}{x}$$

$$p = \frac{y - \sqrt{x^2-y^2}}{x}$$

$$p = \frac{y + \sqrt{x^2-y^2}}{x}$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2-y^2}}{x}$$

Let $\frac{dy}{dx} = f(x,y) = \frac{y + \sqrt{x^2-y^2}}{x}$ — (3)

$$f(kx, ky) = \frac{ky + \sqrt{k^2x^2 - k^2y^2}}{kx}$$

$$f(kx, ky) = \frac{ky \pm k\sqrt{x^2-y^2}}{kx}$$

$$f(kx, ky) = \frac{\cancel{k}\left(y \pm \sqrt{x^2-y^2}\right)}{\cancel{k}x}$$

$$f(kx, ky) = k^0\left[\frac{y \pm \sqrt{x^2-y^2}}{x}\right]$$

$$f(kx, ky) = k^0 f(x,y) \quad [\because (3)]$$

$\therefore$ (3) is a Homogenous DE.

put $\boxed{y = vx}$ — (4) $\Rightarrow \boxed{v = \frac{y}{x}}$

Diff. (4) w.r.to 'x'

$$\frac{dy}{dx} = v(1) + x\frac{dv}{dx} \qquad \left(\because \frac{d}{dx}(uv) = uv' + vu'\right)$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}} \quad \text{— (5)}$$

Sub. ~~(4)~~ & (5) in (3)

$$(3) \Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2-y^2}}{x}$$

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 - v^2x^2}}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2(1-v^2)}}{x}$$

$$v + x\frac{dv}{dx} = \frac{vx + x\sqrt{(1-v^2)}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\cancel{x}\left[v+\sqrt{1-v^2}\right]}{\cancel{x}}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sqrt{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1-v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{1}{x}dx$$ (∵ variable seperable)

$$\Rightarrow \int\frac{1}{\sqrt{1-v^2}}dv = \int\frac{1}{x}dx$$ (∵ $\int$ on b.s)

$$\Rightarrow \sin^{-1}v = \log x + \log c_1$$ $\left[\because \int\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c\right]$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log(xc_1)$$ $\left(\because v = \frac{y}{x}\right)$ from (4)

$$\Rightarrow \boxed{\sin^{-1}(y/x) - \log(xc_1) = 0}$$ $\left[\left(\quad\because\right)\ \sin^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{c_2}{x}\right)\right] = 0$

H.W. (3) solve $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{dy}{dx}\right) + 6 = 0$. $(y-3x-c_1,\ y-2x-c_2) = 0$

$$p^2 - 5p + 6 = 0$$

$$(p-3)(p-2) = 0$$

$p-3=0$ Rough | $p-2=0$

(8). $x^3p^2 + \overset{uvw}{x^2yp} + 4 = 0$ ——(1)

sol:- Q.E 'p'.

$(ap^2+bp+c=0)$ $(\because ax^2+bx+c=0)$

where $a = x^3$; $b = x^2y$; $c = 4$.

$$p = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$p = \frac{-x^2y \pm \sqrt{x^4y^2 - 4x^3(4)}}{2x^3} = \frac{-x^2y \pm x\sqrt{x^2y^2 - 16x}}{2x^3}$$

Solved for 'p'.

Solved for 'y' method.

$$\Rightarrow y = \frac{-4 - x^3p^2}{x^2p}$$

$$y = \frac{-4}{x^2p} - xp.$$

diff. w.r.to 'x' on b.s.

$\frac{dy}{dx} =$

diff. (1) on b.s w.r.to 'x'

$$\Rightarrow x^3\,2p\frac{dp}{dx} + p^2(3x^2) + x^2y\frac{dp}{dx} + yp(2x) + \left[\because \frac{d}{dx}(uvw) = uvw' + vwu' + wuv'\right]$$

$px^2\frac{dy}{}$ ... $= 0$

$$\frac{d}{dx}\ +0=0$$

$$\Rightarrow 2px^3\frac{dp}{dx}+3x^2p^2+x^2y\frac{dp}{dx}+2pxy+px^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dp}{dx}\left(2px^3+x^2y\right)+3x^2p^2+2pxy+p^2x^2=0 \quad \left(\because \frac{dy}{dx}=p\right)$$

$$\Rightarrow \frac{dp}{dx}\left(2px^3+x^2y\right)=-4x^2p^2-2pxy$$

$$\Rightarrow \frac{dp}{dx}=\frac{-4x^2p^2-2pxy}{2px^3+x^2y}$$

$$\frac{dp}{d}\quad \frac{-2p\cancel{x}\cancel{(2xp+y)}}{x^2\cancel{(2px+y)}}$$

$$\Rightarrow \frac{dp}{dx}=\frac{-2p}{x}.$$

separable

$$\Rightarrow \int\frac{dp}{p}=-2\int\frac{1}{x}dx \quad (\because \text{Variable - method})$$

$$\Rightarrow \log p=-2\log x+\log c_1$$

$$\Rightarrow \log p=\log x^{-2}+\log c$$

$$\Rightarrow \log p=\log\left(x^{-2}c\right) \Rightarrow \boxed{p=x^{-2}c} \quad (\text{or}) \quad \boxed{p=\frac{c}{x^2}} \text{ will give G.S.}$$

Now sub. 'p' value in (1)

$$\Rightarrow \cancel{x^3}\frac{c^2}{x^4}+\cancel{x^2}y\frac{c}{\cancel{x^2}}+\varphi=0 \Rightarrow \boxed{\frac{c^2}{x}+cy+\varphi=0}$$

Solvable for 'x' :- $f(x,y,p)=0$ —— (1) both

(1). splitup and linear factors & (1) is of in 'x' (1) can be solved for 'x'.

$x=f(y,p)$ —— (2).

Diff. (2) w.r.to 'y' on b.s. we get

$\varphi(y,p,c)=0.$

$\psi(x,y,c)=0$

① .

② . p^2 D.E.

③ . x^1

④ . Diff. 'y'

⑤ =.

① . $xp^3=a+bp.$ —— (1)

Sol:- 'p' and 'y' solved for 'x'-method.

i.e., $x=\dfrac{a+bp}{p^3} \Rightarrow x=\dfrac{a}{p^3}+\dfrac{b}{p^2}$ —— (2).

Diff. w.r.to 'y' on b.s.

$$\frac{dx}{dy}=a\left(\frac{-3}{p^4}\right)\frac{dp}{dy}+b\left(\frac{-2}{p^3}\right)\frac{dp}{dy} \qquad \left(\because \frac{d}{dp}p^{-3}=-3p^{-3-1}=\frac{-3}{p^4}\right)$$

$$\Rightarrow \frac{1}{p}=\frac{-3a}{p^4}\frac{dp}{dy}-\frac{2b}{p^3}\frac{dp}{dy} \qquad \left(\because \frac{dy}{dx}=p\right)$$