Number Theory – Midterm 1 – Review

Problem 1 Prove or disprove: *There exists integers x, y such that* $x^2 + y^2 \equiv 3 \pmod 4$.

Problem 2 *Show that the equation* $x^2 + 13y^2 = 12z^2 + 3$ *has no integer solutions.*

Problem 3 *Show that* $\{1^{p-2}, 2^{p-2}, \ldots, p^{p-2}\}$ *is a complete residue system modulo* p *for any prime* p. *(What happens if* p *is composite?)*

Problem 4 *Does* $x^4 - x^3 + 1 = 0$ *have any integer solutions? Explain?*

Problem 5 *Which of the following numbers are representable as a sum of two integer squares: 41, 122, 150? Explain?*

Problem 6 *Solve* $x^3 - 2x + 4 \equiv 0 \pmod{3^3}$.

Problem 7 *Find the greatest common divisor of 72 and 231. Write it in the form* $d = 72x + 231y$.

Problem 8 *Calculate* $gcd(x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5, 3x^6 + 5x^4 - 4x^2 + 9x + 21)$.

Problem 9 *Find a number* n *satisfying*

$$\left\{\begin{matrix} n \equiv 5 \pmod 7 \\ n \equiv 2 \pmod 4 \\ n \equiv 2 \pmod 2 \end{matrix}\right\}$$

Problem 10 *If possible, find all solutions for each of the following equations. Explain if not.*

(a) $3x \equiv 5 \pmod 7$

(b) $3x \equiv 9 \pmod 6$

(c) $3x \equiv 5 \pmod 6$

(d) $3x + 5y = 51$

(e) $36x + 52y = 6$

(f) $x^{48} \equiv -1 \pmod{97}$

Learning outcomes:

(g) $x^2 \equiv -1 \pmod{97}$

(h) $x^{2017} \equiv 1 \pmod{97}$

(i) $x^{96} \equiv -1 \pmod{97}$

(j) $x^2 \equiv -1 \pmod{97}$

(k) $x^2 \equiv -1 \pmod{101}$

Problem 11 *Calculate* $45^{90} \pmod{17}$.