Quaternion Octonion Reformulation of Grand Unified Theories

Pushpa[1], P. S. Bisht[1,2], Tianjun Li[2] and O. P. S. Negi[1,2]

March 13, 2015

1.Department of Physics, Kumaun University, S. S. J. Campus, Almora - 263601 (U.K.), India

2. Institute of Theoretical Physics,Chinese Academy of Sciences, Zhong Guan Cun East Street 55, P. O. Box 2735, Beijing 100190, P. R. China.

Email-pushpakalauni60@yahoo.co.in; ps_bisht 123@rediffmail.com; tli@itp.ac.cn; ops_negi@yahoo.co.in

Abstract

In this paper, Grand Unified theories are discussed in terms of quaternions and octonions by using the relation between quaternion basis elements with Pauli matrices and Octonions with Gell Mann λ matrices. Connection between the unitary groups of GUTs and the normed division algebra has been established to re-describe the $SU(5)$ gauge group. We have thus described the $SU(5)$ gauge group and its subgroup $SU(3)_C \times SU(2)_L \times U(1)$ by using quaternion and octonion basis elements. As such the connection between $U(1)$ gauge group and complex number, $SU(2)$ gauge group and quaternions and $SU(3)$ and octonions is established. It is concluded that the division algebra approach to the the theory of unification of fundamental interactions as the case of GUTs leads to the consequences towards the new understanding of these theories which incorporate the existence of magnetic monopole and dyon.

Key Words: Grand unified theories, quaternion and octonion

PACS No.: 12.10 Dm, 12.60.-i, 14.80 Hv.

1

1 Introduction

In spite of all the successes of the Standard Model (SM) of elementary particles, it leaves many unresolved questions to be considered as the complete theory of matter. The next attempt is described as a grand unified theory (GUT) with more symmetry and reduces to the Standard Model at lower energies. It is thus considered as an attempt to describe the physics at higher energies where the three gauge interactions (namely the electromagnetic, weak, and strong interactions) of the Standard Model are merged into one single interaction characterized by one larger gauge symmetry with one unified coupling constant [1]. Here the standard model gauge groups $SU(3) \times SU(2) \times U(1)$ are combined into a single simple gauge group i.e. $SU(5)$. A series of hypothesis were argued [2, 3, 4, 5] in order to leading to the conclusion that a gauge theory based on the $SU(5)$ gauge group provide a unified description of the strong and electroweak interaction. The Georgi – Glashow model [1] was preceded by the Semi simple Lie algebra of Pati – Salam model [6] to unify gauge interactions. On the other hand, the four division algebras ($\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathcal{O}$ (octonions)) also play an important role in physics and mathematics particularly for unification program of fundamental interactions. It is remarkable that the existence of all three exception algebras and sub algebras is mathematically motivated by their connection through the maximum division algebra i.e. algebra of octonions [7]. Octonions are widely used for the understanding of unification structure of successful gauge theory of fundamental interaction. Recently, we [8] have made an attempt to develop the quaternionic formulation of Yang–Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD) by taking magnetic monopoles [9, 10, 11] and dyons (particles carrying electric and magnetic charges) [12, 13, 14] into account. It has been shown that the three quaternion units explain the structure of Yang-Mill's field while the seven octonion units provide the consistent structure of $SU(3)_C$ gauge symmetry of quantum chromo dynamics. Here we apply entirely different approach for the quaternion gauge theory of electroweak interactions and octonion gauge structure for quantum chromo dynamics (QCD) compare to the approaches adopted earlier by Morita [15, 16] and others [17, 18]. It is already explored [8] that the three quaternion units explain the structure of Yang- Mill's field while the seven octonion units provide the consistent structure of $SU(3)_C$ gauge symmetry of quantum chromo dynamics (QCD) as these are connected with the well known $SU(3)$ Gellmann λ matrices. Keeping these facts in mind

and extending our previous work [8], in this paper, we have made an attempt to discuss the Grand Unified Theories and then to reformulate them in terms of algebra of quaternions and octonions in simple and consistent manner. Accordingly, the connection between the unitary groups of GUTs and the normed division algebra has been established to re-describe the $SU(5)$ gauge group and its constituents. So, we have reformulated the GUTs gauge group $SU(5)$ its subgroup $SU(3)_C \times SU(2)_L \times U(1)$ in terms of quaternion and octonion basis elements. Hence, the connection between $U(1)$ gauge group and complex number, $SU(2)$ gauge group and quaternions and $SU(3)$ and octonions has been re-established. It is shown that grand unified theories in terms of quaternion and octonion contain the magnetic monopole. It is concluded that the division algebra approach to the the theory of unification of fundamental interactions as the case of GUTs leads to the consequences towards the new understanding of these theories which incorporate the existence of magnetic monopole and dyon. Three different imaginaries associated octonion formulation may be identified with three different colors (red, blue and green) while the Gell Mann Nishijima λ are described in terms of simple and compact notations of octonion basis elements. The symmetry breaking mechanism of non - Abelian gauge theories in terms of quaternion and octonion opens the window towards the discovery of two type of gauge bosons associated with electric and magnetic charges.

2 SU(5) Gauge Symmetry

Grand Unified Theories [19] are based on the mathematical symmetry group $SU(5)$. The gauge group of the Glashow - Salam - Weinberg theory $SU(2) \times U(1)$ and the $SU(3)$ group of the strong interaction are the constituents of a larger symmetry. The simplest group that incorporates the product $SU(3)_C \times SU(2)_L \times U(1)$ as a subgroup is $SU(5)$. An arbitrary unitary matrix can be represented in terms of an exponential of a Hermitian matrix H as [20],

$$\widehat{U} = e^{i\hat{H}} \, , \widehat{H}^\dagger = H;$$

(1)

where $\hat{H}$ is known as the generating matrix for $\hat{U}$, which can be written as,

$$\hat{U} = \exp\left(i\delta\hat{H}\right) \approx 1 + i\delta\hat{H}. \tag{2}$$

The multiplication of two matrices U_1, U_2 corresponds to the sum of the infinitesimal Hermitian matrices as,

$$\widehat{U_2}\widehat{U_1} \approx 1 + i\left(\delta\hat{H}_2 + \delta\hat{H}_1\right) \tag{3}$$

where quadratic terms are neglected. A complete set of linearly independent Hermitian matrices is termed as a set of generators for the unitary matrices. The unitary condition $UU^\dagger = 1$ and the uni modular condition $\det U = 1$ leave the $5^2 - 1 = 24$ independent matrices for $SU(5)$ symmetry. U may then be described as,

$$U = \exp\left(-i\sum_{a=1}^{24} A_\mu^a L^a\right); \tag{4}$$

where L^a contains 24 generators which are Hermitian and traceless and A_μ^a is a 5×5 matrix. Thus, the $SU(5)$ symmetry splits into the $SU(3)$ symmetry of strong force along with the $SU(2) \times U(1)$ gauge symmetry of electroweak force. The $SU(2) \times U(1)$ gauge symmetry also splits into a $SU(2)$ sub symmetry of the weak interaction and the $U(1)$ sub symmetry of the electromagnetic interaction. Here it is to be cleared that the 5×5 matrices L is taken in such a way that the colour group $SU(3)$ acts on first three rows and columns in terms of octonions [8], while the $SU(2)$ group operates on the last two rows and columns by means of quaternions [8] and $U(1)$ is the singlet associated with complex numbers. Thus the algebra of $SU(5)$ illustrates as $\mathcal{O} \oplus \mathbb{Q} \oplus \mathbb{C}$. $\mathcal{O}$ is used for octonions, $\mathbb{Q}$ for quaternions and $\mathbb{C}$ for complex numbers. $\mathcal{O}$ has the connection with $SU(3)$, $\mathbb{Q}$ describes $SU(2)$ and $\mathbb{C}$ is linked with $U(1)$. This gives the $SU(3) \times SU(2) \times U(1)$ subgroup structure of $SU(5)$ in terms of the constituents of three division algebras namely octonions $\mathcal{O}$, quaternion $\mathbb{Q}$ and

4

the algebra of complex numbers $\mathbb{C}$.

3 Quaternion-Octonion Reformulation of $SU(5)$ gauge Symmetry

Let us break up a 5×5 square matrix $SU(5)$ into four blocks, consisting of two smaller squares and two rectangles. The upper left - hand corner block denotes a 3×3 matrix while the lower right-hand corner block describes a 2×2 matrix. Off-diagonal upper right - hand and lower left - hand corners consist respectively the 3×2 and 2×3 rectangular matrices. The generators are described in such a way that the first eight generators are associated with the generators of $SU(3)$ symmetry as

$$L_a = \begin{pmatrix} \lambda^a & 0 \\ 0 & 0 \end{pmatrix}; \tag{5}$$

where $a = 1,\ 2,\ ....\ 8$ and λ are the well known 3×3 Gell Mann matrices. Here replace [8] Gell Mann λ matrices by octonion basis elements. For the 9th, 10th and 11th generators, we may use quaternion basis elements in the 2×2 block which is related to Pauli matrices σ_j. Let us write the quaternion scalar part as 1 corresponding to the 2×2 unit matrix and three imaginary quaternion units e_1, e_2, e_3 are connected with Pauli - spin matrices as

$$e_0 = 1; \qquad e_j = -i\sigma_j. \tag{6}$$

So, we may write

$$L^{8+j} = \begin{pmatrix} 0 & 0 \\ 0 & ie_j \end{pmatrix}; \tag{7}$$

5

where $j = 1, 2, 3$ belonging to the three generators of $SU(2)$ gauge group. Accordingly the L^{12} describes the hyper charge corresponding to $U(1)$ gauge group associated with the scalar part of a quaternion (or complex), so that

$$L^{12} = \frac{1}{\sqrt{15}} diag\,(-2, -2, -2, 3, 3) = \frac{1}{\sqrt{15}} \begin{bmatrix} -2I_3 & 0 \\ 0 & 3I_3 \end{bmatrix}. \tag{8}$$

For the next set of matrices, it will be convenient to define rectangular matrices A and B [21] as,

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{9}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Thus, the 13th to 24th generators of $SU(5)$ symmetry are expressed as,

$$L^{13,15,17} = L^{11+2k} = e_0 \begin{pmatrix} 0 & A_k \\ A_k^T & 0 \end{pmatrix} = e_0 L^{11+2k};$$

$$L^{14,16,18} = L^{12+2k} = -e_3 L^{11+2k};$$

$$L^{19,21,23} = L^{17+2k} = e_0 \begin{pmatrix} 0 & B_k \\ B_k^T & 0 \end{pmatrix} = e_0 L^{17+2k};$$

$$L^{20,22,24} = L^{18+2k} = e_3 L^{17+2k}; \tag{11}$$

where $k = 1, 2, 3$. Hence we may define the precise association between the vector gauge bosons of $SU(5)$ GUTs , vector gauge bosons of the electro-weak model $(W_\mu^\pm, W_\mu^3, B_\mu)$ and the eight vector gluon G_μ^a of QCD as,

6

$$A_\mu^{1,2,.....8} \Longrightarrow G_\mu^a \ (\forall a = 1, 2, ..., 8);$$
$$A_\mu^{9,10} \Longrightarrow W^\pm;$$
$$A_\mu^{11} \Longrightarrow W_\mu^3;$$
$$A_\mu^{12} \Longrightarrow B_\mu. \tag{12}$$

The new vector bosons, which are specified only by $SU(5)$ and do not contribute to the Standard Model, are then be expressed as,

$$A_\mu^{13,14.......18} = X_\mu^b \ (\forall b = 1, 2, ..., 6);$$
$$A_\mu^{19,20,......24} = Y_\mu^b \ (\forall b = 1, 2, ..., , 6). \tag{13}$$

Generalizing equations (11 to 13), we may write the 5×5 matrix representation for the $SU(5)$ gauge field A_μ in a compact form as,

$$A_\mu = \sqrt{2} \sum_{i=1}^{24} A_\mu^i L^i; \tag{14}$$

Which is also expressed as,

$$A_\mu^i = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}; \tag{15}$$

where

$$P = G_\mu - \frac{2}{\sqrt{3}} B_\mu I \ ; \ Q = \begin{pmatrix} \overline{X}^1 & \overline{Y}^1 \\ \overline{X}^2 & \overline{Y}^2 \\ \overline{X}^3 & \overline{Y}^3 \end{pmatrix}; \tag{16}$$

and

$$R = \begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix} \; ; \; S = \begin{pmatrix} \frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ W^- & \frac{-W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix}. \qquad (17)$$

The symmetry breaking of $SU(5)$ into $SU(3) \times SU(2) \times U(1)$ can be done in a similar way as the breaking of $SU(3)$ into $SU(2) \times U(1)$. The S generator in the adjoint representation of $SU(5)$ commutes with $SU(3) \times SU(2) \times U(1)$. The 24 adjoint representations with vacuum values in the S direction will consequently break $SU(5)$ into $SU(3) \times SU(2) \times U(1)$. Under $SU(3) \times SU(2) \times U(1)$ the decomposition of the adjoint 24 representation may then be expressed as,

$$[24]_5 = (8,\ 1) + (\overline{3},\ 2) + (3,\ 2) + (1,\ 3) + (1,\ 1). \qquad (18)$$

Here, the gauge bosons for strong interactions due to the exchange of quarks and colors are the eight gluon associated with the color gauge group $SU(3)_C$ and for the electro weak interactions $SU(2) \times U(1)$ are the intermediate Weak bosons and photons. Following arguments be made accordingly as

- There is a $SU(3)_C$ octet of gluon $G_\mu^a = (1,\ 2, ... 8) \longmapsto (8,\ 1)$.

- There is an isovector of intermediate bosons , $W_\mu^i\ (i = 1,\ 2,\ 3) \longmapsto (1,\ 3)$.

- There is an isoscalar field of the hyper charged boson $B_\mu \longmapsto (1,\ 1)$.

In addition, to the above three arguments there are twelve more gauge bosons, belonging to the representations $(3,\ 2)$ and $(\overline{3},\ 2)$. These gauge bosons form an isospin doublet of bosons and their antiparticles, which are colored. The simplest representation of $SU(5)$ is the five dimensional fundamental one ψ_5 which may be represented by a column matrix as

$$\psi_5 = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ a^4 \\ a^5 \end{pmatrix}. \tag{19}$$

In the $SU(2)$ symmetry of weak interactions the quaternions occur only on the rows 4 and 5 and we see that a_1, a_2 and a_3 are unaffected by the operation of $SU(2)$ generators i.e. quaternions. For the case of $SU(5)$, the covariant derivative may be written in terms of matrix representation as,

$$D_\mu = \partial_\mu - \frac{ig}{2} A_\mu. \tag{20}$$

Then all 24 gauge bosons A_μ^i $(i = 1, ..., 24)$ are conveniently represented by a 5×5 matrix. Adopting these 24 generators of $SU(5)$, we may write the equation (14) in the following form,

$$A_\mu = \sqrt{2} \sum_{a=1}^{24} A_\mu^a L_a = \left[\sum_{a=1}^{8} G_\mu^a L_a + \sum_{a=9}^{11} A_\mu^a L_a + B_\mu L_{12} + \sum_{a=13}^{24} A_\mu^a L_a \right]. \tag{21}$$

where $\sum_{a=1}^{8} G_\mu^a L_a$ has been expressed in terms of octonion units as,

$$\sum_{a=1}^{8} G_\mu^a L_a = \sum_{a=1}^{7} G_\mu^a e_a + G_\mu^8 e_3; \tag{22}$$

while the second term containing L_9, L_{10}, L_{11} are defined in terms of quaternion units as

$$\sum_{a=1}^{8} \lambda_a \alpha^a (x) = -i \sum_{q=1}^{7} e_q \beta^q (x);\qquad (23)$$

whereas the term $-i \sum_{q=1}^{7} e_q \beta^q (x)$ is expressed in terms of the matrix as,

$$-i \sum_{q=1}^{7} e_q \beta^q (x) = \begin{bmatrix} \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}.\qquad (24)$$

Hence we may write equation (21) as

$$A_\mu = \begin{pmatrix} -i \sum_{q=1}^{7} e_q \beta^q (x) & B \\ C & -i \sum_{i=1}^{3} W_\mu^i e_i \end{pmatrix} + \frac{B_\mu}{2\sqrt{15}} diag\,(-2, -2, -2, 3, 3);\qquad (25)$$

which is further simplified to the following form in terms of 5×5 matrix on substituting the value of P, Q, R, S i.e.

$$A_\mu = \begin{pmatrix} \alpha_3 + \frac{\alpha_8}{\sqrt{3}} - \frac{2B_\mu}{\sqrt{15}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 & \overline{X}^1 & \overline{Y}^1 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \frac{\alpha_8}{\sqrt{3}} - \frac{2B_\mu}{\sqrt{15}} & \alpha_6 - i\alpha_7 & \overline{X}^2 & \overline{Y}^2 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2\alpha_8}{\sqrt{3}} - \frac{2B_\mu}{\sqrt{15}} & \overline{X}^3 & \overline{Y}^3 \\ X_1 & X_2 & X_3 & W_\mu^3 + \frac{3B_\mu}{\sqrt{15}} & W_\mu^1 - iW_\mu^2 \\ Y_1 & Y_2 & Y_3 & W_1^\mu + iW_\mu^2 & -W_\mu^3 + \frac{3B_\mu}{\sqrt{15}} \end{pmatrix}.\qquad (26)$$

As such, the covariant derivative (20) associated with W_μ^3 and B_μ may then be expressed in terms of the coupling of A_μ and Z_μ by extracting the 11th and 12th generators of covariant derivative [22, 23] as,

10

$$D_\mu = \partial_\mu - i\frac{g}{2}\left(W_\mu^3 L^{11} + B_\mu L^{12}\right)$$
$$= \partial_\mu - i\frac{g}{2}\left[A_\mu\left(\sin\theta_W L^{11} + \cos\theta_W L^{12}\right) + Z_\mu\left(\cos\theta_W L^{11} - \sin\theta_W L^{12}\right)\right] \tag{27}$$

where θ_W is defined as the Weinberg angle.

4   Discussion and Conclusion

We have already shown earlier [8] that the quaternion and octonion gauge theories contain the magnetic monopole. As such, the reformulation of grand unified theories (GUTs) in terms of the gauge group $SU(5)$ and its splitting to $SU(3)_C \times SU(2) \times U(1)$ may lead to the simultaneous existence of two different gauge theories associated with electric and magnetic charges (i.e. dyons). This this approach may be used as the milestone for the unification of fundamental interaction at one end and the existence of magnetic monopole at other end so that the unanswered question for the existence of magnetic monopoles can be tackled. Three different imaginaries responsible for the creation of octonion formulation may be identified as the three different colors (red, blue and green) while the matrix form of Gell Mann Nishimijja matrices establish well defined connection with seven octonion elements. The symmetry breaking mechanism of non - Abelian gauge theory in terms of quaternion octonion leads to the existence of massive gauge bosons associated with electric and magnetic charges. The present theory may also provide a variety of compiled phenomenon involving the various particles and forces such as

- The occurrence of particle transition between number of family such as not only between electron and electron neutrino or u, d quarks but also with a monopole and monopole neutrino with monopole quarks. One can

also relate the quarks and neutrinos with the dyons.

- There exists two type of gauge bosons (W and Z) due to electron and monopole in order to transit forces.

- The fact that quarks never appear either alone (a phenomenon called quark confinement) or in combination which have a net color charge. Here we may emphasize that if quarks are considered as dyons the problem may be resolved automatically.

- There exists two types of gluon which transit the strong forces are capable of changing the "color" or chromo magnetic color charges of quarks as dyons which is in agreement with the results of Kühne [24].

As such, the foregoing analysis describes the embedding of $U(1){\times}U(1)$ model in grand unified theories (GUTs) so that one may imagine the underlying group to be $SU(5){\times}\ SU(5)$ where second $SU(5)$ describes the hypothetical magnetic photons, chromo- magnetic gluons and (iso-) magnetic W, Z, X and Y bosons.

References

[1] H. Georgi and S. Glashow, **"Unity of All Elementary-Particle Forces"**, Phys. Rev. Lett., **<u>32</u>** (1974) 438.

[2] Graham G. Ross, **"Grand Unified Theories"**, Benjamin/Cummings, 1985.

[3] R. N. Mohapatra, **"Unification and Supersymmetry: The Frontiers of Quark-Lepton Physics"**, Springer, 1992.

[4] E. Witten, **"Grand unification with and without supersymmetry, in Introduction to supersymmetry in particle and nuclear physics"**, Editors. O. Castanos, A. Frank, L. Urrutia, Plenum Press, 1984, pp. 53–76.

[5] J. Baez and J. Huerta, "**The Algebra of Grand Unified Theories"**, Bull. Am. Math.Soc., **47** (2010) 483: eprint-arXiv:0904.1556 [hep-th].

[6] J. Pati and A. Salam, **"Lepton Number as the Fourth Color"**, Physical Review, **D10** (1974), 275.

[7] J. C. Baez, **"The Octonions"**, Bull. Amer. Math. Soc., **39** (2001), 145.

[8] Pushpa, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Quaternion Octonion Reformulation of Quantum Chromodynamics"**, Int. J. Theor. Phys., 50 (2011), 594.

[9] P. A. M. Dirac, **"Quantised Singularities in the Electromagnetic Field"**, Proc. Royal Society, **A133** (1931), 60.

[10] G. 't Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[11] A. M. Polyakov, **"Particle spectrum in quantum field theory"**, JETP Lett., **20** (1974), 194.

[12] J. Schwinger, "**Dyons Versus Quarks"**, Science, **166** (1969), 690.

[13] D. Zwanzinger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[14] B. Julia and A. Zee , **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11** (1975), 2227.

[15] K. Morita, **"Octonions, Quarks and QCD"**, Prog. Theor. Phys., **65** (1981),787.

[16] K. Morita, **"An Algebraic Explanation for the Family Structure of Quarks and Leptons"**, Prog. Theor. Phys., **66** (1981), 1519.

[17] M. Günaydin and F. Gürsey, **"Quark structure and octonions"**, J. Math. Phys., **14** (1973), 1651.

[18] G. M. Dixon, **"Division Algebras:: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics"**, Springer (1994).

[19] A. J. Buras, J. R. Ellis, M. K. Gaillard, D. V. Nanopoulos, **"Aspects of the grand unification of strong, weak and electromagnetic interactions"**, Nucl. Phys., **B135** (1978), 66.

[20] G. Müller, **"Gauge theory of weak interactions"**, 4th edition, Springer (2009).

[21] M. Kaku, **"Quantum field theory"**, Oxford University Press, (1994).

[22] G. Ross, **"Grand Unified Theories"**, Westview Press, ISBN 978-0-805-36968-7, (1984).

[23] H. Georgi, **"Unified Gauge Theories"**, Proceedings, Theories and Experiments In High Energy Physics, New York (1975), 329.

[24] R. W. Kühne, **"A Model of Magnetic Monopoles"**, Mod. Phys. Letts., **A12** (1997), 40.