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# Norm Estimates for Inverses of Vandermonde Matrices

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*Summary.* Formulas, or close two-sided estimates, are given for the norm of the inverse of a Vandermonde matrix when the constituent parameters are arranged in certain symmetric configurations in the complex plane. The effect of scaling the parameters is also investigated. Asymptotic estimates of the respective condition numbers are derived in special cases.

## 1. Introduction

In an earlier paper [2] we obtained norm inequalities for the inverse $V_n^{-1}$ of a Vandermonde matrix $V(x_1, x_2, \ldots, x_n) \in \mathbb{C}^{n\times n}$, which become equalities if the complex parameters $x_\nu$ are all placed on a ray emanating from the origin. We now obtain equalities for $\|V_n^{-1}\|$ also in the case of parameters located symmetrically with respect to the origin on a straight line through the origin. For parameters which occur in conjugate complex pairs we give close upper and lower bounds for $\|V_n^{-1}\|$. We further examine how scaling of the parameters affects the magnitude of $\|V_n^{-1}\|$. Finally, as an application, we derive asymptotic estimates (for large $n$) for the condition number of Vandermonde matrices for special configurations of the parameters.

## 2. Preliminaries

We denote by $\sigma_m$ the $m$-th elementary symmetric function in $n$ complex variables,

$$\sigma_m = \sigma_m(x_1, x_2, \ldots, x_n) = \sum x_{\nu_1} x_{\nu_2} \ldots x_{\nu_m} \quad (1 \leqq m \leqq n), \qquad \sigma_0 = 1.$$

**Lemma 2.1.** *We have*

$$\sum_{m=0}^{n} |\sigma_m(x_1, x_2, \ldots, x_n)| \leqq \prod_{\nu=1}^{n} (1 + |x_\nu|), \tag{2.1}$$

*where equality holds if and only if* $x_\nu = |x_\nu|\, e^{i\phi}$, $\nu = 1, 2, \ldots, n$.

A proof of Lemma 2.1 is given in [2].

**Lemma 2.2.** *Let* $p_{2n}(x) = \prod_{\nu=1}^{n} (x^2 - x_\nu)$ *and*

$$(s + tx)\, p_{2n}(x) = \sum_{\mu=0}^{2n+1} c_\mu x^{2n-\mu+1}. \tag{2.2}$$

*Then*

$$\sum_{\mu=0}^{2n+1} |c_\mu| \leqq (|s|+|t|) \prod_{\nu=1}^{n} (1+|x_\nu|), \tag{2.3}$$

*where equality holds if and only if* $x_\nu = |x_\nu|\, e^{i\phi}$, $\nu = 1, 2, \ldots, n$.

*Proof.* Since

$$p_{2n}(x) = \sum_{\mu=0}^{n} (-1)^\mu \sigma_\mu(x_1, x_2, \ldots, x_n)\, x^{2n-2\mu},$$

we find for the coefficients $c_\mu$ in (2.2),

$$c_{2\mu} = (-1)^\mu t \sigma_\mu, \qquad c_{2\mu+1} = (-1)^\mu s \sigma_\mu, \qquad \mu = 0, 1, \ldots, n.$$

Consequently, using Lemma 2.1,

$$\sum_{\mu=0}^{2n+1} |c_\mu| = (|s|+|t|) \sum_{m=0}^{n} |\sigma_m(x_1, x_2, \ldots, x_n)| \leqq (|s|+|t|) \prod_{\nu=1}^{n} (1+|x_\nu|),$$

with equality as stated.

**Lemma 2.3.** *Given* $2n$ *real or complex numbers* $x_1, x_2, \ldots, x_{2n}$ *such that*

$$x_{n+\nu} = \bar{x}_\nu, \qquad \nu = 1, 2, \ldots, n, \tag{2.4}$$

*and for all* $\nu$ *either* $\operatorname{Re} x_\nu \geqq 0$, *or* $\operatorname{Re} x_\nu \leqq 0$, *let* $p_{2n}(x) = \prod_{\mu=1}^{2n} (x - x_\mu)$ *and*

$$(s + t x)\, p_{2n}(x) = \sum_{\mu=0}^{2n+1} c_\mu x^{2n-\mu+1}. \tag{2.5}$$

*Then*

$$\big||s|-|t|\big| \prod_{\nu=1}^{n} |1 \pm x_\nu|^2 \leqq \sum_{\mu=0}^{2n+1} |c_\mu| \leqq (|s|+|t|) \prod_{\nu=1}^{n} |1 \pm x_\nu|^2, \tag{2.6}$$

*where the plus sign holds if all* $\operatorname{Re} x_\nu \geqq 0$, *and the minus sign if all* $\operatorname{Re} x_\nu \leqq 0$.

*Proof.* We first observe that in

$$p_{2n}(x) = \sum_{\mu=0}^{2n} (-1)^\mu \sigma_\mu(x_1, x_2, \ldots, x_{2n})\, x^{2n-\mu} \tag{2.7}$$

we have

$$\sigma_\mu \geqq 0 \quad \text{if all } \operatorname{Re} x_\nu \geqq 0, \qquad (-1)^\mu \sigma_\mu \geqq 0 \quad \text{if all } \operatorname{Re} x_\nu \leqq 0.$$

In fact,

$$p_{2n}(x) = \prod_{\nu=1}^{n} [(x - x_\nu)(x - \bar{x}_\nu)] = \prod_{\nu=1}^{n} [x^2 - (2 \operatorname{Re} x_\nu)\, x + |x_\nu|^2],$$

and multiplying out the product on the right yields coefficients which alternate in sign, if all $\operatorname{Re} x_\nu \geqq 0$, and are nonnegative, if all $\operatorname{Re} x_\nu \leqq 0$. Consequently,

$$\sum_{\mu=0}^{2n} |\sigma_\mu(x_1, x_2, \ldots, x_{2n})| = \begin{cases} p_{2n}(-1) = \prod_{\nu=1}^{n} |1 + x_\nu|^2 & \text{if all } \operatorname{Re} x_\nu \geqq 0, \\ p_{2n}(1) = \prod_{\nu=1}^{n} |1 - x_\nu|^2 & \text{if all } \operatorname{Re} x_\nu \leqq 0. \end{cases} \tag{2.8}$$

For the coefficients $c_\mu$ in (2.5) we have

$$c_\mu = (-1)^\mu (t\sigma_\mu - s\sigma_{\mu-1}), \qquad \mu = 0, 1, \ldots, 2n+1,$$

where $\sigma_{-1} = \sigma_{2n+1} = 0$. Therefore,

$$\big||s| - |t|\big| \sum_{\mu=0}^{2n} |\sigma_\mu| \leqq \sum_{\mu=0}^{2n+1} |c_\mu| = \sum_{\mu=0}^{2n+1} |t\sigma_\mu - s\sigma_{\mu-1}| \leqq (|s| + |t|) \sum_{\mu=0}^{2n} |\sigma_\mu|,$$

from which (2.6) follows by virtue of (2.8).

## 3. Inversion of the Vandermonde Matrix

We denote the Vandermonde matrix of order $n$ by

$$V_n = V(x_1, x_2, \ldots, x_n) = \begin{bmatrix} 1 & 1 \ldots 1 \\ x_1 & x_2 \ldots x_n \\ \multicolumn{2}{c}{\dotfill} \\ x_1^{n-1} & x_2^{n-1} \ldots x_n^{n-1} \end{bmatrix}, \tag{3.1}$$

where $x_1, x_2, \ldots, x_n$ are distinct complex numbers and $n > 1$. Its inverse can be obtained by solving the system of linear algebraic equations

$$\begin{aligned} u_1 + \quad u_2 + \cdots + \quad u_n &= v_1 \\ x_1 u_1 + x_2 \; u_2 + \cdots + x_n \; u_n &= v_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ x_1^{n-1} u_1 + x_2^{n-1} u_2 + \cdots + x_n^{n-1} u_n &= v_n. \end{aligned} \tag{3.2}$$

Introducing the elementary Lagrange interpolation polynomials

$$l_\nu(x) = \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n} \frac{x - x_\mu}{x_\nu - x_\mu} = a_{\nu n} x^{n-1} + a_{\nu, n-1} x^{n-2} + \cdots + a_{\nu 1}, \qquad \nu = 1, 2, \ldots, n, \tag{3.3}$$

which satisfy

$$l_\nu(x_\mu) = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu, \end{cases}$$

it is evident that by multiplying the $\mu$-th Eq. (3.2) by $a_{\nu\mu}$, $\mu = 1, 2, \ldots, n$, and adding, we get

$$u_\nu = \sum_{\mu=1}^{n} a_{\nu\mu} v_\mu, \qquad \nu = 1, 2, \ldots, n.$$

Consequently,

$$V_n^{-1} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \multicolumn{2}{c}{\dotfill} \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}. \tag{3.4}$$

## 4. Norm Inequalities for $V_n^{-1}$

We consider throughout the $\infty$-norm of $V_n^{-1}$,

$$\|V_n^{-1}\|_\infty = \max_{1 \leqq \nu \leqq n} \sum_{\mu=1}^{n} |a_{\nu\mu}|.$$

**Theorem 4.1.** *$\|V^{-1}(x_1, x_2, \ldots, x_n)\|_\infty$ is a symmetric function in the variables $x_1, x_2, \ldots, x_n$.*

*Proof.* Interchanging two variables amounts to interchanging two columns of $V_n$, which in turn has the effect of interchanging two rows of $V_n^{-1}$. The value of $\|V_n^{-1}\|_\infty$ remains the same.

**Theorem 4.2.** *Let $\omega \neq 0$ be arbitrary complex, and*

$$V_n(\omega) = V(\omega x_1, \omega x_2, \ldots, \omega x_n).$$

*Then $\|V_n^{-1}(\omega)\|_\infty$ depends only on $|\omega|$ and is strictly decreasing as a function of $|\omega|$.*

*Proof.* Let $V_n = V_n(1)$, $V_n^{-1} = [a_{\nu\mu}]$. Since

$$V_n(\omega) = D(\omega) V_n, \qquad D(\omega) = \operatorname{diag}(1, \omega, \ldots, \omega^{n-1}),$$

we have $V_n^{-1}(\omega) = V_n^{-1} D^{-1}(\omega)$, i.e.,

$$V_n^{-1}(\omega) = \left[\frac{a_{\nu\mu}}{\omega^{\mu-1}}\right], \qquad \nu, \mu = 1, 2, \ldots, n.$$

It is clear, therefore, that the norm of $V_n^{-1}(\omega)$ depends only on $|\omega|$. Furthermore, if $|\omega_1| < |\omega_2|$, we have

$$\|V_n^{-1}(\omega_2)\|_\infty = \max_\nu \sum_{\mu=1}^n \frac{|a_{\nu\mu}|}{|\omega_2|^{\mu-1}} = \sum_{\mu=1}^n \frac{|a_{\nu_0\mu}|}{|\omega_2|^{\mu-1}}$$
$$< \sum_{\mu=1}^n \frac{|a_{\nu_0\mu}|}{|\omega_1|^{\mu-1}} \leqq \max_\nu \sum_{\mu=1}^n \frac{|a_{\nu\mu}|}{|\omega_1|^{\mu-1}} = \|V_n^{-1}(\omega_1)\|_\infty,$$

where strict inequality holds because of

$$a_{\nu_0 n} = \prod_{\mu \neq \nu_0} (x_{\nu_0} - x_\mu)^{-1} \neq 0.$$

This proves Theorem 4.2.

In [2] we have shown that

$$\|V_n^{-1}\|_\infty \leqq \max_{1 \leqq \nu \leqq n} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^n \frac{1 + |x_\mu|}{|x_\nu - x_\mu|}, \tag{4.1}$$

where equality holds if (but not only if) all $x_\nu$ are on the same ray through the origin,

$$x_\nu = |x_\nu| e^{i\phi}, \qquad \nu = 1, 2, \ldots, n. \tag{4.2}$$

In view of Theorem 4.2 we may assume $\phi = 0$ in (4.2), i.e., $x_\nu \geqq 0$, $\nu = 1, 2, \ldots, n$, in which case the equality in (4.1) can be given the alternative form

$$\|V_n^{-1}\|_\infty = \frac{|p_n(-1)|}{\min\limits_{1 \leqq \nu \leqq n} \{(1 + x_\nu) |p_n'(x_\nu)|\}} \qquad (x_\nu \geqq 0), \tag{4.1'}$$

where

$$p_n(x) = \prod_{\mu=1}^n (x - x_\mu). \tag{4.3}$$

We now wish to obtain a result analogous to (4.1′) when the points $x_\nu$ are located symmetrically with respect to the origin on a straight line through the origin. In view of Theorem 4.2 we may assume the straight line to coincide with the real axis.

**Theorem 4.3.** *Let $x_\nu$ be distinct real numbers such that*

$$x_\nu + x_{n+1-\nu} = 0, \qquad \nu = 1, 2, \ldots, n. \tag{4.4}$$

*If $V_n = V(x_1, x_2, \ldots, x_n)$, we then have*

$$\|V_n^{-1}\|_\infty = \begin{cases} \dfrac{1}{2} \max\limits_\nu \left\{ \left(1 + \dfrac{1}{x_\nu}\right) \prod\limits_{\mu \neq \nu} \dfrac{1 + x_\mu^2}{|x_\nu^2 - x_\mu^2|} \right\} & \text{if } n \text{ is even,} \\ \max\limits_\nu \left\{ \varepsilon_\nu (1 + x_\nu) \prod\limits_{\mu \neq \nu} \dfrac{1 + x_\mu^2}{|x_\nu^2 - x_\mu^2|} \right\} & \text{if } n \text{ is odd,} \end{cases} \tag{4.5}$$

*where $\nu$ and $\mu$ vary over all integers for which $x_\nu \geqq 0$ and $x_\mu \geqq 0$, respectively, and where $\varepsilon_\nu = \frac{1}{2}$ when $x_\nu > 0$, and $\varepsilon_\nu = 1$ when $x_\nu = 0$. Alternatively,*

$$\|V_n^{-1}\|_\infty = \frac{|p_n(i)|}{\min\limits_\nu \left\{ \dfrac{1 + x_\nu^2}{1 + x_\nu} |p_n'(x_\nu)| \right\}}, \tag{4.5'}$$

*where $p_n(x)$ is the polynomial in (4.3), and the minimum is taken over all nonnegative abscissas.*

*Proof.* For the sake of definiteness we assume

$$x_\nu > 0 \quad \text{for } \nu = 1, 2, \ldots, [n/2], \qquad x_{(n+1)/2} = 0 \quad \text{if } n \text{ is odd.} \tag{4.6}$$

Let first $n$ be even. The Lagrange polynomials (3.3) then are

$$l_\nu(x) = \frac{x + x_\nu}{2x_\nu} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} \frac{x^2 - x_\mu^2}{x_\nu^2 - x_\mu^2}, \qquad l_{n+1-\nu}(x) = l_\nu(-x), \qquad \nu = 1, 2, \ldots, \frac{n}{2}.$$

It suffices in (3.4) to evaluate the sums $\sum_{\mu=1}^n |a_{\nu\mu}|$ for $1 \leqq \nu \leqq n/2$, the others (for $\nu > n/2$) having the same values. An application of (3.3), (3.4) and Lemma 2.2, in which $n$ is to be replaced by $(n/2) - 1$, and $s$ and $t$ by

$$s = \frac{1}{2 \prod\limits_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} (x_\nu^2 - x_\mu^2)}, \qquad t = \frac{1}{2x_\nu \prod\limits_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} (x_\nu^2 - x_\mu^2)},$$

then gives the first result in (4.5). The second, for $n$ odd, is obtained similarly, noting that

$$l_\nu(x) = \frac{x}{x_\nu} \frac{x + x_\nu}{2x_\nu} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{(n-1)/2} \frac{x^2 - x_\mu^2}{x_\nu^2 - x_\mu^2}, \qquad l_{n+1-\nu}(x) = l_\nu(-x), \qquad \nu = 1, 2, \ldots, \frac{n-1}{2},$$

$$l_{(n+1)/2}(x) = \prod_{\mu=1}^{(n-1)/2} \frac{x^2 - x_\mu^2}{(-x_\mu^2)}.$$

The alternative form (4.5') follows readily from (4.5) by observing that

$$p_n(x) = \prod_{\mu=1}^{n/2} (x^2 - x_\mu^2) \qquad \text{if } n \text{ is even,}$$

and

$$p_n(x) = x \prod_{\mu=1}^{(n-1)/2} (x^2 - x_\mu^2) \quad \text{if } n \text{ is odd.}$$

**Corollary.** *If $n$ is even and $x_\nu$ are symmetric points as in* (4.4), *then* (4.1) *holds with strict inequality.*

*Proof.* The bound in (4.1), again assuming (4.6), is

$$\frac{1}{2} \max_{1 \leqq \nu \leqq n/2} \left\{ \left(1 + \frac{1}{x_\nu}\right) \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} \frac{(1 + x_\mu)^2}{|x_\nu^2 - x_\mu^2|} \right\},$$

which is larger than the top expression in (4.5) because of $(1 + x_\mu)^2 > 1 + x_\mu^2$.

We next consider norm estimates for $V_n^{-1}$ in the case of pairwise conjugate complex abscissas all located in the same half plane.

**Theorem 4.4.** *Let $x_\nu$ be distinct complex numbers such that*

$$x_{n+1-\nu} = \bar{x}_\nu \quad \textit{for } \nu = 1, 2, \ldots, n \quad \textit{and} \quad x_{(n+1)/2} = 0 \quad \textit{if } n \textit{ is odd,} \tag{4.7}$$

*and such that for all $\nu$ either* $\operatorname{Re} x_\nu \geqq 0$ *or* $\operatorname{Re} x_\nu \leqq 0$. *If $V_n = V(x_1, x_2, \ldots, x_n)$, we then have for $n$ even,*

$$\begin{gathered} \max_{1 \leqq \nu \leqq n/2} \left\{ \frac{|1 - |x_\nu||}{|x_\nu - \bar{x}_\nu|} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} \frac{|1 \pm x_\mu|^2}{|x_\nu - x_\mu| \, |x_\nu - \bar{x}_\mu|} \right\} \\ \leqq \|V_n^{-1}\|_\infty \leqq \max_{1 \leqq \nu \leqq n/2} \left\{ \frac{1 + |x_\nu|}{|x_\nu - \bar{x}_\nu|} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n/2} \frac{|1 \pm x_\mu|^2}{|x_\nu - x_\mu| \, |x_\nu - \bar{x}_\mu|} \right\}, \end{gathered} \tag{4.8}$$

*and for $n$ odd,*

$$\begin{gathered} \max_{1 \leqq \nu \leqq (n+1)/2} \left\{ \varepsilon_\nu |1 - |x_\nu|| \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{(n+1)/2} \frac{|1 \pm x_\mu|^2}{|x_\nu - x_\mu| \, |x_\nu - \bar{x}_\mu|} \right\} \\ \leqq \|V_n^{-1}\|_\infty \leqq \max_{1 \leqq \nu \leqq (n+1)/2} \left\{ \varepsilon_\nu (1 + |x_\nu|) \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{(n+1)/2} \frac{|1 \pm x_\mu|^2}{|x_\nu - x_\mu| \, |x_\nu - \bar{x}_\mu|} \right\}, \end{gathered} \tag{4.9}$$

*where plus signs hold if* $\operatorname{Re} x_\nu \geqq 0$, *minus signs if* $\operatorname{Re} x_\nu \leqq 0$, *and where in* (4.9)

$$\varepsilon_{(n+1)/2} = 1, \qquad \varepsilon_\nu = \frac{|x_\nu|}{|x_\nu - \bar{x}_\nu|} \quad \textit{for } 1 \leqq \nu \leqq (n-1)/2. \tag{4.10}$$

*Alternatively,*

$$\frac{|p_n(\mp 1)|}{\min_\nu \left\{ \frac{|1 \pm x_\nu|^2}{|1 - |x_\nu||} \, |p_n'(x_\nu)| \right\}} \leqq \|V_n^{-1}\|_\infty \leqq \frac{|p_n(\mp 1)|}{\min_\nu \left\{ \frac{|1 \pm x_\nu|^2}{1 + |x_\nu|} \, |p_n'(x_\nu)| \right\}}, \tag{4.11}$$

*where $p_n(x)$ is the polynomial in* (4.3), *and the minimum is taken over all $\nu$ with $1 \leqq \nu \leqq n/2$ when $n$ is even, and over all $\nu$ with $1 \leqq \nu \leqq (n+1)/2$ when $n$ is odd.*

We omit the proof of Theorem 4.4, since it is analogous to the proof of Theorem 4.3. Lemma 2.3 now plays the role of Lemma 2.2.

## 5. Scaling of the Abscissas

Let

$$V_n(\omega) = V(\omega x_1, \omega x_2, \ldots, \omega x_n), \qquad \omega > 0.$$

How does the norm of $V_n^{-1}(\omega)$ compare with the norm of $V_n^{-1}(1)$? We shall answer this question first for positive abscissas $x_\nu$, and then for symmetric real abscissas.

**Theorem 5.1.** *Let $x_\nu$ be distinct positive numbers. Then for $\omega > 0$,*

$$\frac{\omega}{\omega+1}\left|\frac{p_n\left(-\frac{1}{\omega}\right)}{p_n(-1)}\right| < \frac{\|V_n^{-1}(\omega)\|_\infty}{\|V_n^{-1}(1)\|_\infty} < (\omega+1)\left|\frac{p_n\left(-\frac{1}{\omega}\right)}{p_n(-1)}\right|, \tag{5.1}$$

*where $p_n(x)$ is the polynomial in* (4.3).

*Proof.* From (4.1′) we obtain

$$\|V_n^{-1}(\omega)\|_\infty = \frac{\left|p_n\left(-\frac{1}{\omega}\right)\right|}{\min\limits_{1\leqq\nu\leqq n}\left\{\left(\frac{1}{\omega}+x_\nu\right)|p_n'(x_\nu)|\right\}}$$

$$= \left|\frac{p_n\left(-\frac{1}{\omega}\right)}{p_n(-1)}\right| \cdot \frac{|p_n(-1)|}{\min\limits_{1\leqq\nu\leqq n}\{g_\omega(x_\nu)(1+x_\nu)|p_n'(x_\nu)|\}},$$

where

$$g_\omega(t) = \frac{\frac{1}{\omega}+t}{1+t}, \qquad 0 \leqq t < \infty.$$

The theorem follows by observing (4.1′) and

$$\frac{1}{\omega+1} < g_\omega(t) < \frac{\omega+1}{\omega}, \qquad 0 \leqq t < \infty.$$

**Theorem 5.2.** *Let $n$ be even and $x_\nu$ be distinct real numbers such that*

$$x_\nu + x_{n+1-\nu} = 0 \qquad \textit{for } \nu = 1, 2, \ldots, n.$$

*Then, for $\omega > 0$,*

$$\frac{2(\sqrt{2}-1)\,\omega}{\omega+1}\left|\frac{p_n\left(\frac{i}{\omega}\right)}{p_n(i)}\right| < \frac{\|V_n^{-1}(\omega)\|_\infty}{\|V_n^{-1}(1)\|_\infty} < \frac{\omega+1}{2(\sqrt{2}-1)}\left|\frac{p_n\left(\frac{i}{\omega}\right)}{p_n(i)}\right|, \tag{5.2}$$

*where $p_n(x)$ is the polynomial in* (4.3).

*Proof.* From (4.5′) we obtain

$$\|V_n^{-1}(\omega)\|_\infty = \left|\frac{p_n\left(\frac{i}{\omega}\right)}{p_n(i)}\right| \cdot \frac{|p_n(i)|}{\min\limits_{\nu}\left\{g_\omega(x_\nu)\,\frac{1+x_\nu^2}{1+x_\nu}\,|p_n'(x_\nu)|\right\}},$$

where now

$$g_\omega(t) = \frac{\frac{1}{\omega^2}+t^2}{1+t^2}\,\frac{1+t}{\frac{1}{\omega}+t}, \qquad 0 \leqq t < \infty.$$

We need to show that

$$\frac{2(\sqrt{2}-1)}{\omega+1} < g_\omega(t) < \frac{\omega+1}{2(\sqrt{2}-1)\,\omega}, \qquad 0 \leqq t < \infty. \tag{5.3}$$

We first note the identities

$$g_\omega(t) = \frac{1}{\omega}\, g_{1/\omega}\left(\frac{1}{t}\right), \qquad g_\omega(t) = \frac{1}{g_{1/\omega}(\omega t)}\,. \tag{5.4}$$

If $0 \leqq t \leqq 1$, the lower bound in (5.3) follows from

$$g_\omega(t) \geqq \frac{\dfrac{1}{\omega^2} + t^2}{\dfrac{1}{\omega} + t} = \frac{1}{\omega}\,\frac{1+\omega^2 t^2}{1+\omega t} > \frac{1}{\omega+1}\,\frac{1+\omega^2 t^2}{1+\omega t},$$

since $(1+y^2)/(1+y)$ for $y>0$ assumes the minimum value $2(\sqrt{2}-1)$ at $y=\sqrt{2}-1$. If $t>1$, we use the first identity in (5.4) to obtain again

$$g_\omega(t) > \frac{1}{\omega}\,\frac{2(\sqrt{2}-1)}{\dfrac{1}{\omega}+1} = \frac{2(\sqrt{2}-1)}{\omega+1}\,.$$

Combining the left inequality in (5.3) just established with the second identity in (5.4) gives the right inequality, and thus proves (5.3).

## 6. Examples

Norm estimates for $V_n^{-1}$ imply estimates for the condition number of $V_n$. These in turn are of interest, e.g., in the study of the condition of polynomial interpolation [3]. In the examples which follow we derive asymptotic estimates for the condition number, assuming typical configurations of interpolation points.

*Example 6.1* (equidistant points). $x_\nu = 1 - \dfrac{2(\nu-1)}{n-1}$, $\nu = 1, 2, \ldots, n$. We assume first $n$ even. From (4.5) we find after some computation that

$$\|V_n^{-1}\|_\infty = \frac{\alpha_n}{\min\limits_{1\leqq\nu\leqq n/2} \pi_\nu},$$

where

$$\alpha_n = \frac{1}{4^{n-2}} \left|\frac{\Gamma(n+i(n-1))}{\Gamma\left(\frac{n}{2}+i\,\frac{n-1}{2}\right)}\right|^2 \left|\frac{\Gamma\left(1+i\,\frac{n-1}{2}\right)}{\Gamma(1+i(n-1))}\right|^2,$$

$$\pi_\nu = [(n-1)^2+(2\nu-1)^2]\left(\frac{n}{2}-\nu\right)!\left(\frac{n}{2}+\nu-2\right)!.$$

Since $\pi_\nu$ is increasing,

$$\min_{1\leqq\nu\leqq n/2} \pi_\nu = \pi_1 = [(n-1)^2+1]\left(\frac{n}{2}-1\right)!^2,$$

and since $|\Gamma(1+iy)|^2 = |iy\Gamma(iy)|^2 = \pi y/\sinh(\pi y)$ for any real $y$, we obtain

$$\|V_n^{-1}\|_\infty = \frac{8}{4^n[(n-1)^2+1]\left(\frac{n}{2}-1\right)!^2}\;\frac{\sinh(\pi(n-1))}{\sinh\left(\pi\,\frac{n-1}{2}\right)} \left|\frac{\Gamma(n+i(n-1))}{\Gamma\left(\frac{n}{2}+i\,\frac{n-1}{2}\right)}\right|^2 \tag{6.1 e}$$

($n$ even).

For $n$ odd, we find similarly,

$$\|V_n^{-1}\|_\infty = \frac{\sinh\left(\pi\frac{n-1}{2}\right)}{\pi\frac{n-1}{2}\left(\frac{n-1}{2}\right)!^2}\left|\Gamma\left(\frac{n+1}{2}+i\frac{n-1}{2}\right)\right|^2 \quad (n \text{ odd}). \tag{6.1o}$$

Since

$$\operatorname{cond}_\infty V_n = \|V_n\|_\infty \|V_n^{-1}\|_\infty = n\|V_n^{-1}\|_\infty, \tag{6.2}$$

using Stirling's formula for the gamma function, and straightforward, but tedious, manipulations, we find from (6.1) that

$$\operatorname{cond}_\infty V_n \sim \frac{1}{\pi} e^{-\frac{\pi}{4}} e^{n\left(\frac{\pi}{4}+\frac{1}{2}\ln 2\right)}, \quad n\to\infty. \tag{6.3}$$

Some numerical values[1] are listed in Table 1.

Table 1. Condition of polynomial interpolation at equidistant points on $[-1, 1]$

| $n$ | $\operatorname{cond}_\infty V_n$ | (6.3) |
|---|---|---|
| 5 | 5.0000 (1) | 4.1668 (1) |
| 10 | 1.3625 (4) | 1.1963 (4) |
| 20 | 1.0535 (9) | 9.8614 (8) |
| 40 | 6.9269 (18) | 6.7007 (18) |
| 80 | 3.1456 (38) | 3.0937 (38) |

*Example 6.2* (Chebyshev points). $x_\nu = \cos\theta_\nu$, $\theta_\nu = \frac{2\nu-1}{2n}\pi$, $\nu = 1, 2, \ldots, n$.

The abscissas $x_\nu$ are the zeros of the Chebyshev polynomial of the first kind, $T_n(x)$. Hence, by (4.5′), since $|T_n'(x_\nu)| = n/\sin\theta_\nu$, we find that

$$\|V_n^{-1}\|_\infty = \frac{|T_n(i)|}{n\cdot\min_\nu f(\theta_\nu)}, \tag{6.4}$$

where

$$f(\theta) = \frac{1+\cos^2\theta}{(1+\cos\theta)\sin\theta}, \quad 0<\theta\leq\pi/2.$$

An elementary calculation shows that $f(\theta)$ has a unique minimum on $[0, \pi/2]$, which is assumed at $\theta=\theta_0$, where

$$\cos\theta_0 = 2-\sqrt{3}, \quad f(\theta_0) = \frac{6-2\sqrt{3}}{3\sqrt{4\sqrt{3}-6}} = 2\cdot 3^{-3/4}.$$

Since the angles $\theta_\nu = \theta_{\nu,n}$ are equidistributed on the arc $0\leq\theta\leq\pi/2$, there exists a sequence of integers $\nu_n$ with $0<\nu_n<n$ such that $\theta_{\nu_n,n}\to\theta_0$ as $n\to\infty$. From

$$f(\theta_0) \leq \min_\nu f(\theta_{\nu,n}) \leq f(\theta_{\nu_n,n})$$

1 The integers in parentheses indicate powers of 10 by which the preceding numbers are to be multiplied.

it then follows that

$$\min_{\nu} f(\theta_{\nu,n}) \to f(\theta_0) \quad \text{as } n \to \infty.$$

Consequently, by (6.4),

$$\|V_n^{-1}\|_\infty \sim \frac{3^{3/4}}{2n} |T_n(i)|, \quad n \to \infty.$$

On the other hand [4, p. 194],

$$|T_n(i)| \sim \tfrac{1}{2}(1+\sqrt{2})^n, \quad n \to \infty,$$

so that, in view of (6.2),

$$\operatorname{cond}_\infty V_n \sim \frac{3^{3/4}}{4}(1+\sqrt{2})^n, \quad n \to \infty. \tag{6.5}$$

Some numerical values are listed in Table 2.

Table 2. Condition of polynomial interpolation at Chebyshev points

| $n$ | $\operatorname{cond}_\infty V_n$ | (6.5) |
|---|---|---|
| 5 | 4.1000 (1) | 4.6737 (1) |
| 10 | 3.7495 (3) | 3.8330 (3) |
| 20 | 2.5727 (7) | 2.5781 (7) |
| 40 | 1.1663 (15) | 1.1663 (15) |
| 80 | 2.3859 (30) | 2.3869 (30) |

*Example 6.3.* $x_\nu = 1 - e^{-i\omega_\nu h}$, $\nu = 1, 2, \ldots, n$ (even), $h > 0$,

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{n/2}, \quad \omega_{\nu+n/2} = -\omega_\nu \quad \text{for } \nu = 1, 2, \ldots, n/2.$$

The interpolation problem corresponding to these complex abscissas arises in the construction of trigonometric multistep methods for ordinary differential equations with almost periodic solutions [1].

A short calculation, based on (4.11), gives the upper bound

$$\|V_n^{-1}\|_\infty \le \frac{\prod_{\mu=1}^{n/2}\left[1+8\sin^2\left(\frac{1}{2}\omega_\mu h\right)\right]}{\min_{1\le\nu\le n/2}\left\{\frac{1+8\sin^2(\frac{1}{2}\omega_\nu h)}{1+2\sin(\frac{1}{2}\omega_\nu h)}\cdot 2\sin(\omega_\nu h)\prod_{\substack{\mu=1\\ \mu\ne\nu}}^{n/2}\left[4\sin\frac{1}{2}(\omega_\nu+\omega_\mu)h\sin\frac{1}{2}|\omega_\nu-\omega_\mu|h\right]\right\}}, \tag{6.6}$$

and a similar lower bound in which $1+2\sin(\frac{1}{2}\omega_\nu h)$ in the denominator of (6.6) is replaced by $1-2\sin(\frac{1}{2}\omega_\nu h)$. For $n$ fixed, and $h \to 0$, we find

$$\|V_n^{-1}\|_\infty \sim \frac{1}{2h^{n-1}\min_{1\le\nu\le n/2}\left\{\omega_\nu \prod_{\substack{\mu=1\\ \mu\ne\nu}}^{n/2}|\omega_\nu^2-\omega_\mu^2|\right\}} \quad (h \to 0). \tag{6.7}$$

The estimate (6.7) can also be obtained by using the approximations $x_\mu \doteq i\omega_\mu h$, rotating the abscissas through an angle of $\pi/2$, and then applying Theorem 4.3 with the simplifying approximation $(1+\omega_\nu h)\prod_{\mu\ne\nu}(1+\omega_\mu^2 h^2) \doteq 1$.

*Example 6.4* (Roots of unity). $x_\nu = e^{2\pi i\nu/n}$, $\nu = 1, 2, \ldots, n$.

Although none of the previous estimates apply, we can obtain the inverse of the Vandermonde matrix directly by observing that the Lagrange interpolation polynomials are

$$l_\nu(x) = \frac{1}{n}\sum_{\mu=1}^{n}\left(\frac{x}{x_\nu}\right)^{\mu-1}, \qquad \nu = 1, 2, \ldots, n.$$

Consequently, by (3.3), (3.4),

$$\|V_n^{-1}\|_\infty = 1, \qquad \operatorname{cond}_\infty V_n = n.$$

Actually, the roots of unity are an optimal point configuration with regard to the spectral condition of Vandermonde matrices [3]. In fact, since $V_n^H V_n = n \cdot I_n$, we have $\operatorname{cond}_2 V_n = 1$.

## References


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