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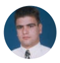

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The relations among k-Fibonacci, k-Lucas and generalized k-Fibonacci numbers and the spectral norms of the matrices of involving these numbers

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Abstract

In this study, we obtain the relations among k-Fibonacci, k-Lucas and generalized k-Fibonacci numbers. Then we define circulant matrices involving k-Lucas and generelized k-Fibonacci numbers. In the last of this study, we investigate the upper and lower bounds for the norms these matrices.

Keywords : k-Lucas number, k-Fibonacci number, generalized k-Fibonacci number, circulant matrix.

AMS Classification : 11B39, 11B65.

1 Introduction

In the last years, it has been studied on Fibonacci, Lucas and generalized Fibonacci sequences. For $n \geq 1$, these sequences are defined by recurrence relations $F_{n+1} = F_n + F_{n-1}$, $(F_0 = 0,\ F_1 = 1)$, $L_{n+1} = L_n + L_{n-1}$, $(L_0 = 2,\ L_1 = 1)$ and $G_{n+1} = G_n + G_{n-1}$, $(G_0 = a,\ G_1 = b)$, $(a, b \in \mathbb{R})$ respectively [3]. In the literature, in [4-7], there are the some generalizations of the Fibonacci and Lucas sequences. For instance, in [4-5], Falcon and Plaza introduce k-Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ by using

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Fibonacci and Pell sequences. Many properties of these numbers were deduced directly from elementary matrix algebra. In [6], it is given k-Lucas sequence $\{L_{k,n}\}_{n=0}^{\infty}$ and obtain some identities related to these sequence. Then, in [7], we defined a new generalization $\{G_{k,n}\}_{n\in\mathbb{N}}$ of k-Fibonacci family. Additionally we also obtained some properties related these numbers. We can also see that it has been investigated the norms of some special matrices with these numbers in many studies. For instance, Solak, in [8], defined the $n\times n$ circulant matrices. Additionally, he investigated the upper and lower bounds of these matrices. Shen and Cen, in [9-10], have found upper and lower bounds for the spectral norms of r-circulant matrices.

In this study, we have given new relations among k-Fibonacci, k-Lucas and Generalized k-Fibonacci numbers. Moreover we define the circulant matrices involving these numbers. Then, we have obtained some bounds for these matrices.

Firstly, let us give well-known preliminaries related to our studies. The circulant matrix $C=[c_{ij}]\in M_{n,n}(\mathbb{C})$ is defined by the form

$$C=\begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_n & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \dots & c_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ c_2 & c_3 & c_4 & \dots & c_1 \end{pmatrix}.$$

For each $i,\ j=1,2,3,...,n$ and $k=0,1,2,...,n-1$, all the elements $(i,\ j)$ such that $(j-i)\equiv s(\operatorname{mod} n)$ have the same value c_s , these elements form the so-called sth stripe of C.

Let us take any matrix $A=[a_{ij}]\in M_{m,n}(\mathbb{C})$. Then, in [1-3], the following properties are hold:

- $\|A\|_F=\left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{\frac{1}{2}}$ (frobenius norm),
- $\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i\left(A^HA\right)}$ (spectral norm), where A^H is the conjugate transpose of matrix A,
- $c_1(A)=\max_j\sqrt{\sum_i|a_{ij}|^2}$ (the maximum column length norm),
- $r_1(A)=\max_i\sqrt{\sum_j|a_{ij}|^2}$ (the maximum row length norm).

The following inequality is hold between the frobenius and spectral norms

$$\frac{1}{\sqrt{n}}\|A\|_F\le\|A\|_2\le\|A\|_F\ . \tag{1}$$

2 Main results

For any positive real number k, Generalized k-Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$ is defined by recurrence relation

$$G_{k,n+1} = kG_{k,n} + G_{k,n-1}, \quad n \geq 1 \tag{2}$$

with initial conditions $G_{k,0} = a,\ G_{k,1} = b \quad (a, b \in \mathbb{R})$ [7]. Generalized k-Fibonacci number is called to each element of Generalized k-Fibonacci sequence. For $a = 0,\ b = 1$ and $a = 2,\ b = 1$, it is obtained k-Fibonacci sequence and k-Lucas sequence, respectively. Also, generalized k-Fibonacci sequence turn into integer number sequence for some special values of k. For example, in generalized k-Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$;

i. If $k = 1$, then we have generalized Fibonacci sequence

$$\{G_{1,n}\} = \{a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ \cdots\}.$$

- For $a = 0,\ b = 1$, it is obtained Fibonacci sequence known as $F_n = \{0, 1, 1, 2, 3, 5, \cdots\}$.
- For $a = 2,\ b = 1$, it is obtained Lucas sequence known as $L_n = \{2, 1, 3, 4, 7, 11, \cdots\}$.

ii. If $k = 2$, then we have generalized Pell sequence

$$\{G_{2,n}\} = \{a,\ b,\ a+2b,\ 2a+5b,\ 5a+12b,\ 12a+29b,\ \cdots\}.$$

- For $a = 0,\ b = 1$, it is obtained Pell sequence known as $P_n = \{0, 1, 2, 5, 12, 29,\ \cdots\}$.
- For $a = 2,\ b = 2$, it is obtained Pell-Lucas sequence known as $P_n = \{2, 2, 6, 14, 34, 82,\ \cdots\}$.

In the following theorems, we give the relations among generalized k-

Fibonacci, k-Fibonacci and k-Lucas numbers.

Theorem 1 *We have the relation between* $\{G_{k,n}\}_{n\in\mathbb{N}}$ *and* $\{F_{k,n}\}_{n\in\mathbb{N}}$:

$$G_{k,\ n} = aF_{k,\ n-1} + bF_{k,\ n}. \tag{3}$$

Proof. Let us use the principle of mathematical induction on m. Since $F_{k,0} = 0,\ F_{k,1} = 1,$ we can write $G_{k,1} = aF_{k,0} + bF_{k,1} = b$, the statement is true for $m = 1$. Assume that the given statement is true for every m, that is

$$G_{k,\ m} = aF_{k,\ m-1} + bF_{k,\ m}.$$

Now, we must show that $G_{k,\ m+1} = aF_{k,\ m} + bF_{k,\ m+1}$. From (2), we can write

$$\begin{aligned} G_{k,\ m+1} &= kG_{k,\ m} + G_{k,\ m-1} \\ &= k(aF_{k,\ m-1} + bF_{k,\ m}) + (aF_{k,\ m-2} + bF_{k,\ m-1}) \\ &= a(kF_{k,\ m-1} + F_{k,\ m-2}) + b(kF_{k,\ m} + F_{k,\ m-1}) \end{aligned}$$

Considering the recurrence equality $F_{k,\ m+1} = kF_{k,\ m} + F_{k,\ m-1}$ for k-Fibonacci numbers, we have

$$G_{k,\ m+1} = aF_{k,\ m} + bF_{k,\ m+1},$$

thus, for every $m \in \mathbb{N}$, the given equality is hold. ■

Theorem 2 *We have the relation between* $\{G_{k,n}\}_{n\in\mathbb{N}}$ *and* $\{L_{k,n}\}_{n\in\mathbb{N}}$:

$$G_{k,\ n} = \frac{a\left(2L_{k,\ n} - L_{k,\ n-1}\right) + b\left(2L_{k,\ n+1} - L_{k,\ n}\right)}{2k+3}. \tag{4}$$

Proof. Let us use the principle of mathematical induction on m. Since $L_{k,0} = 2,\ L_{k,1} = 1$ and $L_{k,2} = k + 2,$ we have

$$G_{k,1} = \frac{a\left(2L_{k,\ 1} - L_{k,\ 0}\right) + b\left(2L_{k,\ 2} - L_{k,\ 1}\right)}{2k+3} = b.$$

Thus the statement is true for $m = 1$. Assume that the given statement is true for every m, that is,

$$G_{k,\ m} = \frac{a\left(2L_{k,\ m} - L_{k,\ m-1}\right) + b\left(2L_{k,\ m+1} - L_{k,\ m}\right)}{2k+3}.$$

Now, we must show that

$$G_{k,\ m+1} = \frac{a\left(2L_{k,\ m+1} - L_{k,\ m}\right) + b\left(2L_{k,\ m+2} - L_{k,\ m+1}\right)}{2k+3}.$$

From (2), we can write

$$\begin{aligned} G_{k,\ m+1} &= kG_{k,\ m} + G_{k,\ m-1} \\ &= k\left(\frac{a\left(2L_{k,\ m} - L_{k,\ m-1}\right) + b\left(2L_{k,\ m+1} - L_{k,\ m}\right)}{2k+3}\right) + \\ &\quad \left(\frac{a\left(2L_{k,\ m-1} - L_{k,\ m-2}\right) + b\left(2L_{k,\ m} - L_{k,\ m-1}\right)}{2k+3}\right). \\ G_{k,\ m+1} &= a\left(\frac{2(kL_{k,\ m} + L_{k,\ m-1}) - (kL_{k,\ m-1} + L_{k,\ m-2})}{2k+3}\right) + \\ &\quad b\left(\frac{2(kL_{k,\ m+1} + L_{k,\ m}) - (kL_{k,\ m} + L_{k,\ m-1})}{2k+3}\right). \end{aligned}$$

Considering the recurrence equality $L_{k,\ m+1} = kL_{k,\ m} + L_{k,\ m-1}$ for k-Lucas numbers, we have

$$G_{k,\ m+1} = \frac{a\left(2L_{k,\ m+1} - L_{k,\ m}\right) + b\left(2L_{k,\ m+2} - L_{k,\ m+1}\right)}{2k+3},$$

as required. ■

Corollary 3 *We have relations between* $\{F_{k,n}\}_{n\in\mathbb{N}}$ *and* $\{L_{k,n}\}_{n\in\mathbb{N}}$:

$$F_{k,\ n} = \frac{2L_{k,\ n+1} - L_{k,\ n}}{2k+3} \quad \text{and} \quad L_{k,\ n} = F_{k,\ n} + 2F_{k,\ n-1}.$$

Proof. If we take $a = 0,\ b = 1$ in (3) and $a = 2,\ b = 1$ in (4) , then we have $G_{k,\ n} = F_{k,\ n}$ and $G_{k,\ n} = L_{k,\ n}$ respectively. In this case, it can be seen clearly the proof of this theorem. ■

The following two theorems give us the upper and lower bounds for the spectral norms of circulant matrix with k-Lucas and generalized k-Fibonacci numbers.

Theorem 4 *Let the* (nxn) *matrix* A *be as* $A = (a_{ij})$ *such that* $a_{ij} = L_{k,\ (\bmod\ (j-i\ ,\ n))}$. *Then we have*

$$\sqrt{\frac{L_{k,n}L_{k,n-1} + 4k - 2}{k}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \frac{\sqrt{L_{k,n}L_{k,n-1} + 4k - 2}\sqrt{L_{k,n}L_{k,n-1} + k - 2}}{k},$$

where $\|.\|_2$ *is the spectral norm and* $L_{k,n}$ *denote* k*-Lucas numbers.*
Proof. *The matrix* A *is of the form*

$$A = \begin{pmatrix} L_{k,0} & L_{k,1} & L_{k,2} & \cdots & L_{k,n-1} \\ L_{k,n-1} & L_{k,0} & L_{k,1} & \cdots & L_{k,n-2} \\ L_{k,n-2} & L_{k,n-1} & L_{k,0} & \cdots & L_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{k,1} & L_{k,2} & L_{k,3} & \cdots & L_{k,0} \end{pmatrix}.$$

From definition of Frobenius norm, we have

$$\begin{aligned} \|A\|_F^2 &= n.\sum_{s=0}^{n-1} L_{k,s}^2 = n.\sum_{s=0}^{n-1} L_{k,s}\left(\frac{L_{k,s+1} - L_{k,s-1}}{k}\right) \\ &= n\sum_{s=0}^{n-1} \frac{L_{k,s}L_{k,s+1}}{k} - \sum_{s=0}^{n-1} \frac{L_{k,s}L_{k,s-1}}{k} \\ &= \frac{n}{k}\left(L_{k,n}L_{k,n-1} - L_{k,0}L_{k,-1}\right). \end{aligned}$$

From (1), $L_{k,0}=2$ *and* $L_{k,-1}=1-2k$*, the following result is obtained*

$$\frac{1}{\sqrt{n}}\left\|A\right\|_F=\sqrt{\frac{1}{k}\left(L_{k,n}L_{k,n-1}+4k-2\right)}\quad\leq\left\|A\right\|_2.$$

On the other hand, let matrices B *and* C *be as*

$$B=(b_{ij})=\begin{cases} b_{ij}=L_{k,\ (\mathrm{mod}(j-i\ ,\ n))} & ,\ i\geq j\\ b_{ij}=1 & ,\ i<j\end{cases}$$

and

$$C=(c_{ij})=\begin{cases} c_{ij}=L_{k,\ (\mathrm{mod}(j-i\ ,\ n))} & ,\ i<j\\ c_{ij}=1 & ,\ i\geq j\end{cases}$$

such that

$$A=B\circ C.$$

Then we can write

$$r_1(B)=\max_i\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{s=0}^{n-1}L_{k,s}^2}=\sqrt{\frac{1}{k}\left(L_{k,n}L_{k,n-1}+4k-2\right)}.$$

and

$$c_1(C)=\max_j\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{1+\sum_{s=1}^{n-1}L_{k,s}^2}=\sqrt{\frac{1}{k}\left(L_{k,n}L_{k,n-1}+k-2\right)}.$$

From the last equalities, we have

$$\left\|A\right\|_2\leq r_1(B).c_1(C)=\frac{1}{k}\sqrt{\left(L_{k,n}L_{k,n-1}+4k-2\right)\left(L_{k,n}L_{k,n-1}+k-2\right)}.$$

Thus the proof of theorem is completed. ■

Corollary 5 *Let us take the matrix* $A=(a_{ij})$ *such that* $a_{ij}=L_{(\mathrm{mod}(j-i,\ n))}$*. If we take* $k=1$ *in Theorem 4, then it is obtained the following inequality*

$$\sqrt{L_nL_{n-1}+2}\leq\left\|A\right\|_2\leq\sqrt{L_nL_{n-1}+2}\sqrt{L_{n-1}L_n-1},$$

where $\left\|.\right\|_2$ *is the spectral norm and* L_n *denote Lucas numbers.*

Corollary 6 *Let us take the matrix* $A=(a_{ij})$ *such that* $a_{ij}=L_{(\mathrm{mod}(j-i,\ n))}$*. If we take* $k=2$ *in Theorem 4, then it is obtained the following inequality*

$$\sqrt{\frac{p_np_{n-1}+6}{2}}\leq\left\|A\right\|_2\leq\frac{1}{2}\sqrt{p_{n-1}p_n(p_np_{n-1}+6)},$$

where $\left\|.\right\|_2$ *is the spectral norm and* p_n *denote Pell-Lucas numbers.*

Theorem 7 *Let the (nxn) matrix A be as $A = (a_{ij})$ such that $a_{ij} = G_{k,\ (\bmod\ (j-i\ ,\ n))}$. Then we have*

$$\sqrt{\frac{G_{k,n}G_{k,n-1} + a^2k - ab}{k}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \frac{1}{k}\sqrt{G_{k,n}G_{k,n-1} + a^2k - ab}\sqrt{G_{k,n}G_{k,n-1} + k - ab},$$

where $\|.\|_2$ is the spectral norm and $G_{k,n}$ denote Generalized k-Fibonacci numbers.

Proof. *We consider the matrix A as follows*

$$A = \begin{pmatrix} G_{k,0} & G_{k,1} & G_{k,2} & \cdots & G_{k,n-1} \\ G_{k,n-1} & G_{k,0} & G_{k,1} & \cdots & G_{k,n-2} \\ G_{k,n-2} & G_{k,n-1} & G_{k,0} & \cdots & G_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{k,1} & G_{k,2} & G_{k,3} & \cdots & G_{k,0} \end{pmatrix}.$$

From definition of Frobenius norm, we can write

$$\begin{aligned} \|A\|_F^2 &= n.\sum_{s=0}^{n-1} G_{k,s}^2 = n.\sum_{s=0}^{n-1} G_{k,s}\left(\frac{G_{k,s+1} - G_{k,s-1}}{k}\right) \\ &= n\sum_{s=0}^{n-1}\frac{G_{k,s}G_{k,s+1}}{k} - \sum_{s=0}^{n-1}\frac{G_{k,s}G_{k,s-1}}{k} \\ &= \frac{n}{k}\left(G_{k,n}G_{k,n-1} - G_{k,0}G_{k,-1}\right). \end{aligned}$$

From (1), $G_{k,0} = a$ and $G_{k,-1} = b - ak$, the following result is obvious

$$\frac{1}{\sqrt{n}}\|A\|_F = \sqrt{\frac{1}{k}\left(G_{k,n}G_{k,n-1} + a^2k - ab\right)} \leq \|A\|_2.$$

On the other hand, let matrices B and C be as

$$B = (b_{ij}) = \begin{cases} b_{ij} = G_{k,\ (\bmod(j-i\ ,\ n))} & ,\ i \geq j \\ b_{ij} = 1 & ,\ i < j \end{cases}$$

and

$$C = (c_{ij}) = \begin{cases} c_{ij} = G_{k,\ (\bmod(j-i\ ,\ n))} & ,\ i < j \\ c_{ij} = 1 & ,\ i \geq j \end{cases}$$

such that

$$A = B \circ C.$$

Then

$$r_1(B) = \max_i \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} G_{k,s}^2} = \sqrt{\frac{1}{k}\left(G_{k,n}G_{k,n-1} + a^2k - ab\right)}.$$

and

$$c_1(C) = \max_j \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{1 + \sum_{s=1}^{n-1} G_{k,s}^2} = \sqrt{\frac{1}{k}\left(G_{k,n}G_{k,n-1} + k - ab\right)}.$$

In this case, we have

$$\|A\|_2 \leq r_1(B).c_1(C) = \frac{\sqrt{\left(G_{k,n}G_{k,n-1} + a^2k - ab\right)\left(G_{k,n}G_{k,n-1} + k - ab\right)}}{k}.$$

Thus the proof of theorem is completed. ■

Corollary 8 *Let the (nxn) matrix A be as $A = (a_{ij})$ such that $a_{ij} = G_{k,\ (\mathrm{mod}\ \ (j-i\ ,\ n))}$. Then we can write*

i)

$$\sqrt{\frac{(aF_{k,n-1} + bF_{k,n})(aF_{k,n-2} + bF_{k,n-1}) + a^2k - ab}{k}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \sqrt{\frac{(aF_{k,n-1} + bF_{k,n})(aF_{k,n-2} + bF_{k,n-1}) + a^2k - ab}{k}}. \sqrt{\frac{(aF_{k,n-1} + bF_{k,n})(aF_{k,n-2} + bF_{k,n-1}) + k - ab}{k}},$$

ii)

$$\sqrt{\frac{(au_n + bu_{n+1})\,(au_{n-1} + bu_n) + (a^2k - ab)(2k+3)^2}{k(2k+3)^2}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \left(\sqrt{\frac{(au_n + bu_{n+1})\,(au_{n+3} + bu_n) + (a^2k - ab)(2k+3)^2}{k(2k+3)^2}}\right) \left(\sqrt{\frac{(au_n + bu_{n+1})\,(au_{n+3} + bu_n) + (a^2k - ab)(2k+3)^2}{k(2k+3)^2}}\right),$$

where $\|\ .\|_2$ is the spectral norm, $u_n = 2L_{k,n} - L_{k,n-1}$, $F_{k,n}$ and $L_{k,n}$ denote k-Fibonacci and k-Lucas numbers respectively.

Proof. From Theorem 7, (3) and (4), the proof of corollary is obvious. ■

Corollary 9 *Let the matrix A be as $A = (a_{ij})$ such that $a_{ij} = G_{(\mathrm{mod}(j-i,\ n))}$. If we take $k = 1$ in Theorem 7, then it is obtained the following inequality*

$$\sqrt{G_n G_{n-1} + a^2 - ab} \le \|A\|_2$$

and

$$\|A\|_2 \le \sqrt{G_n G_{n-1} + a^2 - ab}\sqrt{G_n G_{n-1} + 1 - ab},$$

where generalized Fibonacci number G_n is solution of recurrence relation $G_{n+1} = G_n + G_{n-1}$, with initial values $G_0 = a$, $G_1 = b$, $(a, b \in \mathbb{R})$.

Corollary 10 *Let the matrix A be as $A = (a_{ij})$ such that $a_{ij} = P_{(\mathrm{mod}(j-i,\ n))}$. If we take $k = 2$ in Theorem 7, then it is obtained the following inequality*

$$\sqrt{\frac{P_n P_{n-1} + 2a^2 - ab}{2}} \le \|A\|_2$$

and

$$\|A\|_2 \le \frac{1}{2}\sqrt{P_n P_{n-1} + 2a^2 - ab}\sqrt{P_n P_{n-1} + 2 - ab},$$

where generalized Pell number P_n is solution of recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, with initial values $P_0 = a$, $P_1 = b$, $(a, b \in \mathbb{R})$.

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