Student Information

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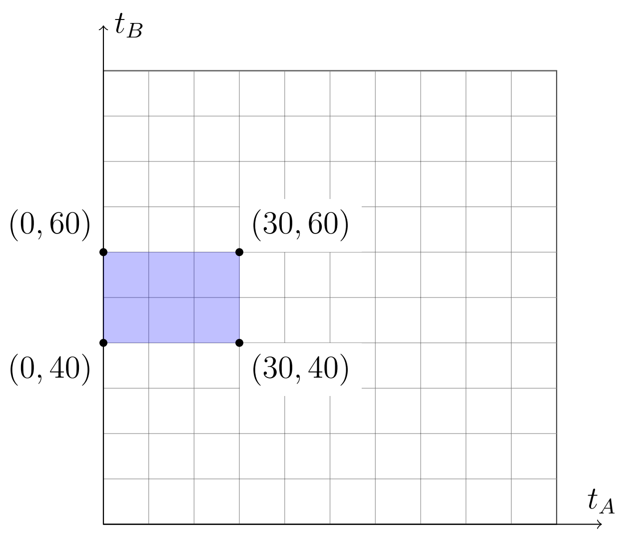
Answer 1

a)

- Since both T_A and T_B are uniformly distributed, their probability density functions are $f_A = f_B = \frac{1}{100}$ with $b = 100$ and $a = 0$. Since they are independent, the joint density function is $f(t_A, t_B) = f_A \cdot f_B = \frac{1}{10{,}000}$.
- The joint cumulative distribution function is $F(t_A, t_B) = \iint \frac{dx \cdot dy}{10{,}000} = \int \frac{x \cdot dy}{10{,}000} = \frac{x \cdot y}{10{,}000}$.

b)

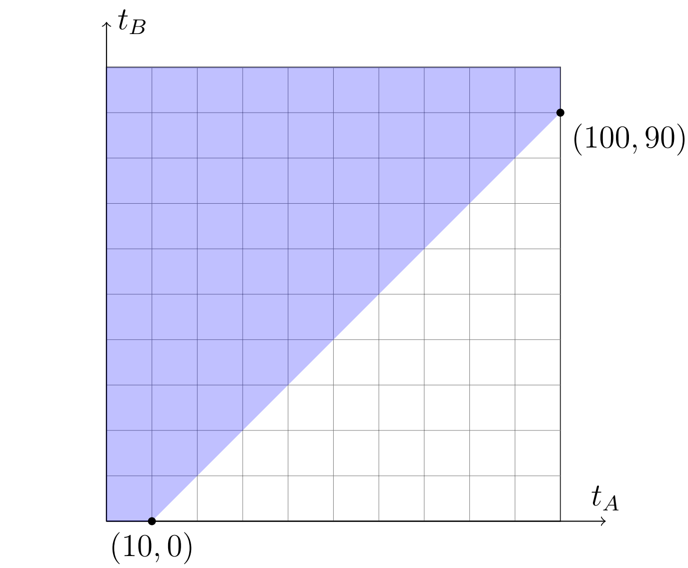
Let's draw a 100×100 square to illustrate the probabilites. Since the probability density function is a constant function, the simple area of a region over ten thousand would give us the probability of an event being inside the region.



The blue region depicts the probability $P\{T_A < 30 \cap 40 < T_B < 60\}$. The probability is equal to the volume of the space underneath it, where the space is bounded by the probability density function f from the above. Since f is constant, the volume simply equals to $30 \cdot 20 \cdot \frac{1}{10000} = \frac{600}{10000} = \frac{3}{50} = 0.06$.

c)

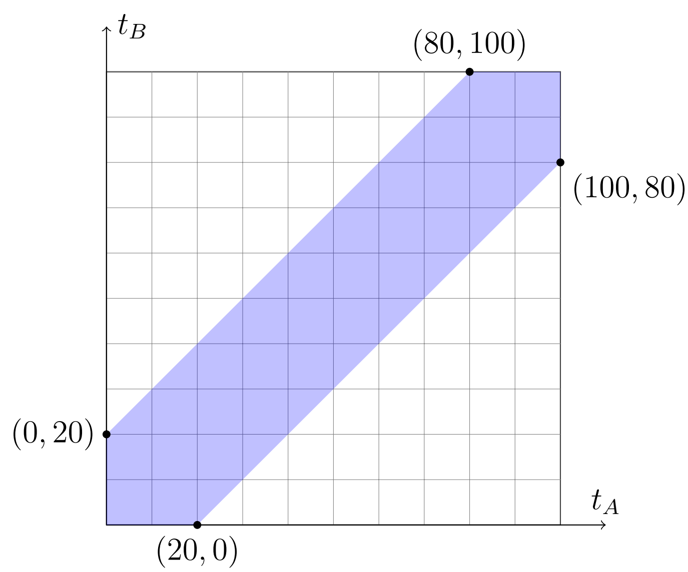
For this question, we must calculate the area of the surface of square with the constraint $t_A < t_B + 10$.



The area of the surface is $100 \cdot 100 - 90 \cdot 90 \cdot \frac{1}{2} = 5950$. The volume of the space is $5950 \cdot \frac{1}{10000} = \frac{5950}{10000} = \frac{119}{200} = 0.595$.

d)

For this question, the inequality $|t_A - t_B| < 20$ must be satisfied. This equals to $-20 < t_A - t_B < 20$ which means $t_B - 20 < t_A < t_B + 20$.



The area of the surface is $100 \cdot 100 - 2 \cdot 80 \cdot 80 \cdot \frac{1}{2} = 3600$. The volume of the space is $3600 \cdot \frac{1}{10000} = \frac{3600}{10000} = \frac{9}{25} = 0.36$.

Answer 2

a)

This probability is binomially distributed with parameters $n = 150$, because 150 people are selected from the population, and $p = 0.6$, because frequent shoppers are 60% of the population. Since n is large, normal approximation to binomial distribution with parameters $\mu = n \cdot p$ and $\sigma = \sqrt{n \cdot p \cdot (1-p)}$ can be used. In that case, $\mu = 90$ and $\sigma = 6$. In order for the 65% of the customers to be frequent shoppers, at least 97.5 of them should be frequent shoppers.

$$\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.65\} &= \boldsymbol{P}\{X > 97.5\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > \frac{97.5-90}{6}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > \frac{7.5}{6}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > 1.25\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X-90}{6} < 1.25\right\} \\
&= 1 - \Phi(1.25) \\
&= \Phi(-1.25) \\
&= 0.1056
\end{aligned}$$

b)

Similarly, this probability is binomially distributed with parameters $n = 150$, and $p = 0.1$. Normal approximation has $\mu = 15$ and $\sigma = \sqrt{13.5} = 3.6742$ as parameters. At most 22.5 of customers must be rare shoppers.

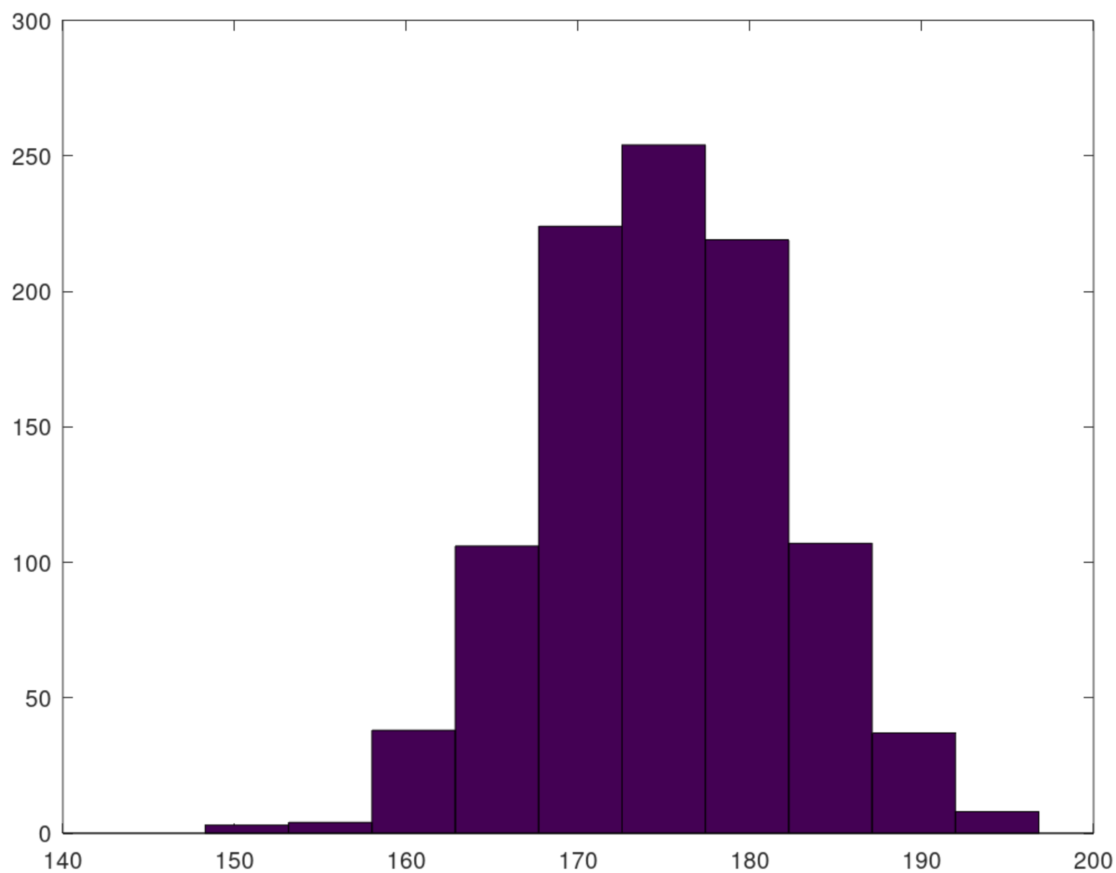
$$\begin{aligned}
\boldsymbol{P}\{X < 150 \cdot 0.15\} &= \boldsymbol{P}\{X < 22.5\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} < \frac{22.5-15}{\sqrt{13.5}}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} < \frac{7.5}{\sqrt{13.5}}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} < 2.0412\right\} \\
&= \Phi(2.0412) \\
&= 0.9794
\end{aligned}$$

Answer 3

$$
\begin{aligned}
\boldsymbol{P}\{170 < X < 180\} &= \boldsymbol{P}\left\{\frac{170-175}{7} < \frac{X-175}{7} < \frac{180-175}{7}\right\} \\
&= \boldsymbol{P}\left\{-\frac{5}{7} < \frac{X-175}{7} < \frac{5}{7}\right\} \\
&= \boldsymbol{P}\left\{-0.7143 < \frac{X-175}{7} < 0.7143\right\} \\
&= \Phi(0.7143) - \Phi(-0.7143) \\
&= 0.7625 - 0.2375 \\
&= 0.5249
\end{aligned}
$$

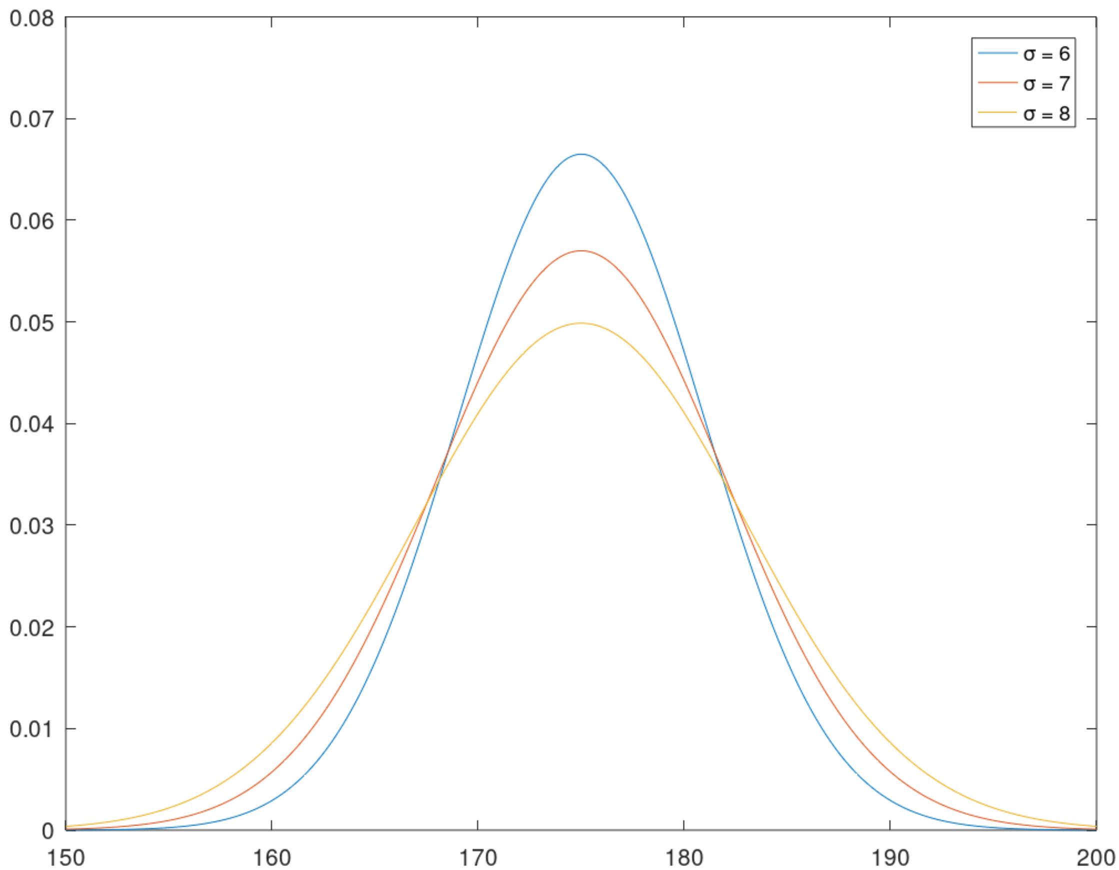
Answer 4

a)



The histogram of the data is similar to the probability density function of the normal distribution, with parameters $\mu = 175$ and $\sigma = 7$. If we were to make more trials and visualize the result with more bars, the resulting chart would be closer to normal PDF. This is essentially Riemann sum of the integral of the PDF.

b)



The probability density functions all have the same mean of 175, while their standard deviations are different. As the standard deviation increases, the probabilites of values closer to the mean decreases and the probabilities of values further from the mean increases.

c)

The probability of an adult having a height between 170 cm and 180 cm is 0.5249 as found in Answer 3. Therefore, among 150 randomly selected adults, the probability that at least 45%, 50%, or 55% of them having heights between 170 cm and 180 cm is binomially distributed, with parameters $n = 150$, and $p = 0.5249$. Normal approximation has parameters $\mu = 78.7424$, and $\sigma = \sqrt{37.4066} = 6.1161$. For continuity correction, in the 50% case, we are going to use 75.5 instead.

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.45\} &= \boldsymbol{P}\{X > 67.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{67.5 - 78.74}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{-11.24}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > -1.84\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} < -1.84\right\} \\
&= 1 - \Phi(-1.84) \\
&= \Phi(1.84) \\
&= 0.9670
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.50\} &= \boldsymbol{P}\{X > 75\} \\
&= \boldsymbol{P}\{X > 75.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{75.5 - 78.74}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{-3.24}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > -0.53\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} < -0.53\right\} \\
&= 1 - \Phi(-0.53) \\
&= \Phi(0.53) \\
&= 0.7020
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.55\} &= \boldsymbol{P}\{X > 82.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{82.5 - 78.74}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{3.76}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > 0.61\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} < 0.61\right\} \\
&= 1 - \Phi(0.61) \\
&= \Phi(-0.61) \\
&= 0.2695
\end{aligned}
$$

This was the theoretical part. Now, we will conduct experiments and compare our results with theoretical estimations. The Octave code is on the next page.

The Octave code is as follows,

```
% Load statistics module for normal probability density function
pkg load statistics;

% Define constants
N = 1000;
mean = 175;
std_dev = 7;

% Generate heights as a column vector
heights = randn(N, 1);
heights *= std_dev;
heights += mean;

% Create and print the histogram of heights
hist(heights, 10);
print("the2_hist", "-dpng");

% Create and print the normal PDF's with various standard deviations
x = 150:0.1:200;
hold off;
plot(x, normpdf(x, mean, 6), ";\\sigma = 6;");
hold on;
plot(x, normpdf(x, mean, 7), ";\\sigma = 7;");
plot(x, normpdf(x, mean, 8), ";\\sigma = 8;");
print("the2_plot", "-dpng");

% Generate heights as a 150x1000 matrix
heights = randn(150, N);
heights *= std_dev;
heights += mean;

% Filter heights and sum the successes
heights = (heights > 170) & (heights < 180);
heights = sum(heights);

% Calculate probabilities
p1 = sum(heights > 67.5)/N
p2 = sum(heights > 75)/N
p3 = sum(heights > 82.5)/N
```

with a screenshot of some outputs,

```
>> the2_code
p1 =  0.96300
p2 =  0.70600
p3 =  0.27400
```

The findings agree with our calculations. As the number of iterations increases, the probabilities are expected to approach their theoretical values.