Student Information

Full Name : Murat Bolu
Id Number : 2521300

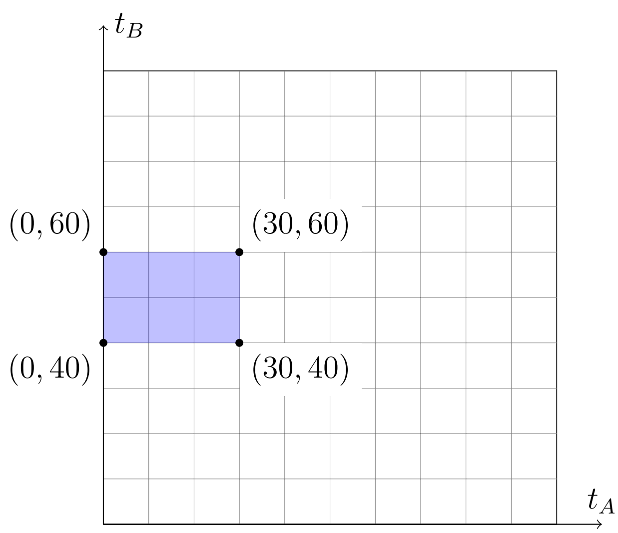
Answer 1

a)

- Since both T_A and T_B are uniformly distributed, their probability density functions are $f_A = f_B = \frac{1}{100}$ with $b = 100$ and $a = 0$. Since they are independent, the joint density function is $f(t_A, t_B) = f_A \cdot f_B = \frac{1}{10{,}000}$.
- The joint cumulative distribution function is $F(t_A, t_B) = \iint \frac{dx \cdot dy}{10{,}000} = \int \frac{x \cdot dy}{10{,}000} = \frac{x \cdot y}{10{,}000}$.

b)

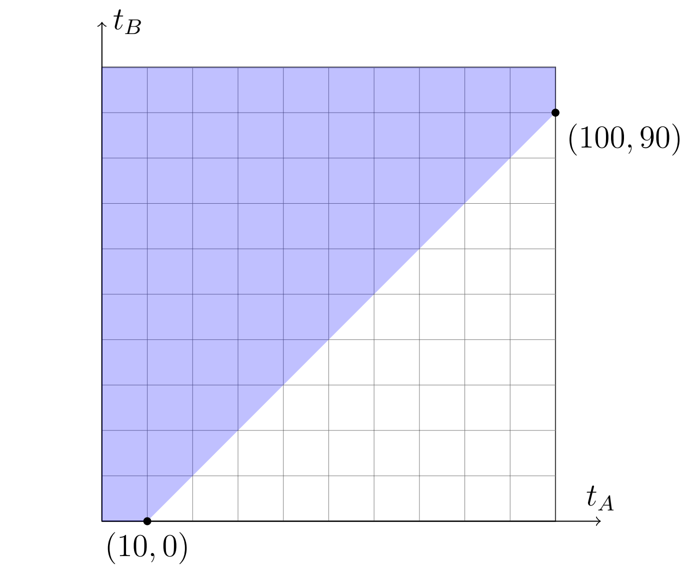
Let's draw a 100×100 square to illustrate the probabilites. Since the probability density function is a constant function, the simple area of a region over ten thousand would give us the probability of an event being inside the region.



The blue region depicts the probability $P\{T_A < 30 \cap 40 < T_B < 60\}$. The probability is equal to the volume of the space underneath it, where the space is bounded by the probability density function f from the above. Since f is constant, the volume simply equals to $30 \cdot 20 \cdot \frac{1}{10000} = \frac{600}{10000} = \frac{3}{50} = 0.06$.

c)

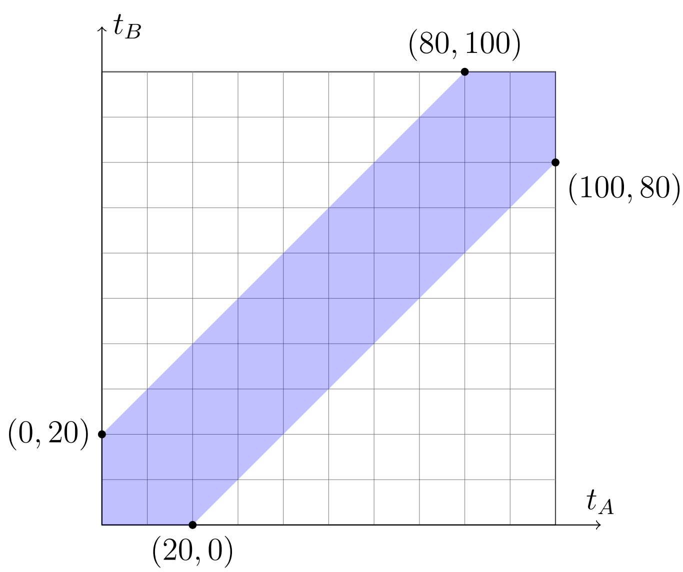
For this question, we must calculate the area of the surface of square with the constraint $t_A < t_B + 10$.



The area of the surface is $100 \cdot 100 - 90 \cdot 90 \cdot \frac{1}{2} = 5950$. The volume of the space is $5950 \cdot \frac{1}{10000} = \frac{5950}{10000} = \frac{119}{200} = 0.595$.

d)

For this question, the inequality $|t_A - t_B| < 20$ must be satisfied. This equals to $-20 < t_A - t_B < 20$ which means $t_B - 20 < t_A < t_B + 20$.



The area of the surface is $100 \cdot 100 - 2 \cdot 80 \cdot 80 \cdot \frac{1}{2} = 3600$. The volume of the space is $3600 \cdot \frac{1}{10000} = \frac{3600}{10000} = \frac{9}{25} = 0.36$.

Answer 2

a)

This probability is binomially distributed with parameters $n = 150$, because 150 people are selected from the population, and $p = 0.6$, because frequent shoppers are 60% of the population. Since n is large, normal approximation to binomial distribution with parameters $\mu = n \cdot p$ and $\sigma = \sqrt{n \cdot p \cdot (1-p)}$ can be used. In that case, $\mu = 90$ and $\sigma = 6$. In order for the 65% of the customers to be frequent shoppers, at least 97.5 of them should be frequent shoppers.

$$\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.65\} &= \boldsymbol{P}\{X > 97.5\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > \frac{97.5-90}{6}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > \frac{7.5}{6}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-90}{6} > 1.25\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X-90}{6} < 1.25\right\} \\
&= 1 - \Phi(1.25) \\
&= \Phi(-1.25) \\
&= 0.1056
\end{aligned}$$

b)

Similarly, this probability is binomially distributed with parameters $n = 150$, and $p = 0.1$. Normal approximation has $\mu = 15$ and $\sigma = \sqrt{13.5} = 3.6742$ as parameters. At most 22.5 of customers must be rare shoppers.

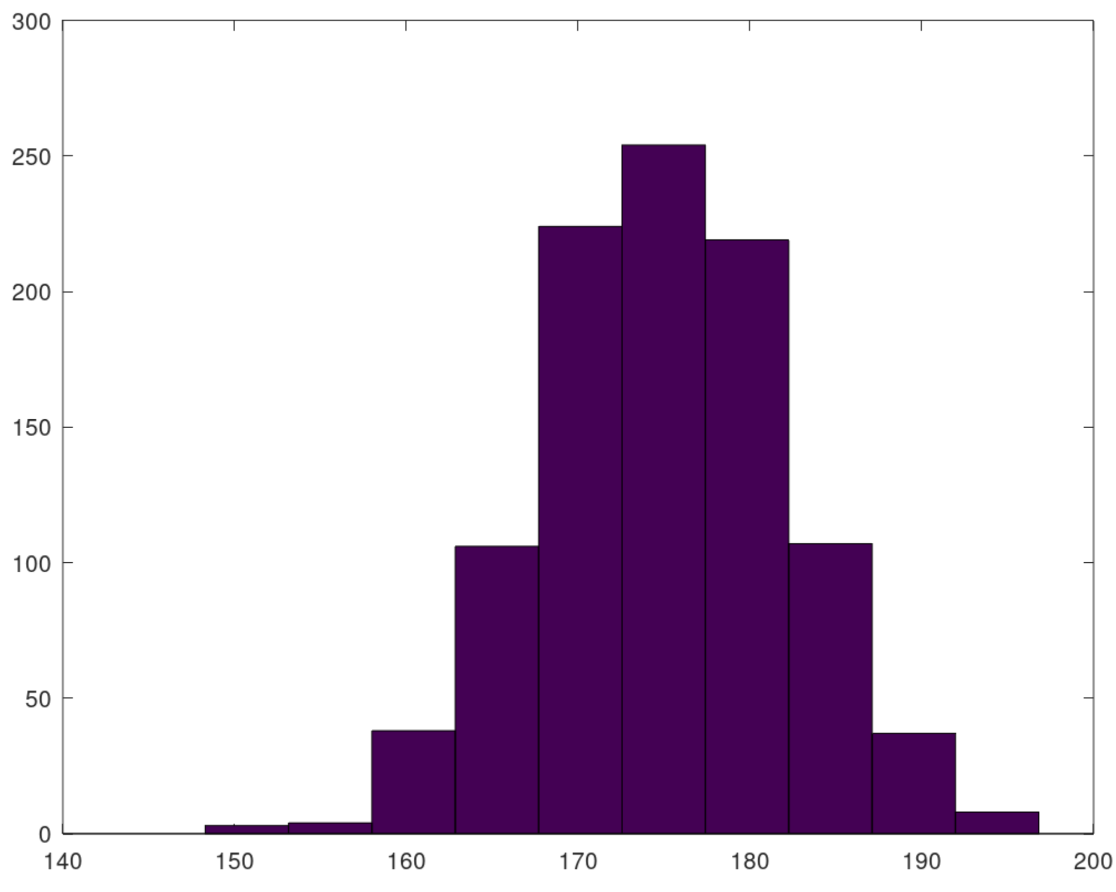
$$\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.15\} &= \boldsymbol{P}\{X > 22.5\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} > \frac{22.5-15}{\sqrt{13.5}}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} > \frac{7.5}{\sqrt{13.5}}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} > 2.0412\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X-15}{\sqrt{13.5}} < 2.0412\right\} \\
&= 1 - \Phi(2.0412) \\
&= \Phi(-2.0412) \\
&= 0.0206
\end{aligned}$$

Answer 3

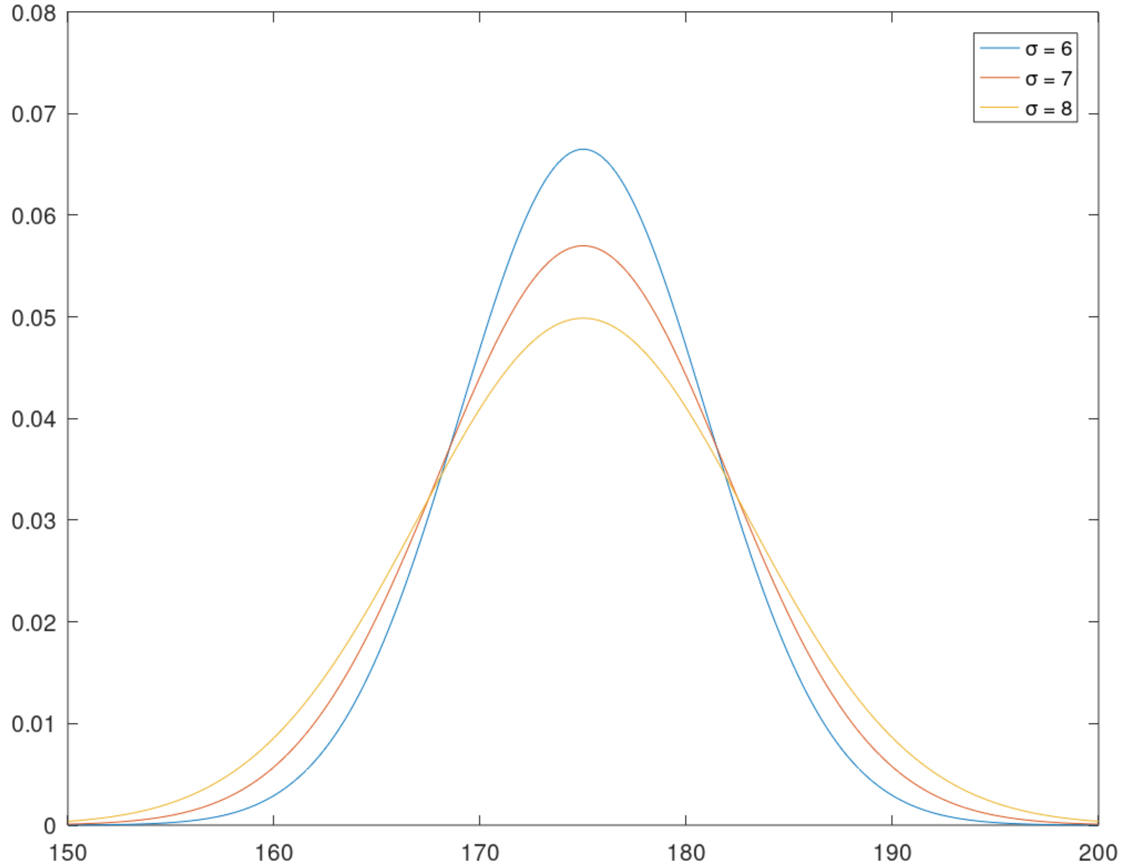
$$\begin{aligned}
\boldsymbol{P}\{170 < X < 180\} &= \boldsymbol{P}\left\{\frac{170-175}{7} < \frac{X-175}{7} < \frac{180-175}{7}\right\} \\
&= \boldsymbol{P}\left\{-\frac{5}{7} < \frac{X-175}{7} < \frac{5}{7}\right\} \\
&= \boldsymbol{P}\left\{-0.7143 < \frac{X-175}{7} < 0.7143\right\} \\
&= \Phi(0.7143) - \Phi(-0.7143) \\
&= 0.7625 - 0.2375 \\
&= 0.5249
\end{aligned}$$

Answer 4

a)



b)



c)

The probability of an adult having a height between 170 cm and 180 cm is 0.5249 as found in Answer 3. Therefore, among 150 randomly selected adults, the probability that at least 45%, 50%, or 55% of them having heights between 170 cm and 180 cm is binomially distributed, with parameters $n = 150$, and $p = 0.5249$. Normal approximation has parameters $\mu = 78.7424$, and $\sigma = \sqrt{37.4066} = 6.1161$. For continuity correction, in the 50% case, we are going to use 75.5 instead.

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.45\} &= \boldsymbol{P}\{X > 67.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{67.5 - 78.74}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{-11.24}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > -1.84\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} < -1.84\right\} \\
&= 1 - \Phi(-1.84) \\
&= \Phi(1.84) \\
&= 0.9670
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.50\} &= \boldsymbol{P}\{X > 75\} \\
&= \boldsymbol{P}\{X > 75.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{75.5 - 78.74}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{-3.24}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > -0.53\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} < -0.53\right\} \\
&= 1 - \Phi(-0.53) \\
&= \Phi(0.53) \\
&= 0.7020
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}\{X > 150 \cdot 0.55\} &= \boldsymbol{P}\{X > 82.5\} \\
&= \boldsymbol{P}\left\{\frac{X - 78.74}{6.12} > \frac{82.5 - 78.74}{6.12}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{P}\left\{\frac{X-78.74}{6.12} > \frac{3.76}{6.12}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-78.74}{6.12} > 0.61\right\} \\
&= 1-\boldsymbol{P}\left\{\frac{X-78.74}{6.12} < 0.61\right\} \\
&= 1-\Phi(0.61) \\
&= \Phi(-0.61) \\
&= 0.2695
\end{aligned}
$$

The Octave code is as follows,

```
% Load statistics module for normal probability density function
pkg load statistics;

% Define constants
N = 1000;
mean = 175;
std_dev = 7;

% Generate heights as a column vector
heights = randn(N, 1);
heights *= std_dev;
heights += mean;

% Create and print the histogram of heights
hist(heights, 10);
print("the2_hist", "-dpng");

% Create and print the normal PDF's with various standard deviations
x = 150:0.1:200;
hold off;
plot(x, normpdf(x, mean, 6), ";\\sigma = 6;");
hold on;
plot(x, normpdf(x, mean, 7), ";\\sigma = 7;");
plot(x, normpdf(x, mean, 8), ";\\sigma = 8;");
print("the2_plot", "-dpng");

% Generate heights as a 150x1000 matrix
heights = randn(150, N);
heights *= std_dev;
heights += mean;

% Filter heights and sum the successes
heights = (heights > 170) & (heights < 180);
heights = sum(heights);

% Calculate probabilities
p1 = sum(heights > 67.5)/N
p2 = sum(heights > 75)/N
p3 = sum(heights > 82.5)/N
```