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Answer 1

Let $\langle a_0, a_1, a_2, a_3, \cdots, a_n, \cdots \rangle$ be the generating function $A(x)$. Then, $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} (3a_{n-1} + 4a_{n-2}) \cdot x^n \\
&= \sum_{n=2}^{\infty} 3a_{n-1}x^n + 4a_{n-2}x^n \\
&= \sum_{n=2}^{\infty} 3a_{n-1}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n \\
&= 3\sum_{n=2}^{\infty} a_{n-1}x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n \\
&= 3x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} \\
&= 3x\sum_{n=1}^{\infty} a_n x^n + 4x^2\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=2}^{\infty} a_n x^n &= 3x\sum_{n=1}^{\infty} a_n x^n + 4x^2\sum_{n=0}^{\infty} a_n x^n \\
A(x) - a_1 \cdot x - a_0 &= 3x \cdot (A(x) - a_0) + 4x^2 \cdot A(x) \\
A(x) - a_1 \cdot x - a_0 &= 3x \cdot A(x) - 3x \cdot a_0 + 4x^2 \cdot A(x) \\
A(x) - 3x \cdot A(x) - 4x^2 \cdot A(x) &= a_1 \cdot x + a_0 - 3x \cdot a_0 \\
A(x) \cdot (1 - 3x - 4x^2) &= a_1 \cdot x + a_0 \cdot (1 - 3x) \\
A(x) &= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{1 - 3x - 4x^2} \\
&= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{(1 - 4x) \cdot (1 + x)}
\end{aligned}$$

Plug in values for a_0 and a_1:

$$\begin{aligned} A(x) &= \frac{x + 1 \cdot (1 - 3x)}{(1-4x)\cdot(1+x)} \\ &= \frac{1-2x}{(1-4x)\cdot(1+x)} \end{aligned}$$

Partial fractions:

$$\begin{aligned} \frac{1-2x}{(1-4x)\cdot(1+x)} &= \frac{\xi}{1-4x} + \frac{\varphi}{1+x} \\ &= \frac{\xi\cdot(1+x) + \varphi\cdot(1-4x)}{(1-4x)\cdot(1+x)} \\ &= \frac{\xi + \xi\cdot x + \varphi - 4\varphi\cdot x}{(1-4x)\cdot(1+x)} \\ \frac{1-2x}{(1-4x)\cdot(1+x)} &= \frac{\xi + \varphi + x\cdot(\xi - 4\varphi)}{(1-4x)\cdot(1+x)} \end{aligned}$$

We can solve the system of linear equations:

$$\begin{aligned} \xi + \varphi &= 1 \\ \xi - 4\varphi &= -2 \\ 5\varphi &= 3 \\ \varphi &= \tfrac{3}{5} \\ \xi &= \tfrac{2}{5} \end{aligned}$$

So the equation becomes:

$$\begin{aligned} A(x) &= \frac{\frac{2}{5}}{1-4x} + \frac{\frac{3}{5}}{1+x} \\ &= \frac{2}{5}\cdot\frac{1}{1-4x} + \frac{3}{5}\cdot\frac{1}{1+x} \end{aligned}$$

$$\begin{aligned} \frac{1}{1-4x} &\leftrightarrow \left\langle 4^0, 4^1, 4^2, \cdots, 4^n, \cdots \right\rangle \\ \frac{2}{5}\cdot\frac{1}{1-4x} &\leftrightarrow \left\langle \tfrac{2}{5}\cdot 4^0, \tfrac{2}{5}\cdot 4^1, \tfrac{2}{5}\cdot 4^2, \cdots, \tfrac{2}{5}\cdot 4^n, \cdots \right\rangle \\ &\leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \cdots, \frac{2^{2n+1}}{5}, \cdots \right\rangle \end{aligned}$$

$$\frac{1}{1+x} \leftrightarrow \langle 1, -1, 1, \cdots, (-1)^n, \cdots \rangle$$

$$\frac{3}{5} \cdot \frac{1}{1+x} \leftrightarrow \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \cdots, \frac{3}{5} \cdot (-1)^n, \cdots \right\rangle$$

$$A(x) \leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \cdots, \frac{2^{2n+1}}{5}, \cdots \right\rangle + \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \cdots, \frac{3}{5} \cdot (-1)^n, \cdots \right\rangle$$

$$A(x) \leftrightarrow \left\langle 1, 1, 7, \cdots, \frac{2^{2n+1} + 3(-1)^n}{5}, \cdots \right\rangle$$

Therefore, $a_n = \dfrac{2^{2n+1} + 3(-1)^n}{5}$.

Answer 2

a)

Let $\langle 2, 5, 11, 29, 83, 245, \cdots \rangle$ be the generating function $F(x)$.

$$\begin{aligned}
F(x) &\leftrightarrow \langle 2, 5, 11, 29, 83, 245, \cdots \rangle \\
&\leftrightarrow \langle 0+2, 3+2, 9+2, 27+2, 81+2, 243+2, \cdots \rangle \\
&\leftrightarrow \langle 0, 3, 9, 27, 81, 243, \cdots \rangle + \langle 2, 2, 2, 2, 2, 2, \cdots \rangle \\
&\leftrightarrow \langle 0, 3, 9, 27, 81, 243, \cdots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \cdots \rangle \\
&\leftrightarrow \langle 1-1, 3, 9, 27, 81, 243, \cdots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \cdots \rangle \\
&\leftrightarrow \langle 1, 3, 9, 27, 81, 243, \cdots \rangle - \langle 1, 0, 0, 0, 0, 0, \cdots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \cdots \rangle
\end{aligned}$$

$$\begin{aligned}
F(x) &= \frac{1}{1-3x} - 1 + 2 \cdot \frac{1}{1-x} \\
&= \frac{1 - 1 + 3x}{1-3x} + \frac{2}{1-x} \\
&= \frac{3x}{1-3x} + \frac{2}{1-x} \\
&= \frac{3x \cdot (1-x) + 2 \cdot (1-3x)}{(1-3x) \cdot (1-x)} \\
&= \frac{3x - 3x^2 + 2 - 6x}{1 - 4x + 3x^2} \\
F(x) &= \frac{2 - 3x - 3x^2}{1 - 4x + 3x^2}
\end{aligned}$$

b)

$$\begin{aligned}G(x) &= \frac{7-9x}{1-3x+2x^2}\\ &= \frac{7-9x}{(1-2x)(1-x)}\end{aligned}$$

$$\begin{aligned}\frac{7-9x}{(1-2x)(1-x)} &= \frac{\xi}{1-2x}+\frac{\varphi}{1-x}\\ &= \frac{\xi\cdot(1-x)+\varphi\cdot(1-2x)}{(1-2x)(1-x)}\\ &= \frac{\xi-\xi\cdot x+\varphi-2\varphi\cdot x}{(1-2x)(1-x)}\\ &= \frac{\xi+\varphi-\xi\cdot x-2\varphi\cdot x}{(1-2x)(1-x)}\\ \frac{7-9x}{(1-2x)(1-x)} &= \frac{\xi+\varphi+x\cdot(-\xi-2\varphi)}{(1-2x)(1-x)}\end{aligned}$$

$$\begin{aligned}\xi+\varphi &= 7\\ -\xi-2\varphi &= -9\\ -\varphi &= -2\\ \varphi &= 2\\ \xi &= 5\end{aligned}$$

$$\begin{aligned}G(x) &= \frac{5}{1-2x}+\frac{2}{1-x}\\ &= 5\cdot\frac{1}{1-2x}+2\cdot\frac{1}{1-x}\end{aligned}$$

$$\begin{aligned}\frac{1}{1-2x} &\leftrightarrow \langle 1,2,4,8,\cdots,2^n,\cdots\rangle\\ 5\cdot\frac{1}{1-2x} &\leftrightarrow \langle 5,10,20,40,\cdots,5\cdot 2^n,\cdots\rangle\\ \frac{1}{1-x} &\leftrightarrow \langle 1,1,1,1,\cdots,1,\cdots\rangle\\ 2\cdot\frac{1}{1-x} &\leftrightarrow \langle 2,2,2,2,\cdots,2,\cdots\rangle\end{aligned}$$

$$5 \cdot \frac{1}{1-2x} + 2 \cdot \frac{1}{1-x} \leftrightarrow \langle 5, 10, 20, 40, \cdots, 5 \cdot 2^n, \cdots \rangle + \langle 2, 2, 2, 2, \cdots, 2, \cdots \rangle$$
$$G(x) \leftrightarrow \langle 7, 12, 22, 42, \cdots, 5 \cdot 2^n + 2, \cdots \rangle$$

Therefore, $g_n = 5 \cdot 2^n + 2$.

Answer 3

a)

R can be a equivalence relation if it is reflexive, symmetric and transitive. R is not reflexive, it is symmetric and it is not transitive. R is not reflexive since $1 \in \mathbb{Z}$, but $1R1$ is not true since there does not exist a right triangle with two edges of size one and the other edge being an integer. In a right triangle, hypotenuse is always the longest side of the triangle. If the hypotenuse is 1, by Pythagorean theorem $x^2 + 1^2 = 1^2 \to x^2 = 0 \to x = 0$, therefore one of the sides is zero and the shape cannot be a triangle. If the hypotenuse is not 1, by Pythagorean theorem $1^2 + 1^2 = x^2 \to 2 = x^2$ and by quadratic formula $x = \sqrt{2}$ since sides are always positive. Therefore, x is not an integer. The relation $1R1$ is not satisfied either way. Since $1R1$ is not satisfied, R is not reflexive, and since R is not reflexive, it cannot be an equivalence relation.

b)

R can be a equivalence relation if it is reflexive, symmetric and transitive. R is reflexive, symmetric, and transitive.

- R is reflexive since for all pairs,

$$\begin{aligned}(x_1, y_1)R(x_1, y_1) &\leftrightarrow 2x_1 + y_1 = 2x_1 + y_1 \\ &\leftrightarrow 0 = 0 \\ &\leftrightarrow T\end{aligned}$$

  Which means $(x_1, y_1)R(x_1, y_1)$ is always true.

- R is symmetric since for all pairs,

$$\begin{aligned}(x_1, y_1)R(x_2, y_2) &\to 2x_1 + y_1 = 2x_2 + y_2 \\ &\to 2x_2 + y_2 = 2x_1 + y_1 \\ &\to (x_2, y_2)R(x_1, y_1)\end{aligned}$$

- R is transitive since for all pairs,

$$\begin{aligned}(x_1, y_1)R(x_2, y_2) \wedge (x_2, y_2)R(x_3, y_3) &\rightarrow (2x_1 + y_1 = 2x_2 + y_2) \wedge (2x_2 + y_2 = 2x_3 + y_3)\\ &\rightarrow 2x_1 + y_1 = 2x_2 + y_2 = 2x_3 + y_3\\ &\rightarrow 2x_1 + y_1 = 2x_3 + y_3\\ &\rightarrow (x_1, y_1)R(x_3, y_3)\end{aligned}$$

For the equivalence class of $(1, -2)$:

$$\begin{aligned}(1, -2)R(x, y) &\leftrightarrow 2 \cdot 1 - 2 = 2x + y\\ &\leftrightarrow 0 = 2x + y\\ &\leftrightarrow -2x = y\end{aligned}$$

Therefore, $(1, -2)R(x, -2x)$ for all $x \in \mathbb{R}$. The equivalence class $[(1, -2)]$ is equal to the set $\{(x, -2x) \mid x \in \mathbb{R}\}$. If we take the pairs like points on a Cartesian coordinate system, this equivalence class represents the line passing through $(1, -2)$ with the slope -2. We can find the slope like this:

$$\begin{aligned}(x_1, y_1)R(x_2, y_2) &\leftrightarrow 2x_1 + y_1 = 2x_2 + y_2\\ &\leftrightarrow y_1 - y_2 = 2x_2 - 2x_1\\ &\leftrightarrow -(y_2 - y_1) = 2(x_2 - x_1)\\ &\leftrightarrow \frac{y_2 - y_1}{x_2 - x_1} = -2\end{aligned}$$

In fact, this relation R partitions $\mathbb{R}^2$ into lines with slope -2.

Answer 4