Student Information

Full Name : Murat Bolu
Id Number : 2521300

Answer 1

A function $f : A \to B$ is surjective if and only if $\forall b \in B,\ \exists a \in A$ such that $f(a) = b$.
A function $f : A \to B$ is injective if and only if $\forall a_1 \in A,\ \forall a_2 \in A,\ f(a_1) = f(a_2) \to a_1 = a_2$.
The set of nonnegative real numbers $\overline{\mathbb{R}}^+ = \{0\} \cup \mathbb{R}^+$.

a)

- f_1 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_1(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.
- f_1 is not injective since $f_1(2) = f_1(-2) = 4$, and $2 \neq -2$.

b)

- f_2 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_2(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.

- f_2 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_2(a_1) = f_2(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. I will prove this by contradiction. Suppose that a_1 is not zero but a positive real number. This would mean that $a_1 + a_2$ is strictly greater than a_2: $a_1 + a_2 > a_2$. That would mean zero is strictly greater than a_2: $0 > a_2$. That is not possible since $a_2 \in \{0\} \cup \mathbb{R}^+$ and every value of a_2 is equal to or greater than zero. This leads to a contradiction, therefore a_1 must be zero since $a_1 \in \{0\} \cup \mathbb{R}^+$. Since $a_1 = 0$, $a_1 + a_2 = 0 \to 0 + a_2 = 0 \to a_2 = 0$. Therefore, if $a_1 + a_2 = 0$, $a_1 = a_2$. We have already shown that if $a_1 - a_2 = 0$, $a_1 = a_2$. Therefore, $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+,\ f_2(a_1) = f_2(a_2) \to a_1 = a_2$.

c)

- f_3 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \mathbb{R}$ such that $a = \sqrt{b}$ and $f_3(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \mathbb{R}$ such that $f_3(a) = b$.

- f_3 is not injective since $f_3(2) = f_3(-2) = 4$, and $2 \neq -2$.

d)

- f_4 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \overline{\mathbb{R}}^+$ such that $a = \sqrt{b}$ and $f_4(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \overline{\mathbb{R}}^+$ such that $f_4(a) = b$.

- f_4 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+$, $\forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_4(a_1) = f_4(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. The proof is similar to above.

Answer 2

a) Since $f : A \subset \mathbb{Z} \subset \mathbb{R} \to \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

The Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = |x_1|$.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon)$$

I claim that for all ε, taking $\delta < 1$ works, which would mean $|x - x_0|$ must also be less than one.

$$\forall x \in A \, (|x - x_0| < 1 \to |f(x) - f(x_0)| < \varepsilon)$$

In that case, either $x = x_0$ or $x \neq x_0$.

- In the first case, $x = x_0 \to f(x) = f(x_0) \to f(x) - f(x_0) = 0$. Additionally, $x = x_0 \to x - x_0 = 0$. Since the both sides of the implication are true, the result is true.

- In the second case, $|x - x_0|$ must take positive integer values like $\{1, 2, 3, \dots\}$ since $x, x_0 \in \mathbb{Z}$. In that case $|x - x_0| < 1$ is always false, and the result is vacuously true.

Therefore, $\delta < 1$ satisfies the definition of continuity for all ε, x and x_0. Since we assumed nothing about x_0, the function f is continuous for all x_0 in A.

b) Since $f : \mathbb{R} \to \mathbb{Z} \subset \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in \mathbb{R} \, (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

Similarly, the Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = | x_1 |$.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in \mathbb{R} \, (| x - x_0 | < \delta \to | f(x) - f(x_0) | < \varepsilon)$$

The question is to show that f being a constant function is a necessary and sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous (at every point). This means that the truth of one implies the other.

- The first part is to show that f being a constant function is a sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means f being a constant function implies $f : \mathbb{R} \to \mathbb{Z}$ is continuous. Since f is constant, $f(x) = f(x_0)$ for all $x \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, which would imply $| f(x) - f(x_0) | = | 0 | = 0 < e$ for all $\varepsilon \in \mathbb{R}^+$. Since the right hand side of the implication is true for all ε, x and x_0, and it is independent of δ, it is always true. δ can be chosen arbitrarily.

- The second part is to show that f being a constant function is a necessary condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means $f : \mathbb{R} \to \mathbb{Z}$ being continuous implies f is a constant function. The proof will involve Intermediate Value Theorem, which says that a continuous function f whose domain contains the interval $[a, b]$ takes on all values between $f(a)$ and $f(b)$ in that interval. I will pick two numbers and show that $f(a) = f(b)$ for all a and b.

  - If a and b are equal, $f(a) = f(b)$ trivially by the virtue of f being a function.
  - If a and b are different, assume that $f(a) \neq f(b)$. That would mean f needs to take a non-integer value between $f(a)$ and $f(b)$, since there always exist (uncountably infinitely many) real numbers between any two different integers. Since the codomain of f is the set of integers, that is not possible, thus $f(a)$ cannot be different from $f(b)$.

  Since we assumed nothing about a and b, they can be chosen arbitrarily among all real numbers, showing that f has the same value throughout its domain, i.e. f is constant.

Answer 3

a) Proof by mathematical induction:

1. Basis: $X_2 = A_1 \times A_2$ is countable since $A_1 = \{a_{11}, a_{12}, \dots\}$ and $A_2 = \{a_{21}, a_{22}, \dots\}$ can be enumerated by the sum of i and j in indices of a_{1i} and a_{2j}.

$$\begin{array}{rr}
(a_{11}, a_{21}) & i+j=2 \\
(a_{11}, a_{22}), (a_{12}, a_{21}) & i+j=3 \\
(a_{11}, a_{23}), (a_{12}, a_{22}), (a_{13}, a_{21}) & i+j=4 \\
\dots & \dots \\
(a_{11}, a_{2(n-1)}), (a_{12}, a_{2(n-2)}), \dots, (a_{1(n-2)}, a_{22}), (a_{1(n-1)}, a_{21}) & i+j=n \\
\dots & \dots
\end{array}$$

2. Inductive Step: Assuming $X_k = A_1 \times A_2 \times \dots \times A_k$ is countable, I will show that $X_{k+1} = A_1 \times A_2 \times \dots \times A_k \times A_{k+1}$ is countable.

b)

Answer 4

I will first show that for functions f, g, and h, if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

$$\begin{aligned}
f \in O(g) &\leftrightarrow \exists c_1 \, \exists k_1 \, (\forall x \geq k_1 \, (|f(x)| \leq c_1 \cdot |g(x)|)) \\
g \in O(h) &\leftrightarrow \exists c_2 \, \exists k_2 \, (\forall x \geq k_2 \, (|g(x)| \leq c_2 \cdot |h(x)|)) \\
f \in O(h) &\leftrightarrow \exists c_3 \, \exists k_3 \, (\forall x \geq k_3 \, (|f(x)| \leq c_3 \cdot |h(x)|))
\end{aligned}$$

For x values bigger than both k_1 and k_2, both equations are satisfied. We can multiply the second equation by c_1.

$$\begin{aligned}
(f \in O(g)) \land (g \in O(h)) \leftrightarrow \exists c_1 \, \exists k_1 \, \exists c_2 \, \exists k_2 \, \forall x \\
(((x \geq k_1) \land (x \geq k_2)) \rightarrow (|f(x)| \leq c_1 \cdot |g(x)| \leq c_1 \cdot c_2 \cdot |h(x)|))
\end{aligned}$$

Therefore, there exists a c_3 value such that $c_3 = c_1 \cdot c_2$ and there exists a k_3 value such that $k_3 = \max\{k_1, k_2\}$ that makes $|f(x)| \leq c_3 \cdot |h(x)|$ for all x values greater than k_3.

$$\begin{aligned}
(f \in O(g)) \land (g \in O(h)) &\leftrightarrow \exists c_3 \, \exists k_3 \, (\forall x \geq k_3 \, (|f(x)| \leq c_3 \cdot |h(x)|)) \\
(f \in O(g)) \land (g \in O(h)) &\leftrightarrow f \in O(h)
\end{aligned}$$

a)

$$O((n!)^2) \ni (5^n) \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|5^x| \leq c \cdot |(x!)^2|))$$

Let $c = \frac{5^5}{5!}$ and $k = 6$. Since for $k \geq 6$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
5^x &\leq 5^5 \cdot 5^{x-5} && \forall x \geq 6 \\
&\leq 5^5 \cdot (5 \cdot 5 \cdot \ldots \cdot 5) && \forall x \geq 6 \\
&\leq 5^5 \cdot (6 \cdot 7 \cdot \ldots \cdot x) && \forall x \geq 6 \\
&\leq 5^5 \cdot \tfrac{x!}{5!} && \forall x \geq 6 \\
&\leq \tfrac{5^5}{5!} \cdot x! \cdot x! && \forall x \geq 6 \\
&\leq \tfrac{5^5}{5!} \cdot (x!)^2 && \forall x \geq 6
\end{aligned}$$

b)

$$O(5^n) \ni (2^n) \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|2^x| \leq c \cdot |5^x|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
2^x &\leq 2 \cdot 2 \cdot \ldots \cdot 2 && \forall x \geq 1 \\
&\leq 5 \cdot 5 \cdot \ldots \cdot 5 && \forall x \geq 1 \\
&\leq 5^x && \forall x \geq 1
\end{aligned}$$

Before these four proofs I will have to prove that $\log_2 x \leq \frac{x}{51}$, $\forall x \geq 1$, $\log_2 x \leq x$, $\forall x \geq 1$, and $\log_2 x \leq \sqrt{x}$, $\forall x \geq 16$.

c)

$$O(2^n) \ni (n^{51} + n^{49}) \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|n^{51} + n^{49}| \leq c \cdot |2^x|))$$

Let $c = 2$ and $k = 500$. Since for $k \geq 500$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
x^{51} + x^{49} &\leq x^{51} + x^{51} && \forall x \geq 500 \\
&\leq 2 \cdot x^{51} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{\log_2 (x^{51})} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{51 \cdot \log_2 x} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{51 \cdot \frac{x}{51}}{}^{*} && \forall x \geq 500 \\
&\leq 2 \cdot 2^x && \forall x \geq 500
\end{aligned}$$

*Since $\log_2 x \leq \frac{x}{51}$, $\forall x \geq 500$.

d)

$$O(n^{51} + n^{49}) \ni (n^{50}) \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|n^{50}| \leq c \cdot |n^{51} + n^{49}|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned} x^{50} &\leq x^{51} && \forall x \geq 1 \\ &\leq x^{51} + x^{49} && \forall x \geq 1 \end{aligned}$$

For these two proofs I am assuming $\log n$ is of base 2, however, proofs are similar for different bases.

e)

$$O(n^{50}) \ni (\sqrt{n} \cdot \log n) \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|\sqrt{n} \cdot \log n| \leq c \cdot |n^{50}|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned} \sqrt{x} \cdot \log x &\leq \sqrt{x} \cdot x^{\dagger} && \forall x \geq 1 \\ &\leq x^{\frac{1}{2}} \cdot x && \forall x \geq 1 \\ &\leq x^{\frac{3}{2}} && \forall x \geq 1 \\ &\leq x^{50} && \forall x \geq 1 \end{aligned}$$

f)

$$O(\sqrt{n} \cdot \log n) \ni (\log n)^2 \leftrightarrow \exists c\, \exists k\ (\forall x \geq k\ (|(\log n)^2| \leq c \cdot |\sqrt{n} \cdot \log n|))$$

Let $c = 1$ and $k = 16$. Since for $k \geq 16$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned} (\log x)^2 &\leq (\log x) \cdot (\log x) && \forall x \geq 16 \\ &\leq \sqrt{x}^{\ddagger} \cdot \log x && \forall x \geq 16 \end{aligned}$$

Answer 5

a)

b)

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† Since $\log x \leq x,\ \forall x \geq 1$.

‡ Since $\log x \leq \sqrt{x},\ \forall x \geq 16$.