Student Information

Full Name : Murat Bolu
Id Number : 2521300

Answer 1

A function $f : A \to B$ is surjective if and only if $\forall b \in B,\ \exists a \in A$ such that $f(a) = b$.
A function $f : A \to B$ is injective if and only if $\forall a_1 \in A,\ \forall a_2 \in A,\ f(a_1) = f(a_2) \to a_1 = a_2$.
The set of nonnegative real numbers $\overline{\mathbb{R}}^+ = \{0\} \cup \mathbb{R}^+$.

a)

- f_1 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_1(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.
- f_1 is not injective since $f_1(2) = f_1(-2) = 4$, and $2 \neq -2$.

b)

- f_2 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_2(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.

- f_2 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_2(a_1) = f_2(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. I will prove this by contradiction. Suppose that a_1 is not zero but a positive real number. This would mean that $a_1 + a_2$ is strictly greater than a_2: $a_1 + a_2 > a_2$. That would mean zero is strictly greater than a_2: $0 > a_2$. That is not possible since $a_2 \in \{0\} \cup \mathbb{R}^+$ and every value of a_2 is equal to or greater than zero. This leads to a contradiction, therefore a_1 must be zero since $a_1 \in \{0\} \cup \mathbb{R}^+$. Since $a_1 = 0$, $a_1 + a_2 = 0 \to 0 + a_2 = 0 \to a_2 = 0$. Therefore, if $a_1 + a_2 = 0$, $a_1 = a_2$. We have already shown that if $a_1 - a_2 = 0$, $a_1 = a_2$. Therefore, $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+,\ f_2(a_1) = f_2(a_2) \to a_1 = a_2$.

c)

- f_3 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \mathbb{R}$ such that $a = \sqrt{b}$ and $f_3(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \mathbb{R}$ such that $f_3(a) = b$.

- f_3 is not injective since $f_3(2) = f_3(-2) = 4$, and $2 \neq -2$.

d)

- f_4 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \overline{\mathbb{R}}^+$ such that $a = \sqrt{b}$ and $f_4(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \overline{\mathbb{R}}^+$ such that $f_4(a) = b$.

- f_4 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+$, $\forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_4(a_1) = f_4(a_2) &\rightarrow {a_1}^2 = {a_2}^2 \\ &\rightarrow {a_1}^2 - {a_2}^2 = 0 \\ &\rightarrow (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \rightarrow a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. The proof is similar to above.

Answer 2

a) Since $f : A \subset \mathbb{Z} \subset \mathbb{R} \rightarrow \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (\|x - x_0\| < \delta \rightarrow \|f(x) - f(x_0)\| < \varepsilon)$$

The Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = | x_1 |$.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (| x - x_0 | < \delta \rightarrow | f(x) - f(x_0) | < \varepsilon)$$

I claim that for all ε, taking $\delta < 1$ works, which would mean $| x - x_0 |$ must also be less than one.

$$\forall x \in A \, (| x - x_0 | < 1 \rightarrow | f(x) - f(x_0) | < \varepsilon)$$

In that case, either $x = x_0$ or $x \neq x_0$.

- In the first case, $x = x_0 \rightarrow f(x) = f(x_0) \rightarrow f(x) - f(x_0) = 0$. Additionally, $x = x_0 \rightarrow x - x_0 = 0$. Since the both sides of the implication are true, the result is true.

- In the second case, $| x - x_0 |$ must take positive integer values like $\{1, 2, 3, \dots\}$ since $x, x_0 \in \mathbb{Z}$. In that case $| x - x_0 | < 1$ is always false, and the result is vacuously true.

Therefore, $\delta < 1$ satisfies the definition of continuity for all ε, x and x_0. Since we assumed nothing about x_0, the function f is continuous for all x_0 in A.

b) Since $f : \mathbb{R} \to \mathbb{Z} \subset \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} \; (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

Similarly, the Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = | x_1 |$.

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} \; (| x - x_0 | < \delta \to | f(x) - f(x_0) | < \varepsilon)$$

The question is to show that f being a constant function is a necessary and sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous (at every point). This means that the truth of one implies the other.

- The first part is to show that f being a constant function is a sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means f being a constant function implies $f : \mathbb{R} \to \mathbb{Z}$ is continuous. Since f is constant, $f(x) = f(x_0)$ for all $x \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, which would imply $| f(x) - f(x_0) | = | 0 | = 0 < e$ for all $\varepsilon \in \mathbb{R}^+$. Since the right hand side of the implication is true for all ε, x and x_0, and it is independent of δ, it is always true. δ can be chosen arbitrarily.

- The second part is to show that f being a constant function is a necessary condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means $f : \mathbb{R} \to \mathbb{Z}$ being continuous implies f is a constant function. The proof will involve Intermediate Value Theorem, which says that a continuous function f whose domain contains the interval $[a, b]$ takes on all values between $f(a)$ and $f(b)$ in that interval. I will pick two numbers and show that $f(a) = f(b)$ for all a and b.

  - If a and b are equal, $f(a) = f(b)$ trivially by the virtue of f being a function.
  - If a and b are different, assume that $f(a) \neq f(b)$. That would mean f needs to take a non-integer value between $f(a)$ and $f(b)$, since there always exist (uncountably infinitely many) real numbers between any two different integers. Since the codomain of f is the set of integers, that is not possible, thus $f(a)$ cannot be different from $f(b)$.

  Since we assumed nothing about a and b, they can be chosen arbitrarily among all real numbers, showing that f has the same value throughout its domain, i.e. f is constant.

Answer 3

a) Proof by mathematical induction:

1. Basis: $X_2 = A_1 \times A_2$ is countable since $A_1 = \{a_1, a_2, \dots\}$ and $A_2 = \{b_1, b_2, \dots\}$ can be enumerated by the sum of indices of a_i and b_j.

$$
\begin{array}{rr}
(a_1, b_1) & i + j = 2 \\
(a_1, b_2), (a_2, b_1) & i + j = 3 \\
(a_1, b_3), (a_2, b_2), (a_3, b_1) & i + j = 4 \\
\dots & \dots \\
(a_1, b_{n-1}), (a_2, b_{n-2}), \dots, (a_{n-2}, b_2), (a_{n-1}, b_1) & i + j = n \\
\dots & \dots
\end{array}
$$

2. Inductive Step: Assuming $X_k = A_1 \times A_2 \times \dots \times A_k$ is countable, I will show that $X_{k+1} = A_1 \times A_2 \times \dots \times A_k \times A_{k+1}$ is countable.

b)

Answer 4

a)

b)

c)

d)

e)

f)

Answer 5

a)

b)