# Student Information

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# Answer 1

**Basis:** For $n = 1$, $6^{2n} - 1 = 6^2 - 1 = 35$, and $5 \mid 35$, and $7 \mid 35$.

**Inductive Step:** Assuming that $6^{2n} - 1$ is divisible by both 5 and 7, one can show that $6^{2(n+1)} - 1 = 6^{2n+2} - 1$ is divisible by 5 and 7 for $n \in \{1, 2, 3, \dots\}$. Since $5 \mid 6^{2n} - 1$ and $7 \mid 6^{2n} - 1$, that means $35 \mid 6^{2n} - 1$ which is shown below.

$$
\begin{aligned}
6^{2n} - 1 &= 5 \cdot x && \text{for some } x \in \mathbb{N} \\
6^{2n} - 1 &= 7 \cdot y && \text{for some } y \in \mathbb{N} \\
5 \cdot x &= 7 \cdot y
\end{aligned}
$$

Since $x$ and $y$ are natural numbers, $x$ must be divisible by 7 and $y$ must be divisible by 5, and we can rewrite them as $x = 7 \cdot k$ and $y = 5 \cdot l$. The other possibility is that they are both zero, which is obviously not possible since $6^{2n} - 1$ is at least 35.

$$
\begin{aligned}
6^{2n} - 1 &= 5 \cdot 7 \cdot k && \text{for some } k \in \mathbb{N} \\
6^{2n} - 1 &= 7 \cdot 5 \cdot l && \text{for some } l \in \mathbb{N} \\
6^{2n} - 1 &= 35 \cdot k \\
6^{2n} - 1 &= 35 \cdot l
\end{aligned}
$$

There exists $k$ or $l$ such that they are natural numbers, and that means $6^{2n} - 1$ is divisible by 35.

$$
\begin{aligned}
6^{2n} - 1 &\equiv 0 && \pmod{35} \\
6^{2n} &\equiv 1 && \pmod{35} \\
6^{2n+2} &\equiv 36 && \pmod{35} \\
6^{2n+2} &\equiv 1 && \pmod{35} \\
6^{2n+2} - 1 &\equiv 0 && \pmod{35}
\end{aligned}
$$

Therefore, there exists a natural number $a \in \mathbb{N}$ such that $6^{2n+2} - 1 = 35 \cdot a$. It is obvious that $6^{2n+2} - 1$ is divisible by both 5 and 7 since $5 \mid 35a$ and $7 \mid 35a$.

$\blacksquare$

# Answer 2

**Basis:**

- For $n = 0$, $H_0 = 1 \leq 1 = 9^0$.
- For $n = 1$, $H_1 = 5 \leq 9 = 9^1$.
- For $n = 2$, $H_2 = 7 \leq 81 = 9^2$.
- For $n = 3$, $H_3 = 8H_2 + 8H_1 + 9H_0 = 8 \cdot 7 + 8 \cdot 5 + 9 \cdot 1 = 105 \leq 729 = 9^3$.

**Inductive Step:** Assuming that $H_n \leq 9^n$, $\forall n \leq k$, one can show that $H_{k+1}$ is less than or equal to $9^{k+1}$. We can replace every $H_n$ with something greater than or equal to itself.

$$
\begin{aligned}
H_{k+1} &= 8H_k + 8H_{k-1} + 9H_{k-2} \\
H_{k+1} &\leq 8 \cdot 9^k + 8H_{k-1} + 9H_{k-2} \\
H_{k+1} &\leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9H_{k-2} \\
H_{k+1} &\leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9 \cdot 9^{k-2} \\
H_{k+1} &\leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9^{k-1} \\
H_{k+1} &\leq 8 \cdot 9^k + 9 \cdot 9^{k-1} \\
H_{k+1} &\leq 8 \cdot 9^k + 9^k \\
H_{k+1} &\leq 9 \cdot 9^k \\
H_{k+1} &\leq 9^{k+1}
\end{aligned}
$$

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# Answer 3

Let $A$ denote the set that contains 8 digit bit strings that contain four consecutive zeros and $B$ denote the set that contains 8 digit bit strings that contain four consecutive ones. It it clear that the answer we are looking for is $|A \cup B|$ which is equal to the commonly known identity $|A| + |B| - |A \cap B|$. It can be seen that $A \cap B = \{00001111, 11110000\}$ and $|A \cap B| = 2$. Then, we need to find $|A|$ and $|B|$.
Let $a_n$ denote the number of strings of size $n$ that contain four consecutive zeros and $b_n$ denote the number of strings of size $n$ that contain four consecutive ones. One can see that $|A| = a_8$ and $|B| = b_8$.
$a_n$ can be written as a recurrence relation where $a_n = 2 \cdot a_{n-1} + 2^{n-5} - a_{n-5}$ and $a_0 = a_1 = a_2 = a_3 = 0$, $a_4 = 1$. The initial conditions are obvious, and the reason for recurrence relation is that we can append zero or one to a valid digit of size $n-1$ to get a valid digit of size $n$, hence $2 \cdot a_{n-1}$. We can also take a invalid digit of size $n-5$ and append 10000 to it, invalid digits are all digits minus the valid digits, hence $2^{n-5} - a_{n-5}$. One can calculate $a_5 = 3$, $a_6 = 8$, $a_7 = 20$, and $a_8 = 48$. The same argument applies to $b_n$ where the bits are flipped, so $b_8 = 48$. Therefore $|A| = 48$, $|B| = 48$, and $|A \cup B| = 48 + 48 - 2 = 94$.

## Answer 4

One can choose the stars, habitable planets, and inhabitable planets first and order them later. One needs to choose 1 star from 10 stars, 2 habitable planets from 20 habitable planets, 8 inhabitable planets from 80 inhabitable planets. Therefore, there are $\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8} = 55{,}076{,}320{,}585{,}000$ possible choices for different stars and planets. For ordering, if we consider the habitable planets same with each other and inhabitable planets same with each other, there are 6 possibilities.

$$\begin{array}{ll} s & uuhuuuuuuh \\ s & uhuuuuuuuh \\ s & huuuuuuuuh \\ s & uhuuuuuuhu \\ s & huuuuuuuhu \\ s & huuuuuuhuu \end{array}$$

Thus, there are 6 different ways to order the planets. In total, there are $\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8} \cdot 6 = 330{,}457{,}923{,}510{,}000$ different ways to create a galaxy.

## Answer 5

**a)**

**b)**

**c)**