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Answer 1

Basis: For $n = 1$, $6^{2n} - 1 = 6^2 - 1 = 35$, and $5 \mid 35$, and $7 \mid 35$.

Inductive Step: Assuming that $6^{2n} - 1$ is divisible by both 5 and 7, one can show that $6^{2(n+1)} - 1 = 6^{2n+2} - 1$ is divisible by 5 and 7 for $n \in \{1, 2, 3, \dots\}$. Since $5 \mid 6^{2n} - 1$ and $7 \mid 6^{2n} - 1$, that means $35 \mid 6^{2n} - 1$ which is shown below.

$$6^{2n} - 1 = 5 \cdot x \qquad \text{for some } x \in \mathbb{N}$$
$$6^{2n} - 1 = 7 \cdot y \qquad \text{for some } y \in \mathbb{N}$$
$$5 \cdot x = 7 \cdot y$$

Since x and y are natural numbers, x must be divisible by 7 and y must be divisible by 5, and we can rewrite them as $x = 7 \cdot k$ and $y = 5 \cdot l$. The other possibility is that they are both zero, which is obviously not possible since $6^{2n} - 1$ is at least 35.

$$6^{2n} - 1 = 5 \cdot 7 \cdot k \qquad \text{for some } k \in \mathbb{N}$$
$$6^{2n} - 1 = 7 \cdot 5 \cdot l \qquad \text{for some } l \in \mathbb{N}$$
$$6^{2n} - 1 = 35 \cdot k$$
$$6^{2n} - 1 = 35 \cdot l$$

There exists k or l such that they are natural numbers, and that means $6^{2n} - 1$ is divisible by 35.

$$6^{2n} - 1 \equiv 0 \qquad (\text{mod } 35)$$
$$6^{2n} \equiv 1 \qquad (\text{mod } 35)$$
$$6^{2n+2} \equiv 36 \qquad (\text{mod } 35)$$
$$6^{2n+2} \equiv 1 \qquad (\text{mod } 35)$$
$$6^{2n+2} - 1 \equiv 0 \qquad (\text{mod } 35)$$

Therefore, there exists a natural number $a \in \mathbb{N}$ such that $6^{2n+2} - 1 = 35 \cdot a$. It is obvious that $6^{2n+2} - 1$ is divisible by both 5 and 7 since $5 \mid 35a$ and $7 \mid 35a$.

$\blacksquare$

Answer 2

Basis:

- For $n = 0$, $H_0 = 1 \leq 1 = 9^0$.
- For $n = 1$, $H_1 = 5 \leq 9 = 9^1$.
- For $n = 2$, $H_2 = 7 \leq 81 = 9^2$.
- For $n = 3$, $H_3 = 8H_2 + 8H_1 + 9H_0 = 8 \cdot 7 + 8 \cdot 5 + 9 \cdot 1 = 105 \leq 729 = 9^3$.

Inductive Step: Assuming that $H_n \leq 9^n$, $\forall n \leq k$, one can show that H_{k+1} is less than or equal to 9^{k+1}. We can replace every H_n with something greater than or equal to itself.

$$H_{k+1} = 8H_k + 8H_{k-1} + 9H_{k-2}$$
$$H_{k+1} \leq 8 \cdot 9^k + 8H_{k-1} + 9H_{k-2}$$
$$H_{k+1} \leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9H_{k-2}$$
$$H_{k+1} \leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9 \cdot 9^{k-2}$$
$$H_{k+1} \leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9^{k-1}$$
$$H_{k+1} \leq 8 \cdot 9^k + 9 \cdot 9^{k-1}$$
$$H_{k+1} \leq 8 \cdot 9^k + 9^k$$
$$H_{k+1} \leq 9 \cdot 9^k$$
$$H_{k+1} \leq 9^{k+1}$$

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Answer 3

Let A denote the set that contains 8 digit bit strings that contain four consecutive zeros and B denote the set that contains 8 digit bit strings that contain four consecutive ones. It it clear that the answer we are looking for is $|A \cup B|$ which is equal to the commonly known identity $|A| + |B| - |A \cap B|$. It can be seen that $A \cap B = \{00001111, 11110000\}$ and $|A \cap B| = 2$. Then, we need to find $|A|$ and $|B|$.

Let a_n denote the number of strings of size n that contain four consecutive zeros and b_n denote the number of strings of size n that contain four consecutive ones. One can see that $|A| = a_8$ and $|B| = b_8$.

a_n can be written as a recurrence relation where $a_n = 2 \cdot a_{n-1} + 2^{n-5} - a_{n-5}$ and $a_0 = a_1 = a_2 = a_3 = 0$, $a_4 = 1$. The initial conditions are obvious, and the reason for recurrence relation is that we can append zero or one to a valid digit of size $n - 1$ to get a valid digit of size n, hence $2 \cdot a_{n-1}$. We can also take a invalid digit of size $n - 5$ and append 10000 to it, invalid digits are all digits minus the valid digits, hence $2^{n-5} - a_{n-5}$. One can calculate $a_5 = 3$, $a_6 = 8$, $a_7 = 20$, and $a_8 = 48$. The same argument applies to b_n where the bits are flipped, so $b_8 = 48$. Therefore $|A| = 48$, $|B| = 48$, and $|A \cup B| = 48 + 48 - 2 = 94$.

Answer 4

One can choose the stars, habitable planets, and inhabitable planets such that they are chosen in the order the closest to the star to the farthest. One needs to choose 1 star from 10 stars, 2 habitable planets from 20 habitable planets, 8 inhabitable planets from 80 inhabitable planets in order. Therefore, there are $^{10}P_1 \cdot {}^{20}P_2 \cdot {}^{80}P_8 = 4{,}441{,}354{,}491{,}974{,}400{,}000$ possible choices for different stars and planets. For different orderings between planets, if we consider the habitable planets same with each other and inhabitable planets same with each other, there are 6 possibilities.

$$s \quad uuhuuuuuuh$$
$$s \quad uhuuuuuuuh$$
$$s \quad huuuuuuuuh$$
$$s \quad uhuuuuuuhu$$
$$s \quad huuuuuuuhu$$
$$s \quad huuuuuuhuu$$

One may also consider the block $huuuuuuh$ with six u's in the middle unmoving and two remaining u's moving. In that case, h's act like separators and two remaining u's can be placed in any three positions. There are 4 objects in total, two separators and two u's. Both pairs have identical objects, so the permutation of these objects must be divided by 2! two times. Thus, the possible orderings are $\frac{4!}{2! \cdot 2!} = 6$.

$$s \quad \_|\_|\_$$
$$s \quad \_|uu|\_$$

Thus, there are 6 different ways to order the planets. In total, there are $^{10}P_1 \cdot {}^{20}P_2 \cdot {}^{80}P_8 \cdot 6 = 26{,}648{,}126{,}951{,}846{,}400{,}000$ different ways to create a galaxy.

Answer 5

Let the robot start from position 0, much like array indexing.

a) Let a_n be the number of ways the robot can reach the nth cell. The recurrence relation for a_n is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = 1$, $a_2 = 2$.

b) Initial conditions are $a_0 = 1$, $a_1 = 1$, $a_2 = 2$ since there is only one way to reach 0th and 1st cell, by standing still and jumping one cell. There are two ways to reach 2nd cell, by jumping one by one two times and by jumping two cells at once.

c) The robot can move to the 9th cell in a_9 different ways, which can be calculated bottom-up.

$$a_3 = a_2 + a_1 + a_0 = 2 + 1 + 1 = 4$$
$$a_4 = a_3 + a_2 + a_1 = 4 + 2 + 1 = 7$$
$$a_5 = a_4 + a_3 + a_2 = 7 + 4 + 2 = 13$$
$$a_6 = a_5 + a_4 + a_3 = 13 + 7 + 4 = 24$$
$$a_7 = a_6 + a_5 + a_4 = 24 + 13 + 7 = 44$$
$$a_8 = a_7 + a_6 + a_5 = 44 + 24 + 13 = 81$$
$$a_9 = a_8 + a_7 + a_6 = 81 + 44 + 24 = 149$$

The robot can move to the 9th cell in 149 different ways.