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Answer 1

A function $f : A \to B$ is surjective if and only if $\forall b \in B,\ \exists a \in A$ such that $f(a) = b$.
A function $f : A \to B$ is injective if and only if $\forall a_1 \in A,\ \forall a_2 \in A,\ f(a_1) = f(a_2) \to a_1 = a_2$.
The set of nonnegative real numbers $\overline{\mathbb{R}}^+ = \{0\} \cup \mathbb{R}^+$.

a)

- f_1 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_1(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.
- f_1 is not injective since $f_1(2) = f_1(-2) = 4$, and $2 \neq -2$.

b)

- f_2 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_2(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.

- f_2 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_2(a_1) = f_2(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. I will prove this by contradiction. Suppose that a_1 is not zero but a positive real number. This would mean that $a_1 + a_2$ is strictly greater than a_2: $a_1 + a_2 > a_2$. That would mean zero is strictly greater than a_2: $0 > a_2$. That is not possible since $a_2 \in \{0\} \cup \mathbb{R}^+$ and every value of a_2 is equal to or greater than zero. This leads to a contradiction, therefore a_1 must be zero since $a_1 \in \{0\} \cup \mathbb{R}^+$. Since $a_1 = 0$, $a_1 + a_2 = 0 \to 0 + a_2 = 0 \to a_2 = 0$. Therefore, if $a_1 + a_2 = 0$, $a_1 = a_2$. We have already shown that if $a_1 - a_2 = 0$, $a_1 = a_2$. Therefore, $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+,\ f_2(a_1) = f_2(a_2) \to a_1 = a_2$.

c)

- f_3 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \mathbb{R}$ such that $a = \sqrt{b}$ and $f_3(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \mathbb{R}$ such that $f_3(a) = b$.

- f_3 is not injective since $f_3(2) = f_3(-2) = 4$, and $2 \neq -2$.

d)

- f_4 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \overline{\mathbb{R}}^+$ such that $a = \sqrt{b}$ and $f_4(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \overline{\mathbb{R}}^+$ such that $f_4(a) = b$.

- f_4 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+$, $\forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_4(a_1) = f_4(a_2) &\rightarrow {a_1}^2 = {a_2}^2 \\ &\rightarrow {a_1}^2 - {a_2}^2 = 0 \\ &\rightarrow (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \rightarrow a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. The proof is similar to above.

Answer 2

a) Since $f : A \subset \mathbb{Z} \subset \mathbb{R} \rightarrow \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (\|x - x_0\| < \delta \rightarrow \|f(x) - f(x_0)\| < \varepsilon)$$

The Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = | x_1 |$.

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \, (| x - x_0 | < \delta \rightarrow | f(x) - f(x_0) | < \varepsilon)$$

I claim that for all ε, taking $\delta < 1$ works, which would mean $| x - x_0 |$ must also be less than one.

$$\forall x \in A \, (| x - x_0 | < 1 \rightarrow | f(x) - f(x_0) | < \varepsilon)$$

In that case, either $x = x_0$ or $x \neq x_0$.

- In the first case, $x = x_0 \rightarrow f(x) = f(x_0) \rightarrow f(x) - f(x_0) = 0$. Additionally, $x = x_0 \rightarrow x - x_0 = 0$. Since the both sides of the implication are true, the result is true.

- In the second case, $| x - x_0 |$ must take positive integer values like $\{1, 2, 3, \ldots\}$ since $x, x_0 \in \mathbb{Z}$. In that case $| x - x_0 | < 1$ is always false, and the result is vacuously true.

Therefore, $\delta < 1$ satisfies the definition of continuity for all ε, x and x_0. Since we assumed nothing about x_0, the function f is continuous for all x_0 in A.

b) Since $f : \mathbb{R} \to \mathbb{Z} \subset \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} \; (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

Similarly, the Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself, since $\| x \| = \sqrt{\sum_{i=1}^{n} {x_i}^2} = \sqrt{\sum_{i=1}^{1} {x_i}^2} = \sqrt{{x_1}^2} = | x_1 |$.

$$\forall \varepsilon \in \mathbb{R}^+ \; \exists \delta \in \mathbb{R}^+ \; \forall x \in \mathbb{R} \; (| x - x_0 | < \delta \to | f(x) - f(x_0) | < \varepsilon)$$

The question is to show that f being a constant function is a necessary and sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous (at every point). This means that the truth of one implies the other.

- The first part is to show that f being a constant function is a sufficient condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means f being a constant function implies $f : \mathbb{R} \to \mathbb{Z}$ is continuous. Since f is constant, $f(x) = f(x_0)$ for all $x \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, which would imply $| f(x) - f(x_0) | = | 0 | = 0 < e$ for all $\varepsilon \in \mathbb{R}^+$. Since the right hand side of the implication is true for all ε, x and x_0, and it is independent of δ, it is always true. δ can be chosen arbitrarily.

- The second part is to show that f being a constant function is a necessary condition for the function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous. That means $f : \mathbb{R} \to \mathbb{Z}$ being continuous implies f is a constant function. The proof will involve Intermediate Value Theorem, which says that a continuous function f whose domain contains the interval $[a, b]$ takes on all values between $f(a)$ and $f(b)$ in that interval. I will pick two numbers and show that $f(a) = f(b)$ for all a and b.

  - If a and b are equal, $f(a) = f(b)$ trivially by the virtue of f being a function.
  - If a and b are different, assume that $f(a) \neq f(b)$. That would mean f needs to take a non-integer value between $f(a)$ and $f(b)$, since there always exist (uncountably infinitely many) real numbers between any two different integers. Since the codomain of f is the set of integers, that is not possible, thus $f(a)$ cannot be different from $f(b)$.

  Since we assumed nothing about a and b, they can be chosen arbitrarily among all real numbers, showing that f has the same value throughout its domain, i.e. f is constant.

Answer 3

a) Proof by mathematical induction:

1. Basis: $X_2 = A_1 \times A_2$ is countable since $A_1 = \{a_{11}, a_{12}, \dots\}$ and $A_2 = \{a_{21}, a_{22}, \dots\}$ can be enumerated by the sum of i and j in indices of a_{1i} and a_{2j}.

$$
\begin{array}{rr}
(a_{11}, a_{21}) & i+j=2 \\
(a_{11}, a_{22}), (a_{12}, a_{21}) & i+j=3 \\
(a_{11}, a_{23}), (a_{12}, a_{22}), (a_{13}, a_{21}) & i+j=4 \\
\dots & \dots \\
(a_{11}, a_{2(n-1)}), (a_{12}, a_{2(n-2)}), \dots, (a_{1(n-2)}, a_{22}), (a_{1(n-1)}, a_{21}) & i+j=n \\
\dots & \dots
\end{array}
$$

2. Inductive Step: Assuming $X_k = A_1 \times A_2 \times \dots \times A_k$ is countable, I will show that $X_{k+1} = A_1 \times A_2 \times \dots \times A_k \times A_{k+1}$ is countable. Let $x_{k\,i} \in X_k$ denote the elements of X_k where $i \in \mathbb{N}$. Let $a_{(k+1)\,j}$ denote the elements of A_{k+1} where $j \in \mathbb{N}$. The elements of X_{k+1} can we enumerated by the sum of i and j in indices of $x_{k\,i}$ and $a_{(k+1)\,j}$, in a similar way to the base case above.

$$
\begin{array}{rr}
(x_{k\,1}, a_{(k+1)\,1}) & i+j=2 \\
(x_{k\,1}, a_{(k+1)\,2}), (x_{k\,2}, a_{(k+1)\,1}) & i+j=3 \\
(x_{k\,1}, a_{(k+1)\,3}), (x_{k\,2}, a_{(k+1)\,2}), (x_{k\,3}, a_{(k+1)\,1}) & i+j=4 \\
\dots & \dots \\
(x_{k\,1}, a_{(k+1)\,(n-1)}), (x_{k\,2}, a_{(k+1)\,(n-2)}), \dots, (x_{k\,(n-2)}, a_{(k+1)\,2}), (x_{k\,(n-1)}, a_{(k+1)\,1}) & i+j=n \\
\dots & \dots
\end{array}
$$

b) Let $S = X \times X \times \dots$ be the set of infinite countable product of the set $X = \{0, 1\}$. Then, every element of S can be represented by an infinite sequence of ones and zeros, since they will be in the form $(1, 0, 0, 1, \dots)$. I will use Cantor's Diagonal Argument to show that S is uncountably infinite. Assuming S is countable, let $s_i \in S$ where $i \in \mathbb{N}$. Then, let s_{new} be the representation such that i^{th} digit of s_{new} is 0 if i^{th} digit of s_i is 1, and 1 if i^{th} digit of s_i is 0. Since s_{new} is different from every $s_i \in S$, it isn't in S. Since it is not in enumeration, enumeration is not complete. This contradicts the assumption that the enumeration was complete, i.e. S is countable. Thus, S is uncountable.

$$
\begin{aligned}
s_1 &= 1011010\dots \\
s_2 &= 0110101\dots \\
s_3 &= 0001111\dots \\
s_4 &= 1100001\dots \\
&\vdots \\
s_{\text{new}} &= 0011\dots
\end{aligned}
$$

Answer 4

I will first show that for functions f, g, and h, if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

$$\begin{aligned}
f \in O(g) &\leftrightarrow \exists c_1\, \exists k_1\, (\forall x \geq k_1\, (|f(x)| \leq c_1 \cdot |g(x)|))\\
g \in O(h) &\leftrightarrow \exists c_2\, \exists k_2\, (\forall x \geq k_2\, (|g(x)| \leq c_2 \cdot |h(x)|))\\
f \in O(h) &\leftrightarrow \exists c_3\, \exists k_3\, (\forall x \geq k_3\, (|f(x)| \leq c_3 \cdot |h(x)|))
\end{aligned}$$

For x values bigger than both k_1 and k_2, both equations are satisfied. We can multiply the second equation by c_1.

$$\begin{aligned}
(f \in O(g)) \wedge (g \in O(h)) \leftrightarrow{} & \exists c_1\, \exists k_1\, \exists c_2\, \exists k_2\, \forall x\\
& (((x \geq k_1) \wedge (x \geq k_2)) \to (|f(x)| \leq c_1 \cdot |g(x)| \leq c_1 \cdot c_2 \cdot |h(x)|))
\end{aligned}$$

Therefore, there exists a c_3 value such that $c_3 = c_1 \cdot c_2$ and there exists a k_3 value such that $k_3 = \max\{k_1, k_2\}$ that makes $|f(x)| \leq c_3 \cdot |h(x)|$ for all x values greater than k_3.

$$\begin{aligned}
&(f \in O(g)) \wedge (g \in O(h)) \leftrightarrow \exists c_3\, \exists k_3\, (\forall x \geq k_3\, (|f(x)| \leq c_3 \cdot |h(x)|))\\
&(f \in O(g)) \wedge (g \in O(h)) \leftrightarrow f \in O(h)
\end{aligned}$$

a)

$$O((n!)^2) \ni (5^n) \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|5^x| \leq c \cdot |(x!)^2|))$$

Let $c = \frac{5^5}{5!}$ and $k = 6$. Since for $k \geq 6$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
5^x &\leq 5^5 \cdot 5^{x-5} && \forall x \geq 6\\
&\leq 5^5 \cdot (5 \cdot 5 \cdot \ldots \cdot 5) && \forall x \geq 6\\
&\leq 5^5 \cdot (6 \cdot 7 \cdot \ldots \cdot x) && \forall x \geq 6\\
&\leq 5^5 \cdot \tfrac{x!}{5!} && \forall x \geq 6\\
&\leq \tfrac{5^5}{5!} \cdot x! \cdot x! && \forall x \geq 6\\
&\leq \tfrac{5^5}{5!} \cdot (x!)^2 && \forall x \geq 6
\end{aligned}$$

b)

$$O(5^n) \ni (2^n) \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|2^x| \leq c \cdot |5^x|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
2^x &\leq 2 \cdot 2 \cdot \ldots \cdot 2 && \forall x \geq 1\\
&\leq 5 \cdot 5 \cdot \ldots \cdot 5 && \forall x \geq 1\\
&\leq 5^x && \forall x \geq 1
\end{aligned}$$

Before these four proofs I will have to prove that $\log_2 x \leq \frac{x}{51}$, $\forall x \geq 500$, $\log_2 x \leq x$, $\forall x \geq 1$, and $\log_2 x \leq \sqrt{x}$, $\forall x \geq 16$. I will show this by showing $\lim_{x\to\infty} \frac{\log_2 x}{ax^b} = 0$ for all a, b where $a > 0, b > 0$, which implies $\log_2 x \in O(ax^b)$ for all a, b where $a > 0, b > 0$.

$$\begin{aligned}
\lim_{x\to\infty} \tfrac{\log_2 x}{ax^b} &= \lim_{x\to\infty} \tfrac{\ln x}{\ln 2 \cdot ax^b} \\
&= \lim_{x\to\infty} \tfrac{1/x}{\ln 2 \cdot a \cdot b \cdot x^{b-1}} && \text{By L'Hôpital's rule, since } [\tfrac{0}{0}] \text{ indeterminacy} \\
&= \lim_{x\to\infty} \tfrac{1}{\ln 2 \cdot a \cdot b \cdot x^{b}} \\
&= 0 && \text{Due to } [\tfrac{1}{\infty}] \text{ form}
\end{aligned}$$

c)

$$O(2^n) \ni (n^{51} + n^{49}) \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|n^{51} + n^{49}| \leq c \cdot |2^x|))$$

Let $c = 2$ and $k = 500$. Since for $k \geq 500$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
x^{51} + x^{49} &\leq x^{51} + x^{51} && \forall x \geq 500 \\
&\leq 2 \cdot x^{51} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{\log_2 (x^{51})} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{51 \cdot \log_2 x} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{51 \cdot \frac{x}{51}}{}^{*} && \forall x \geq 500 \\
&\leq 2 \cdot 2^{x} && \forall x \geq 500
\end{aligned}$$

d)

$$O(n^{51} + n^{49}) \ni (n^{50}) \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|n^{50}| \leq c \cdot |n^{51} + n^{49}|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
x^{50} &\leq x^{51} && \forall x \geq 1 \\
&\leq x^{51} + x^{49} && \forall x \geq 1
\end{aligned}$$

For these two proofs I am assuming $\log n$ is of base 2, however, proofs are similar for different bases.

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*Since $\log_2 x \leq \frac{x}{51}$, $\forall x \geq 500$.

e)

$$O(n^{50}) \ni (\sqrt{n} \cdot \log n) \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|\sqrt{n} \cdot \log n| \leq c \cdot |n^{50}|))$$

Let $c = 1$ and $k = 1$. Since for $k \geq 1$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
\sqrt{x} \cdot \log x &\leq \sqrt{x} \cdot x^{\dagger} && \forall x \geq 1 \\
&\leq x^{\frac{1}{2}} \cdot x && \forall x \geq 1 \\
&\leq x^{\frac{3}{2}} && \forall x \geq 1 \\
&\leq x^{50} && \forall x \geq 1
\end{aligned}$$

f)

$$O(\sqrt{n} \cdot \log n) \ni (\log n)^2 \leftrightarrow \exists c\, \exists k\, (\forall x \geq k\, (|(\log n)^2| \leq c \cdot |\sqrt{n} \cdot \log n|))$$

Let $c = 1$ and $k = 16$. Since for $k \geq 16$ both of these functions are always positive, we can get rid of absolute value signs.

$$\begin{aligned}
(\log x)^2 &\leq (\log x) \cdot (\log x) && \forall x \geq 16 \\
&\leq \sqrt{x}^{\ddagger} \cdot \log x && \forall x \geq 16
\end{aligned}$$

Answer 5

a)

$$\begin{aligned}
\text{gdc}(94, 134) &= \text{gdc}(134, 94) \\
&= \text{gdc}(94, 134 \bmod 94) \\
&= \text{gdc}(94, 40) \\
&= \text{gdc}(40, 94 \bmod 40) \\
&= \text{gdc}(40, 14) \\
&= \text{gdc}(14, 40 \bmod 14) \\
&= \text{gdc}(14, 12) \\
&= \text{gdc}(12, 14 \bmod 12) \\
&= \text{gdc}(12, 2) \\
&= \text{gdc}(2, 12 \bmod 2) \\
&= \text{gdc}(2, 0) \\
&= 2
\end{aligned}$$

†Since $\log x \leq x$, $\forall x \geq 1$.

‡Since $\log x \leq \sqrt{x}$, $\forall x \geq 16$.

b) We will have to show it both ways. First, assume Goldbach's conjecture and reach to the conclusion given in the question. Second, assume the conclusion given in the question is correct and reach Goldbach's conjecture.

1. Assuming Goldbach's conjecture, where $n = p_1 + p_2$, $n = 2k$, $k \in \{2, 3, 4, \dots\}$, and p_1 and p_2 are primes. Then, we can add 2, a prime number, making the equation $n + 2 = p_1 + p_2 + 2$. Since n is even, $n + 2$ is even, therefore $n + 2 = 2l$, $l \in \{3, 4, 5, \dots\}$. This shows that every even integer greater than 5 can be written as a sum of three primes. Or, we can add 3, a prime number, making the equation $n + 3 = p_1 + p_2 + 3$. Since n is even, $n + 3$ is odd, therefore $n + 3 = 2m + 1$, $m \in \{3, 4, 5, \dots\}$. Therefore, every odd integer greater than 5 can be written as a sum of three primes. Therefore, every integer greater than 5 can be written as a sum of three primes.

2. Assuming that every integer greater than 5 can be written as a sum of three primes, where $n = p_1 + p_2 + p_3$, $n \in \{6, 7, 8, \dots\}$ and p_1, p_2 and p_3 are primes. The only even prime is 2, so if one of p_i's are 2, n must be even, since the sum of an even number, an odd number and an odd number is an even number. Without loss of generality, we can assume that $p_3 = 2$. Then, we can subtract 2, making the equation $n - 2 = p_1 + p_2$. Since n is even, $n - 2$ is even, therefore $n - 2 = 2k$, $k \in \{2, 3, 4, \dots\}$. If all three primes are odd, then n must be odd, since the sum of three odd numbers is odd. Additionally, since the smallest odd prime is 3, n is at least 9. Since all p_i's are odd, we can sum any two of them to get an even number. Without loss of generality, we can choose p_2 and p_3. Every number greater than 5 can be written as a sum of three primes, and since our primes are at least 3 and 3, every sum of p_2 and p_3 can be written as a sum of three primes. Since the sum is even, one of the primes must be 2, since only the sum of one even number and two odd numbers can make an even number, remembering the fact that 2 is the only prime number. Therefore, $n = p_1 + p_4 + p_5 + 2$. We can subtract 3, making the equation $n - 3 = p_1 + p_4 + p_5 - 1$.