

# Student Information

Full Name : Murat Bolu

Id Number : 2521300

## Answer 1

Let  $\langle a_0, a_1, a_2, a_3, \dots, a_n, \dots \rangle$  be the generating function  $A(x)$ . Then,  $A(x) = \sum_{n=0}^{\infty} a_n x^n$ .

$$\begin{aligned} \sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} (3a_{n-1} + 4a_{n-2}) \cdot x^n \\ &= \sum_{n=2}^{\infty} 3a_{n-1} x^n + 4a_{n-2} x^n \\ &= \sum_{n=2}^{\infty} 3a_{n-1} x^n + \sum_{n=2}^{\infty} 4a_{n-2} x^n \\ &= 3 \sum_{n=2}^{\infty} a_{n-1} x^n + 4 \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\ &= 3x \sum_{n=1}^{\infty} a_n x^n + 4x^2 \sum_{n=0}^{\infty} a_n x^n \\ \sum_{n=2}^{\infty} a_n x^n &= 3x \sum_{n=1}^{\infty} a_n x^n + 4x^2 \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

$$A(x) - a_1 \cdot x - a_0 = 3x \cdot (A(x) - a_0) + 4x^2 \cdot A(x)$$

$$A(x) - a_1 \cdot x - a_0 = 3x \cdot A(x) - 3x \cdot a_0 + 4x^2 \cdot A(x)$$

$$A(x) - 3x \cdot A(x) - 4x^2 \cdot A(x) = a_1 \cdot x + a_0 - 3x \cdot a_0$$

$$A(x) \cdot (1 - 3x - 4x^2) = a_1 \cdot x + a_0 \cdot (1 - 3x)$$

$$\begin{aligned} A(x) &= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{1 - 3x - 4x^2} \\ &= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{(1 - 4x) \cdot (1 + x)} \end{aligned}$$

Plug in values for  $a_0$  and  $a_1$ :

$$\begin{aligned} A(x) &= \frac{x + 1 \cdot (1 - 3x)}{(1 - 4x) \cdot (1 + x)} \\ &= \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} \end{aligned}$$

Partial fractions:

$$\begin{aligned} \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} &= \frac{\xi}{1 - 4x} + \frac{\varphi}{1 + x} \\ &= \frac{\xi \cdot (1 + x) + \varphi \cdot (1 - 4x)}{(1 - 4x) \cdot (1 + x)} \\ &= \frac{\xi + \xi \cdot x + \varphi - 4\varphi \cdot x}{(1 - 4x) \cdot (1 + x)} \\ \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} &= \frac{\xi + \varphi + x \cdot (\xi - 4\varphi)}{(1 - 4x) \cdot (1 + x)} \end{aligned}$$

We can solve the system of linear equations:

$$\begin{aligned} \xi + \varphi &= 1 \\ \xi - 4\varphi &= -2 \\ 5\varphi &= 3 \\ \varphi &= \frac{3}{5} \\ \xi &= \frac{2}{5} \end{aligned}$$

So the equation becomes:

$$\begin{aligned} A(x) &= \frac{\frac{2}{5}}{1 - 4x} + \frac{\frac{3}{5}}{1 + x} \\ &= \frac{2}{5} \cdot \frac{1}{1 - 4x} + \frac{3}{5} \cdot \frac{1}{1 + x} \\ \frac{1}{1 - 4x} &\leftrightarrow \langle 4^0, 4^1, 4^2, \dots, 4^n, \dots \rangle \\ \frac{2}{5} \cdot \frac{1}{1 - 4x} &\leftrightarrow \langle \frac{2}{5} \cdot 4^0, \frac{2}{5} \cdot 4^1, \frac{2}{5} \cdot 4^2, \dots, \frac{2}{5} \cdot 4^n, \dots \rangle \\ &\leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \dots, \frac{2^{2n+1}}{5}, \dots \right\rangle \end{aligned}$$

$$\frac{1}{1+x} \leftrightarrow \langle 1, -1, 1, \dots, (-1)^n, \dots \rangle$$

$$\frac{3}{5} \cdot \frac{1}{1+x} \leftrightarrow \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \dots, \frac{3}{5} \cdot (-1)^n, \dots \right\rangle$$

$$A(x) \leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \dots, \frac{2^{2n+1}}{5}, \dots \right\rangle + \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \dots, \frac{3}{5} \cdot (-1)^n, \dots \right\rangle$$

$$A(x) \leftrightarrow \left\langle 1, 1, 7, \dots, \frac{2^{2n+1} + 3(-1)^n}{5}, \dots \right\rangle$$

Therefore,  $a_n = \frac{2^{2n+1} + 3(-1)^n}{5}$ .

## Answer 2

a)

Let  $\langle 2, 5, 11, 29, 83, 245, \dots \rangle$  be the generating function  $F(x)$ .

$$\begin{aligned} F(x) &\leftrightarrow \langle 2, 5, 11, 29, 83, 245, \dots \rangle \\ &\leftrightarrow \langle 0 + 2, 3 + 2, 9 + 2, 27 + 2, 81 + 2, 243 + 2, \dots \rangle \\ &\leftrightarrow \langle 0, 3, 9, 27, 81, 243, \dots \rangle + \langle 2, 2, 2, 2, 2, 2, \dots \rangle \\ &\leftrightarrow \langle 0, 3, 9, 27, 81, 243, \dots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \dots \rangle \\ &\leftrightarrow \langle 1 - 1, 3, 9, 27, 81, 243, \dots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \dots \rangle \\ &\leftrightarrow \langle 1, 3, 9, 27, 81, 243, \dots \rangle - \langle 1, 0, 0, 0, 0, 0, \dots \rangle + 2 \cdot \langle 1, 1, 1, 1, 1, 1, \dots \rangle \end{aligned}$$

$$\begin{aligned} F(x) &= \frac{1}{1-3x} - 1 + 2 \cdot \frac{1}{1-x} \\ &= \frac{1-1+3x}{1-3x} + \frac{2}{1-x} \\ &= \frac{3x}{1-3x} + \frac{2}{1-x} \\ &= \frac{3x \cdot (1-x) + 2 \cdot (1-3x)}{(1-3x) \cdot (1-x)} \\ &= \frac{3x - 3x^2 + 2 - 6x}{1 - 4x + 3x^2} \\ F(x) &= \frac{2 - 3x - 3x^2}{1 - 4x + 3x^2} \end{aligned}$$

b)

$$\begin{aligned} G(x) &= \frac{7-9x}{1-3x+2x^2} \\ &= \frac{7-9x}{(1-2x)(1-x)} \end{aligned}$$

$$\begin{aligned} \frac{7-9x}{(1-2x)(1-x)} &= \frac{\xi}{1-2x} + \frac{\varphi}{1-x} \\ &= \frac{\xi \cdot (1-x) + \varphi \cdot (1-2x)}{(1-2x)(1-x)} \\ &= \frac{\xi - \xi \cdot x + \varphi - 2\varphi \cdot x}{(1-2x)(1-x)} \\ &= \frac{\xi + \varphi - \xi \cdot x - 2\varphi \cdot x}{(1-2x)(1-x)} \\ \frac{7-9x}{(1-2x)(1-x)} &= \frac{\xi + \varphi + x \cdot (-\xi - 2\varphi)}{(1-2x)(1-x)} \end{aligned}$$

$$\xi + \varphi = 7$$

$$-\xi - 2\varphi = -9$$

$$-\varphi = -2$$

$$\varphi = 2$$

$$\xi = 5$$

$$\begin{aligned} G(x) &= \frac{5}{1-2x} + \frac{2}{1-x} \\ &= 5 \cdot \frac{1}{1-2x} + 2 \cdot \frac{1}{1-x} \end{aligned}$$

$$\frac{1}{1-2x} \leftrightarrow \langle 1, 2, 4, 8, \dots, 2^n, \dots \rangle$$

$$5 \cdot \frac{1}{1-2x} \leftrightarrow \langle 5, 10, 20, 40, \dots, 5 \cdot 2^n, \dots \rangle$$

$$\frac{1}{1-x} \leftrightarrow \langle 1, 1, 1, 1, \dots, 1, \dots \rangle$$

$$2 \cdot \frac{1}{1-x} \leftrightarrow \langle 2, 2, 2, 2, \dots, 2, \dots \rangle$$

$$5 \cdot \frac{1}{1-2x} + 2 \cdot \frac{1}{1-x} \leftrightarrow \langle 5, 10, 20, 40, \dots, 5 \cdot 2^n, \dots \rangle + \langle 2, 2, 2, 2, \dots, 2, \dots \rangle$$

$$G(x) \leftrightarrow \langle 7, 12, 22, 42, \dots, 5 \cdot 2^n + 2, \dots \rangle$$

Therefore,  $g_n = 5 \cdot 2^n + 2$ .

## Answer 3

a)

$R$  can be an equivalence relation if it is reflexive, symmetric and transitive.  $R$  is not reflexive, it is symmetric and it is not transitive.  $R$  is not reflexive since  $1 \in \mathbb{Z}$ , but  $1R1$  is not true since there does not exist a right triangle with two edges of size one and the other edge being an integer. In a right triangle, hypotenuse is always the longest side of the triangle. If the hypotenuse is 1, by Pythagorean theorem  $x^2 + 1^2 = 1^2 \rightarrow x^2 = 0 \rightarrow x = 0$ , therefore one of the sides is zero and the shape cannot be a triangle. If the hypotenuse is not 1, by Pythagorean theorem  $1^2 + 1^2 = x^2 \rightarrow 2 = x^2$  and by quadratic formula  $x = \sqrt{2}$  since sides are always positive. Therefore,  $x$  is not an integer. The relation  $1R1$  is not satisfied either way. Since  $1R1$  is not satisfied,  $R$  is not reflexive, and since  $R$  is not reflexive, it cannot be an equivalence relation.

b)

## Answer 4