Student Information

Full Name : Murat Bolu
Id Number : 2521300

Answer 1

Let $\langle a_0, a_1, a_2, a_3, \cdots, a_n, \cdots \rangle$ be the generating function $A(x)$. Then, $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} (3a_{n-1} + 4a_{n-2}) \cdot x^n \\
&= \sum_{n=2}^{\infty} 3a_{n-1}x^n + 4a_{n-2}x^n \\
&= \sum_{n=2}^{\infty} 3a_{n-1}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n \\
&= 3\sum_{n=2}^{\infty} a_{n-1}x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n \\
&= 3x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} \\
&= 3x\sum_{n=1}^{\infty} a_n x^n + 4x^2\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=2}^{\infty} a_n x^n &= 3x\sum_{n=1}^{\infty} a_n x^n + 4x^2\sum_{n=0}^{\infty} a_n x^n \\
A(x) - a_1 \cdot x - a_0 &= 3x \cdot (A(x) - a_0) + 4x^2 \cdot A(x) \\
A(x) - a_1 \cdot x - a_0 &= 3x \cdot A(x) - 3x \cdot a_0 + 4x^2 \cdot A(x) \\
A(x) - 3x \cdot A(x) - 4x^2 \cdot A(x) &= a_1 \cdot x + a_0 - 3x \cdot a_0 \\
A(x) \cdot (1 - 3x - 4x^2) &= a_1 \cdot x + a_0 \cdot (1 - 3x) \\
A(x) &= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{1 - 3x - 4x^2} \\
&= \frac{a_1 \cdot x + a_0 \cdot (1 - 3x)}{(1 - 4x) \cdot (1 + x)}
\end{aligned}
$$

Plug in values for a_0 and a_1:

$$\begin{aligned} A(x) &= \frac{x + 1 \cdot (1 - 3x)}{(1 - 4x) \cdot (1 + x)} \\ &= \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} \end{aligned}$$

Partial fractions:

$$\begin{aligned} \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} &= \frac{\xi}{1 - 4x} + \frac{\varphi}{1 + x} \\ &= \frac{\xi \cdot (1 + x) + \varphi \cdot (1 - 4x)}{(1 - 4x) \cdot (1 + x)} \\ &= \frac{\xi + \xi \cdot x + \varphi - 4\varphi \cdot x}{(1 - 4x) \cdot (1 + x)} \\ \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)} &= \frac{\xi + \varphi + x \cdot (\xi - 4\varphi)}{(1 - 4x) \cdot (1 + x)} \end{aligned}$$

We can solve the system of linear equations:

$$\begin{aligned} \xi + \varphi &= 1 \\ \xi - 4\varphi &= -2 \\ 5\varphi &= 3 \\ \varphi &= \tfrac{3}{5} \\ \xi &= \tfrac{2}{5} \end{aligned}$$

So the equation becomes:

$$\begin{aligned} A(x) &= \frac{\frac{2}{5}}{1 - 4x} + \frac{\frac{3}{5}}{1 + x} \\ &= \frac{2}{5} \cdot \frac{1}{1 - 4x} + \frac{3}{5} \cdot \frac{1}{1 + x} \end{aligned}$$

$$\begin{aligned} \frac{1}{1 - 4x} &\leftrightarrow \left\langle 4^0, 4^1, 4^2, \cdots, 4^n, \cdots \right\rangle \\ \frac{2}{5} \cdot \frac{1}{1 - 4x} &\leftrightarrow \left\langle \tfrac{2}{5} \cdot 4^0, \tfrac{2}{5} \cdot 4^1, \tfrac{2}{5} \cdot 4^2, \cdots, \tfrac{2}{5} \cdot 4^n, \cdots \right\rangle \\ &\leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \cdots, \frac{2^{2n+1}}{5}, \cdots \right\rangle \end{aligned}$$

$$\frac{1}{1+x} \leftrightarrow \langle 1, -1, 1, \cdots, (-1)^n, \cdots \rangle$$
$$\frac{3}{5} \cdot \frac{1}{1+x} \leftrightarrow \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \cdots, \frac{3}{5} \cdot (-1)^n, \cdots \right\rangle$$

$$A(x) \leftrightarrow \left\langle \frac{2}{5}, \frac{8}{5}, \frac{32}{5}, \cdots, \frac{2^{2n+1}}{5}, \cdots \right\rangle + \left\langle \frac{3}{5}, -\frac{3}{5}, \frac{3}{5}, \cdots, \frac{3}{5} \cdot (-1)^n, \cdots \right\rangle$$
$$A(x) \leftrightarrow \left\langle 1, 1, 7, \cdots, \frac{2^{2n+1} + 3(-1)^n}{5}, \cdots \right\rangle$$

Therefore, $a_n = \dfrac{2^{2n+1} + 3(-1)^n}{5}$.

Answer 2

a)

Let $\langle 2, 5, 11, 29, 83, 245, \cdots \rangle$ be the generating function $F(x)$.

$$F(x) \leftrightarrow \langle 2, 5, 11, 29, 83, 245, \cdots \rangle$$
$$F(x) - 2 \leftrightarrow \langle 0, 3, 9, 27, 81, 243, \cdots \rangle$$

b)

Answer 3

a)

b)

Answer 4