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Answer 1

A function $f : A \to B$ is surjective if and only if $\forall b \in B,\ \exists a \in A$ such that $f(a) = b$.
A function $f : A \to B$ is injective if and only if $\forall a_1 \in A,\ \forall a_2 \in A,\ f(a_1) = f(a_2) \to a_1 = a_2$.
The set of nonnegative real numbers $\overline{\mathbb{R}}^+ = \{0\} \cup \mathbb{R}^+$.

a)

- f_1 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_1(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.
- f_1 is not injective since $f_1(2) = f_1(-2) = 4$, and $2 \neq -2$.

b)

- f_2 is not surjective since $-1 \in \mathbb{R}$, and there does not exists an a such that $f_2(a) = -1$, since $\forall x \in \mathbb{R},\ x^2 \geq 0 > -1$.

- f_2 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_2(a_1) = f_2(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. I will prove this by contradiction. Suppose that a_1 is not zero but a positive real number. This would mean that $a_1 + a_2$ is strictly greater than a_2: $a_1 + a_2 > a_2$. That would mean zero is strictly greater than a_2: $0 > a_2$. That is not possible since $a_2 \in \{0\} \cup \mathbb{R}^+$ and every value of a_2 is equal to or greater than zero. This leads to a contradiction, therefore a_1 must be zero since $a_1 \in \{0\} \cup \mathbb{R}^+$. Since $a_1 = 0$, $a_1 + a_2 = 0 \to 0 + a_2 = 0 \to a_2 = 0$. Therefore, if $a_1 + a_2 = 0$, $a_1 = a_2$. We have already shown that if $a_1 - a_2 = 0$, $a_1 = a_2$. Therefore, $\forall a_1 \in \overline{\mathbb{R}}^+,\ \forall a_2 \in \overline{\mathbb{R}}^+,\ f_2(a_1) = f_2(a_2) \to a_1 = a_2$.

c)

- f_3 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \mathbb{R}$ such that $a = \sqrt{b}$ and $f_3(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \mathbb{R}$ such that $f_3(a) = b$.

- f_3 is not injective since $f_3(2) = f_3(-2) = 4$, and $2 \neq -2$.

d)

- f_4 is surjective since $\forall b \in \overline{\mathbb{R}}^+$, $\exists a \in \overline{\mathbb{R}}^+$ such that $a = \sqrt{b}$ and $f_4(\sqrt{b}) = (\sqrt{b})^2 = |b| = b$ since $b \geq 0$ since $b \in \{0\} \cup \mathbb{R}^+$. Therefore, $\forall b \in \overline{\mathbb{R}}^+$, $\exists a = \sqrt{b} \in \overline{\mathbb{R}}^+$ such that $f_4(a) = b$.

- f_4 is injective since $\forall a_1 \in \overline{\mathbb{R}}^+$, $\forall a_2 \in \overline{\mathbb{R}}^+$:

$$\begin{aligned} f_4(a_1) = f_4(a_2) &\to {a_1}^2 = {a_2}^2 \\ &\to {a_1}^2 - {a_2}^2 = 0 \\ &\to (a_1 - a_2)(a_1 + a_2) = 0 \end{aligned}$$

  This means that either $a_1 - a_2 = 0 \to a_1 = a_2$ or $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, $a_1 = a_2 = 0$ since $a_1 \in \{0\} \cup \mathbb{R}^+$ and $a_2 \in \{0\} \cup \mathbb{R}^+$. The proof is similar to above.

Answer 2

a) Since $f : A \subset \mathbb{Z} \subset \mathbb{R} \to \mathbb{R}$, we can take n and m to be 1 and apply the definition of continuity.

$$\forall \varepsilon \in \mathbb{R}^+ \ \exists \delta \in \mathbb{R}^+ \ \forall x \in A \ (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

The Euclidean norm in $\mathbb{R}^n = \mathbb{R}$ simply becomes the absolute value of the number itself.

$$\forall \varepsilon \in \mathbb{R}^+ \ \exists \delta \in \mathbb{R}^+ \ \forall x \in A \ (|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon)$$

I claim that for all ε, taking $\delta = \frac{1}{2}$ works.

$$\forall \varepsilon \in \mathbb{R}^+ \ \forall x \in A \ (|x - x_0| < \tfrac{1}{2} \to |f(x) - f(x_0)| < \varepsilon)$$

Either $x = x_0$ or $x \neq x_0$. In the first case $x = x_0 \to f(x) = f(x_0) \to f(x) - f(x_0) = 0$ which would make the right hand side of the implication true since $|f(x) - f(x_0)| = |0| = 0 < \varepsilon$, $\forall \varepsilon \in \mathbb{R}^+$ and therefore the result true. Taking any δ would work, such as $\delta = \frac{1}{2}$. In the second case the minimum of $|x - x_0|$ is 1 since $x, x_0 \in \mathbb{Z}$. In that case taking $\delta = \frac{1}{2}$ would work since the left hand side of the equation will always be false, and the result will always be true.

b)

Answer 3

a)

b)

Answer 4

a)

b)

c)

d)

e)

f)

Answer 5

a)

b)