Assignment 2
IB3702 Mathematics for Machine Learning
2025/2026

Deadline: November 7, 2025

The purpose of this assignment is to assess your understanding and ability to apply advanced linear algebra. It is important to thoroughly review the material and ensure that you have a firm grasp of the concepts in learning units 7 to 9. By working through the problems provided, you can evaluate your comprehension and proficiency in utilising geometric, theoretical, and abstract linear algebra in different scenarios.

Study advice: First and foremost, it is essential to recognise that you possess the capability to solve all the problems presented in this assignment. However, achieving this requires effective studying techniques. Most importantly, it is crucial to be honest with yourself and acknowledge any areas that may not be entirely clear to you. Practice is key, so make sure to engage with the provided exercises from the Savov book, study guide, and lectures. Whenever encountering new concepts or struggling to grasp them, take the initiative to explore and delve deeper into those topics. The study guide offers valuable tips on effective approaches for this purpose (see Introduction/Section 3.1). Additionally, it is highly recommended to engage in discussions with others to gain further insights and perspectives. As you work on the assignment, it will provide you with a comprehensive overview of your knowledge and skills. If you encounter difficulties with any part of the assignment, it is advisable to revisit the corresponding study material for further clarification.

Requirements: You have to solve each problem thoroughly. Clearly articulate your thought process and explain all steps involved, incorporating logical reasoning or references to course materials and external sources whenever necessary. Dedicate ample effort and perseverance to your work, as it is during this process that the most substantial learning takes place. Please bear in mind that approval of both assignments is a requirement for passing the entire course. Lastly, use the provided template when composing your assignment. If you opt to include digital photos or scans of your hand-written work, ensure that they are clear and legible in your document.

Important: While collaboration is encouraged, it is important to note that you must write the assignment independently. This means that you should provide a personal account of how you perceive each problem, outline the steps you took to reach a solution, and discuss any challenges you encountered along the way. By including these aspects in your assignment, you enhance its depth and richness, as it showcases your individual thought process and problem-solving approach.

Problem 1

Given two matrices A and Q:

$$A = \begin{pmatrix} -2 & -0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \qquad Q = \begin{pmatrix} -a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}$$

Advice: We strongly encourage you to maximize the use of pen and paper for computations in this problem. Engaging in manual calculations allows for greater observation and learning opportunities. Save the computer usage primarily for verifying your results rather than relying on it for the entire process.
Q is said to orthogonally diagonalize A.

(a) The eigenvalues of A are 25, -3, and -50. If the eigendecomposition of A is $Q\Lambda Q^{-1}$, where Λ is a diagonal matrix containing the eigenvalues of A, find the values of a, and b.

(b) Using the definition in the Savov book, prove that Q is an orthogonal matrix. (The inverse of an orthogonal matrix is equal to its transpose).

(c) Show that the column vectors of Q are pairwise orthogonal.

(d) Using the eigendecomposition of A, find $A^{1/3}$. Describe also how you would use this method for finding A^k where k is greater than 5?

(e) Consider eigen value equation, $Ax = \lambda x$, how would you relate the null space of a square matrix and its eigenvalues? Then how would you show that a square matrix does not have an inverse if at least one of its eigenvalues have zero value?

Problem 2

In this problem, we explore how polynomials can be seen as vectors and what we can do with them.

(a) Prove that $\{-1+x, x+x^2, 1+x+x^2\}$ is a basis for the vector space of polynomials of degree at most two over the real numbers. For clarity, this vector space contains all polynomials of the form $a_0+a_1x+a_2x^2$, where $a_0, a_1, a_2 \in \mathbb{R}$.

Hint: Remember that to prove that a set of vectors form a basis, you need to show that the vectors are linearly independent and they span the entire vector space.

(b) On page 327 in the Savov book, the following inner product is defined for real-valued functions f and g over the interval $[-1, 1]$:

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

Using the definition at the beginning of Section 6.4, show that this is indeed an inner product.

Hint: As suggested above, the definition can be found in the introduction of Section 6.4 on pages 322–323. Read this carefully! You have to show three properties. Furthermore, it is a good idea to first understand the three properties of an inner product by familiarising yourself with the regular dot product.

(c) Continuing with the inner product from (b), note that to prove that the four polynomials given on page 327 are *mutually* orthogonal, you would need to perform six computations. From these six, you have to perform here two:

1. Show that the polynomials $P_0(x) = 1, P_1(x) = x$ are orthogonal.
2. Show that the polynomials $P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$ are orthogonal.

(d) Still continuing the polynomials from (c), we consider the polynomial: $\frac{x}{2} + 3x^2 + \frac{5}{2}x^3$.

Find the linear combination of $\{P_0, P_1, P_2, P_3\}$ for the polynomial $\frac{x}{2}+3x^2+\frac{5}{2}x^3$. Verify the results.

(e) In terms of the inner product from (b), compute the norm of the polynomial $x^2 + 3x^3$.

Problem 3

In this problem, we consider linear transformations in $\mathbb{R}^2$.

(a) We consider a two-step transformation; first reflection with respect to y = x line and second reflection across x axis. How would you find one transformation that has the same final results with the above two transformations, conducted in the respective order and what is that transformation?

(b) Now we consider the linear transformation matrix that rotates a given vector 75 degrees counterclockwise and denote that matrix as B.

- Find the matrix B as described above "without explicitly using the values of $cos(75)$ and $sin(75)$".
- What is the kernel and what is the image of B?
- What is B^{12}?
- What is the matrix of B^{-1}?

Hint: This problem does not contain much computation. It is more about knowing and understanding the concepts!

Problem 4

Construct a 3×3 matrix A such that $\det(A) = 5$. Then, consider the *Invertible matrix theorem* on page 288 in the Savov book.

(a) Describe how you constructed matrix A.

(b) Show the validity of each of the 10 equivalent statements by instantiating each of them using your matrix A. It is sufficient to show the crucial steps (ideas and relevant computations); you don't need to write down all details of the computation here.

Hint: You can choose the order for showing the validity of these statements. This can shorten your work, as you can refer to steps that you have already taken. For instance, the REF or the RREF that you have already computed can help at another statement.

Problem 5

Which problem or sub-problem from the two assignments in this course did you find the most interesting and why? (This is not an essay question; it is enough to write one or two sentences. Nevertheless, it is important that you explain your choice – we are very curious.)