A stochastic optimization formulation for demand uncertainty

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February 15, 2024

1 Problem formulation

The demand uncertainty is represented by a random variable D with a known(estimated) probability distribution P, and the optimal production quantity x is chosen to maximize the expected profit

$$\max_x \mathbb{E}[\pi(x, D)] \tag{1}$$

Let $i \in P$ where P is the set of products with $|P| = 500$. Let x_i and $demand_i$ be the product i to be produced and a variable in the optimization for the demand respectively (Here $demand_i$ is not a random variable or its value, it is a decision variable in the optimization, its value would be known if the demands were known.) Since there is a loss of profit if $x_i < demand_i$, the profit in this case would be $m_i demand_i - (x_i - demand_i)cogs_i$. If $x_i \geq demand_i$, then the profit is $m_i x_i$ (m_i is the margin for product i). Next introduce an auxiliary variable q_i for product i so that $q_i \leq m_i x_i$ and $q_i \leq m_i demand_i - (x_i - demand_i)cogs_i$, while we maximize $\sum_{i \in P} q_i$. At optimality q_i would be $\max\{m_i x_i, (m_i demand_i - (x_i - demand_i)cogs_i)\}$.

Now consider the decision variable $demand_i$. As stated above, if the actual demands would be known, $demand_i$ would be those values. Hence it is important to impose constraints on $demand_i$ that are reflected in the distribution of demand D_i and its values d_i's. One way to impose such constraints would be of the form $U = \{demand_i \in \mathbb{R}^n_+ | l_i \leq demand_i \leq u_i\}$ for some upper and lower bounds u_i, l_i. If such static bounds would always be valid bounds for the demand distributions for each product i, then they would be overly-conservative since it is not likely for multiple uncertain parameters to take on their extreme values simultaneously. To alleviate that [1] proposes ellipsoid uncertainty set, which lead to a convex quadratic constraint optimization. [2, 3] proposes a linear constraint of the form $U = \{demand_i \in \mathbb{R}^n_+ | \sum_i \frac{|demand_i - hist\_demand_i|}{hist\_demand_i} \leq \tau\}$. In the current solution, we adopt an analogous constraint.

Now introduce some slack variables $0 \leq s_i \leq 1$ and let

$$demand_i = s_i hist\_demand_i(F_i(1 - \epsilon)) \tag{2}$$

Here $(F_i(1-\epsilon)$ is the $(1-\epsilon)th$ percentile of the demand distribution i, i.e., $P(demand_i \leq F_i) \leq 1 - epsilon$. In this model we take $\epsilon := 1e-6$. We constraint the slack variables as $\sum_i s_i \leq 500\tau$ and allow $0 \leq \tau \leq 1$ be a an adjustable parameter determining the conservatism in the model.

We also impose the business constraints of the form $0 \leq x_i \leq u_i$ and $\sum_i x_i \leq (1+p)\sum_i(hist\_demand_i)$ where $0.1 \leq p \leq 0.5$. u_i's are determined by Capacity for each product in the Excel sheet.

1.1 Group constraints

To be able to exploit substitution of product j by i within a group, sufficient total product amounts should be maintained. There are different ways to handle that. In this solution, we use

$$\sum_{i \in G_i} x_i \geq group\_lower\_bound\_percentage \sum_{i \in G_i} hist\_demand_i \tag{3}$$

Here $group\_lower\_bound\_percentage$ is an adjustable parameter and G_i is the substitutability group for product i. Another way to handle this would be to use a slack variable $0 \leq t_i$ with

$$\sum_{i \in G_i} x_i - t_i \geq \sum_{i \in G_i} hist\_demand_i \tag{4}$$

and penalize t_i in the objective with an appropriate coefficient.
Finally the objective is $max \sum_i q_i$. All the constraints in the (1st stage optimization) model are stated above.

1.2 Substitutions within a group and 2nd stage optimization

When we optimize the 1st stage problem above, x_i's are determined. Once the actual demands become known, in order to exploit the substitutability, one needs to determine substitution of product j by i. This is done with a 2nd stage optimization problem. Let $y_{i,j}$ be the amount of substitution of product j by i. Let $m_{i,j}$ be $m_j + cogs_j - cogs_i$. Note that once all demands d_i's become known, one must have

$$\sum_{j \in G_i} y_{i,j} \leq x_i \tag{5}$$

for all $i \in G_i$ and

$$\sum_{i \in G_i} y_{i,j} \leq d_j \tag{6}$$

for all $j \in G_i$. ($G_i = G_j$ whenever i and j are in the same group). The objective to maximize is $\sum_{i,j} m_{i,j} y_{i,j}$.

This is an (unbalanced) transportation problem that can be solved for each group separately. Since the size of the problems for each group is small, we use the same linear optimization solver.

1.3 Optimizing

We used the linear optimization solver from Ortools/Google. The solver can handle both the 1st and 2nd stage problems easily.

1.4 Solution evaluation and details

To evaluate the quality of the solutions produced, we simulate demands from the estimated demand distrubutions burr12. Note that once *total_constraint_percentage* is given, there are two parameters determining the quality of the solutions: τ and *group_lower_bound_percentage*. All these parameters can be controlled from the JSON file in the repo. To maintain an efficient search space leading to higher optimal profit, the last two parameters should be tuned. The general flow is as follows:

1. Fix the parameters above and get optimal x_i's from the 1st stage optimization.
2. Simulate demands d_i for each product k times ($k = 60$ by default within the code).
3. For each simulation with known x_i's and d_i's, solve the 2nd stage optimization. 2nd stage optimization involves optimizing the substitutions within each group (done in a loop).
4. Create a histogram of the optimal objective from the 2nd stage.

The given JSON file has 3 parameters for illustration and the following figures are the corresponding profit distributions. The red vertical lines from left to right are 25th, 50th and 75th percentiles of the profits. Note that the accepted solution will be determined by one's risk tolerance and/or conservatism.

1.4.1 Run-time

Once the 1st stage optimization problem is solved (in seconds), the demands are simulated k times, which takes a couple of minutes. This is repeated for each value of the parameters, so the run-times are reasonable given that this is a long-termm production planning problem.

1.4.2 Discussion of tuned parameters

Simulation shows the last figure has the best profit is achieved when a higher *group_lower_bound_percentage* is used. This suggests that substitution within a group indeed yield better profits. Note that

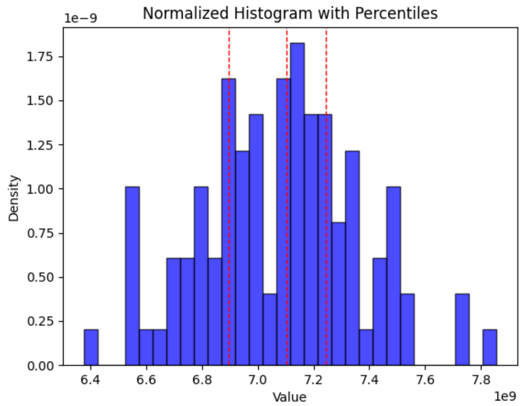

Figure 1: $total\_constraint\_percentage = 0.2,\ \tau = 0.4,\ group\_lower\_bound\_percentage = 0.7$

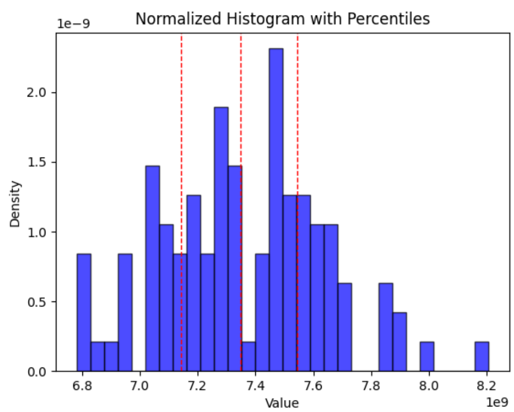

Figure 2: $total\_constraint\_percentage = 0.2,\ \tau = 0.3,\ group\_lower\_bound\_percentage = 0.8$

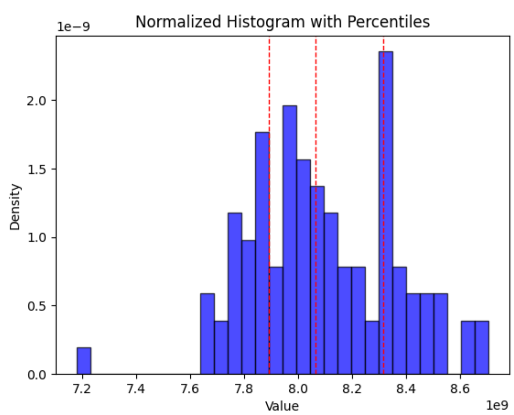

Figure 3: $total\_constraint\_percentage = 0.2,\ \tau = 0.15,\ group\_lower\_bound\_percentage = 1.0$

a larger τ would be a more relaxed 1st stage optimization problem leading to higher production x_i variables. But the cost incurred when x_i is large leads to potential losses. So empirically the best value for τ seems to be within the interval $[0.1, 0.2]$

References

[1] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of uncertain linear programs. *Operations research letters*, 25(1):1–13, 1999.

[2] Dimitris Bertsimas and Melvyn Sim. The price of robustness. *Operations research*, 52(1):35–53, 2004.

[3] Mengshi Lu and Zuo-Jun Max Shen. A review of robust operations management under model uncertainty. *Production and Operations Management*, 30(6):1927–1943, 2021.