

We now turn to a particularly interesting type of systems of differential equations, known formally as a *gradient system*.

Definition. A *gradient system* on \mathbb{R}^n is a system of differential equations of the form

$$X' = -\text{grad}V(X) \quad (1)$$

for $X \in \mathbb{R}^n$, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with

$$\text{grad}V = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right) \quad (2)$$

The minus sign in (1) is merely a convention, and as such we need not worry of its significance. As displayed in (2), the gradient of a functions is simply an ordered list of its partial derivatives. However, its physical significance severely outweighs what it portrays to be. The gradient of a point in a vector field is a measure of the *direction and rate of fastest increase*.

The following example is question 12, chapter 9 from Hirsch, Smale and Devaney's *Differential Equations, Dynamical Systems, and an Introduction to Chaos*.

Example 1.0. Let T be the torus defined as the square $0 \leq \theta_1, \theta_2 \leq 2\pi$ with opposite sides identified. Let $F(\theta_1, \theta_2) = \cos \theta_1 + \cos \theta_2$. Sketch the phase portrait for the system $-\text{grad}F$ in T . Sketch a three-dimensional representation of this phase portrait with T represented as the surface of the doughnut.

Let $X = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$. Then we have the following system of equations;

$$X' = \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = \begin{pmatrix} -\partial F / \partial \theta_1 \\ -\partial F / \partial \theta_2 \end{pmatrix} = \begin{pmatrix} \sin \theta_1 \\ \sin \theta_2 \end{pmatrix} \quad (3)$$

A first order, nonlinear uncoupled system. In order to understand the way this system behaves, we must solve for the equilibrium points of the system (namely, the pairs (θ_1, θ_2) such that $X' = 0$), and then linearize the system to understand the local behaviour near these points.

The reader may notice, however, that there are in fact an infinite number of pairs (θ_1, θ_2) which solve $X' = 0$: Any pair of the form $(n\pi, m\pi)$ for $n, m \in \mathbb{Q}$ will do. To make the task easier for us, we restrict our view to the square $[0, 2\pi]_{\theta_1} \times [0, 2\pi]_{\theta_2}$. In which case, we are left with 9 distinct equilibrium which can be summarized as $\{(n\pi, m\pi) : n, m \in \{0, 1, 2\}\}$. Now that we have our equilibrium points, we want to understand the local behaviour of solutions to (3) in a neighbourhood of the equilibrium points. To do so, we take the Jacobian of X' to get a linear planar system of the form $Y' = DF_X Y$;

$$DF_X = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix} \quad (4)$$

If we consider the equilibrium point $(0, 0)$, we find a linear system given by

$$Y' = DF_{(0,0)} Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y \quad (5)$$

A planar system with repeated, positive eigenvalues. This indicates that $(0,0)$ is a source, and locally, solutions project directly outwards from this point. The identification of the other equilibrium are handled similarly, and are left to the reader. Figure 1 shows the phase portrait of the system.

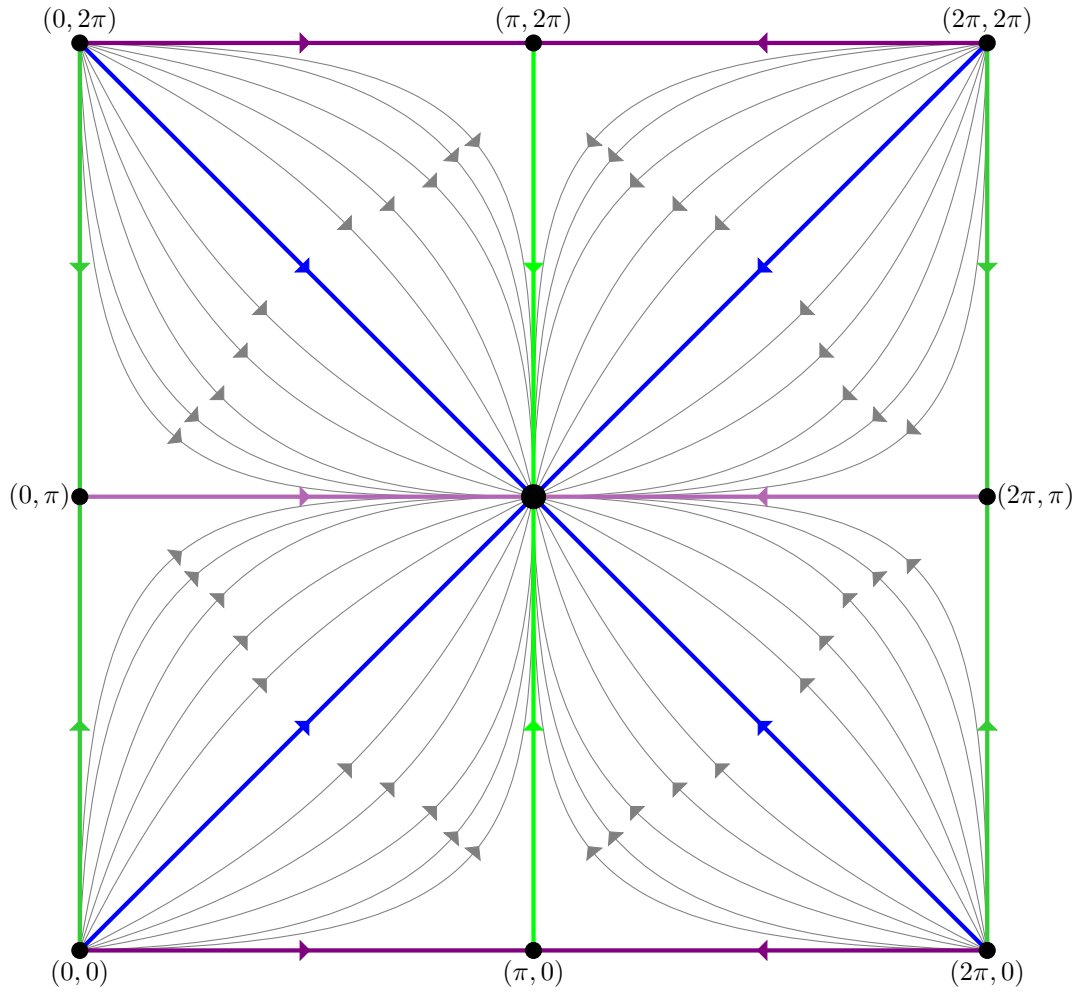


Figure 1: Phase portrait of (3) on $[0, 2\pi] \times [0, 2\pi]$

For a more interactive view of this phase portrait, visit [Field Play](#) . As is shown, at each of the four corners, solutions flow out from a source, each of which falls into the sink located at (π, π) . Moreover, the midpoints of the perimeter each take the form of a saddle, with stable solutions flowing from the four corner, and the unstable lines again flowing directly towards the sink.

Moreover, we can map this phase portrait onto the torus. While we can certainly parameterize the surface of the torus using the map $T : [0, 2\pi]_u \times [0, 2\pi]_v \rightarrow \mathbb{R}^3$ defined by

$$T(u, v) = [(2 + \cos u) \sin u, (2 + \cos uv) \cos u, \sin u] \quad (6)$$

It is, in my opinion, far more motivating (and interesting) to visualize the topological construction of the torus. More explicitly, we will construct the torus from a rectangular subset of \mathbb{R}^2 . See Appendix A??.

The phase portrait mapped onto the torus can be seen in Figure 2, with colour coordinated solutions which align with Figure 1. Only a sample of the lines have been plotted to allow for an easier viewing.

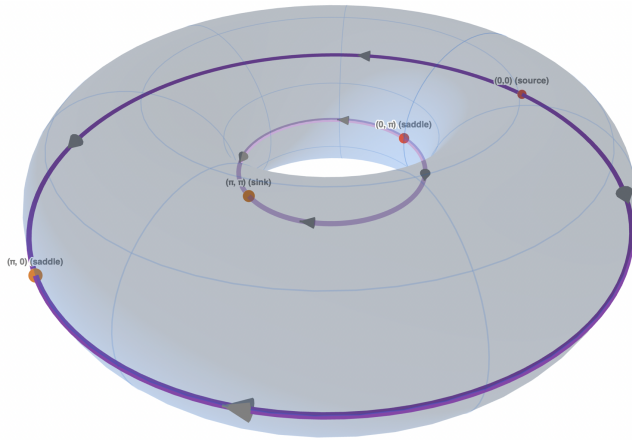
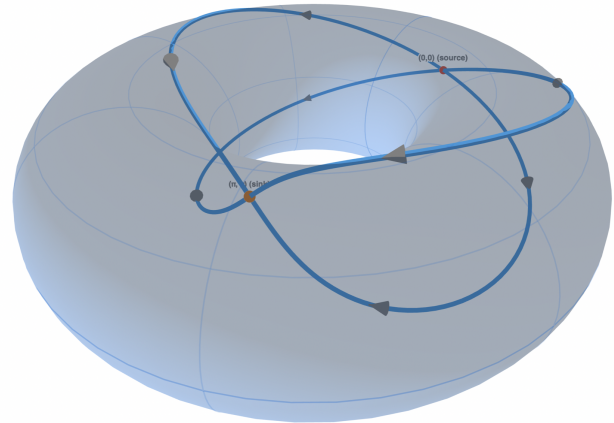



Figure 2: a) Purple Curves



b) Blue Curves

For a more visual view of this torus, visit [Math3D](#) .