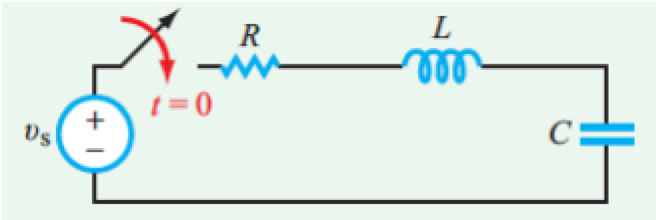
Lecture 6 Second-Order Circuits

> *What if we have both L and C in the circuit*

Series RLC Circuits



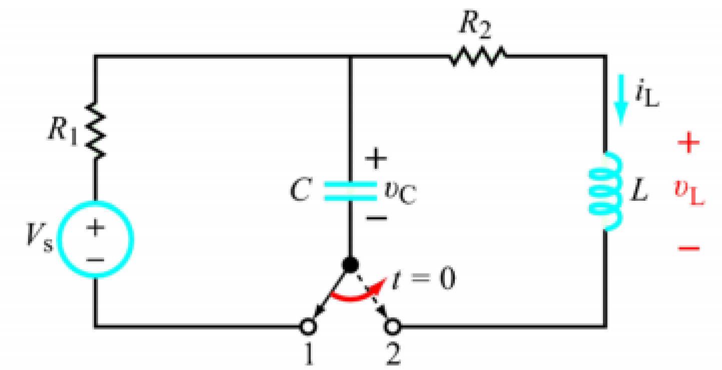
when $i_L(0_-)=0, v_c(0_-)=0$ find $V_c(t)$

$$\begin{cases} i_L(t)=c\frac{dv_c(t)}{dt} \\ V_s-R\cdot i_L(t)-L\frac{di_l(t)}{dt}-v_c(t)=0 \end{cases} \Rightarrow RC\frac{dv_c(t)}{dt}+LC\frac{d^2v_c(t)}{dt^2}+v_c(t)=v_s$$

$$\frac{d^2v_c(t)}{dt^2}+\frac{R}{L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=\frac{v_s}{LC}$$

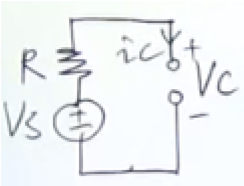
这是一个二阶方程，所以也是二阶电路。second order circuit

Initial and Final condition



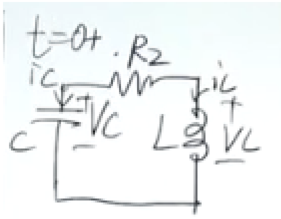
Find $v_c(0_-), i_c(0_-), i_L(0_-), v_L(0_-),$
$v_c(0_+), i_c(0_+), i_L(0_+), v_L(0_+),$
$v_c(\infty), i_c(\infty), i_L(\infty), v_L(\infty)$

1. $t=0_-$ steady state capacitor likes open circuit inductor like short circuit



$v_c(0_-) = v_s$
$i_c(0_-) = 0$
$i_L(0_-) = 0$
$v_L(0_-) = 0$

2. $t = 0_+$ 电容电压不突变，电感电流不突变



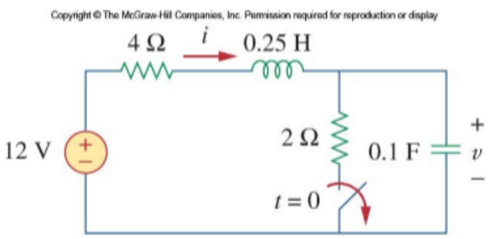
$v_c(0_+) = v_s$
$i_c(0_+) = 0$
$i_L(0_+) = 0$
$v_L(0_+) = v_s$

3. $t = \infty$

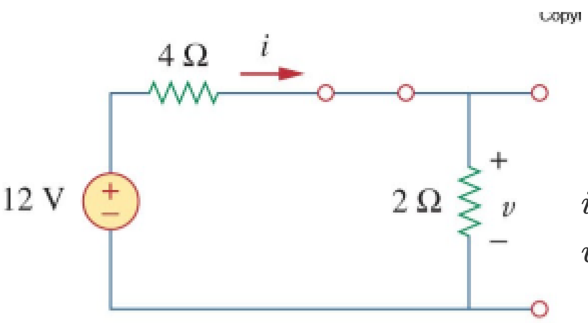
$v_c(\infty) = 0$
$i_c(\infty) = 0$
$i_L(\infty) = 0$
$v_L(\infty) = 0$

Example

- The switch has been closed for a long time. It is open at $t = 0$. Find
  - $i(0^+)$, $v(0^+)$
  - $di(0^+)/dt$, $dv(0^+)/dt$
  - $i(\infty)$, $v(\infty)$



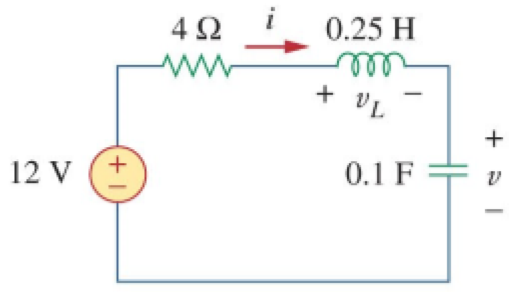
1. $t = 0_-$



(a)

$i(0_-) = 12/6 = 2A$
$v(0_-) = 2 \cdot 2 = 4V$

2. $t = 0_+$

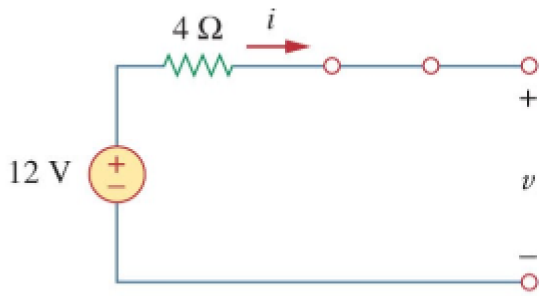


(b)

$$i(0_+) = i(0_-) = 2A$$
$$v(0_+) = v(0_-) = 4V$$

$$\frac{di(0_+)}{dt} = \frac{1}{L}V_L(0_+) = 12 - 2*4 - 4 = 0A/s$$
$$\frac{dv(0_+)}{dt} = \frac{1}{c}i_C(0_+) = 2/0.1 = 20V/s$$

3. $t = \infty$

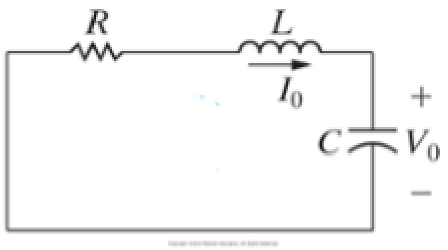


(c)

$$i(\infty) = 0A$$
$$v(\infty) = 12V$$

Natural Response自然响应，零输入响应

series RLC circuit



$V_c(0_-) = V_0, i_L(0_-) = I_0$ Find $v_c(t), (t \geq 0)$

$$\begin{cases} i_L(t) = c\frac{dv_c(t)}{dt} \\ -R \cdot i_L(t) - L\frac{di_l(t)}{dt} - v_c(t) = 0 \end{cases} \Rightarrow RC\frac{dv_c(t)}{dt} + LC\frac{d^2v_c(t)}{dt^2} + v_c(t) = 0$$

$$\frac{d^2v_c(t)}{dt^2} + p\frac{dv_c(t)}{dt} + qv_c(t) = 0, \text{ where } p = \frac{R}{L}, q = \frac{1}{LC}$$

解上述二阶常系数微分方程（2^{nd} order ordinary differential equation/ODE）就可以解除这个电路。

Let $V_C(t) = Ae^{st}$, then $Ae^{st}(s^2 + ps + q) = 0$. Solve equation $s^2 + ps + q = 0$ about s

$$s=\frac{-p\pm\sqrt{p^2-4q}}{2}=\frac{R}{2L}\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}$$

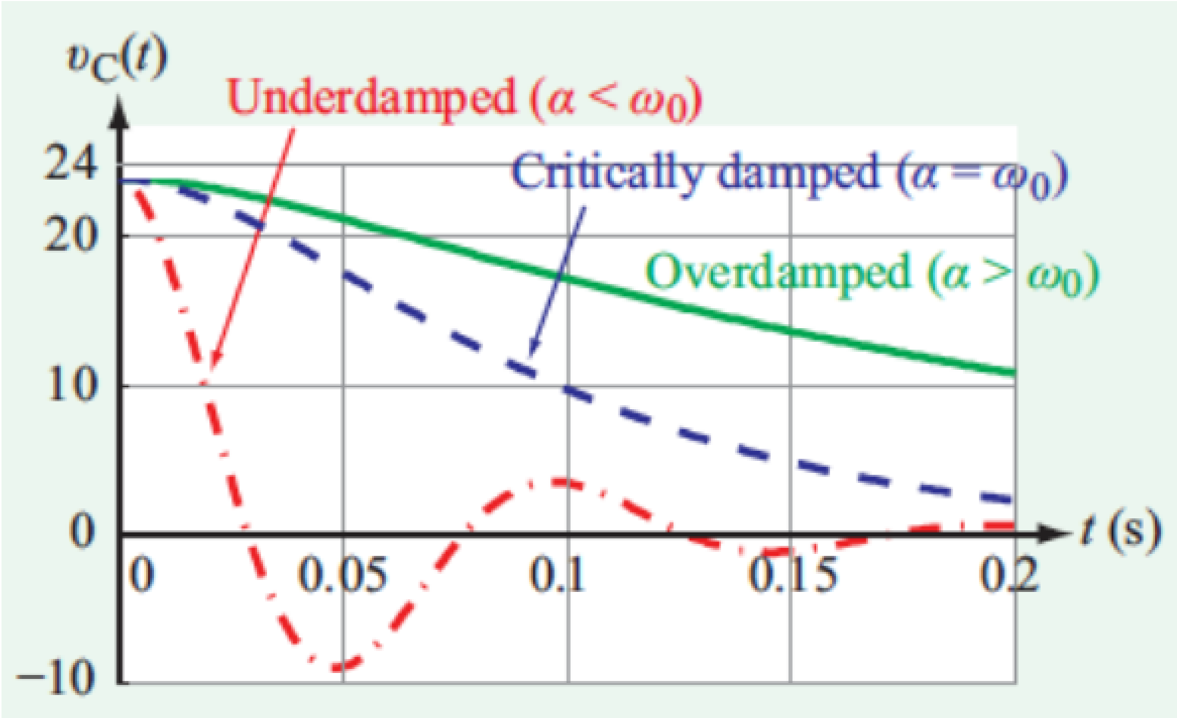
So we have 3 cases here:

1. ***overdamped*** 过阻尼 s have two different real solution
   where $(\frac{R}{2L})^2-\frac{1}{LC}>0$, let $\alpha=\frac{R}{2L},\omega_0=\frac{1}{\sqrt{LC}}$ so the equation about the circuit can be interpret into $s^2+2\alpha s+\omega_0^2=0$, so that $s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$.
   Therefore $V_C(t)=A_1e^{s_1t}+A_2e^{s_2t}$,where A_1,A_2 can be found using two initial conditions.
2. ***critically damped*** 临界阻尼 s have only one real solution
   where $(\frac{R}{2L})^2-\frac{1}{LC}=0$, let $\alpha=\frac{R}{2L},\omega_0=\frac{1}{\sqrt{LC}},\alpha=\omega_0$ $s=s_1=s_2=-\alpha$
   Therefore $V_C(t)=(A_1t+A_2)e^{st}$,where A_1,A_2 can be found using two initial conditions.
3. ***underdamped*** 欠阻尼 s_1,s_2 are complex
   where $(\frac{R}{2L})^2-\frac{1}{LC}<0$, let $\alpha=\frac{R}{2L},\omega_0=\frac{1}{\sqrt{LC}}$ so the equation about the circuit can be interpret into $s^2+2\alpha s+\omega_0^2=0$, so that $s_{1,2}=-\alpha\pm\sqrt{\omega_0^2-\alpha^2}i$, seeing that $V_C(t)\in\mathbb{R}$
   $$c_1e^{s_1t}+c_2e^{s_2t}=c_1e^{-(\alpha+j\omega_di)t}+c_2e^{-(\alpha-j\omega_di)t}$$
   $$(c_1e^{s_1t}+c_2e^{s_2t})^*=c_1^*e^{-(\alpha-j\omega_di)t}+c_2^*e^{-(\alpha+j\omega_di)t}$$
   so that $c_1=c_2^*$, then
   $$v_c(t)=e^{-\alpha t}[A_1cos\omega_dt+A_2sin\omega_dt]$$

Summary *series RLC network*

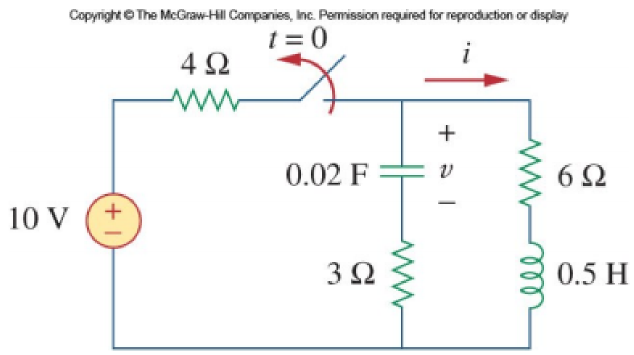
Case		$\alpha=\frac{R}{2L},\omega_0=\sqrt{\frac{1}{LC}}$	
1	overdamped	$\alpha>\omega\Leftrightarrow R>2\sqrt{\frac{L}{C}}$	$V_C(t)=A_1e^{s_1t}+A_2e^{s_2t}$ $s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$
2	critically damped	$\alpha=\omega\Leftrightarrow R=2\sqrt{\frac{L}{C}}$	$V_C(t)=(A_1t+A_2)e^{st}$ $s=-\alpha$
3	underdamped	$\alpha<\omega\Leftrightarrow R<2\sqrt{\frac{L}{C}}$	$V_C(t)=e^{-\alpha t}[A_1cos\omega_dt+A_2sin\omega_dt]$ $\omega_0=\sqrt{\omega_0^2-\alpha^2}$

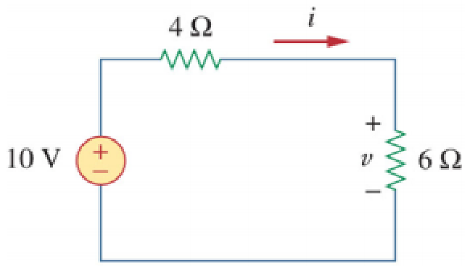
Image



Example

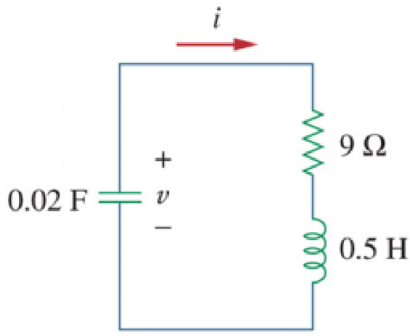
Find $v(t)$ &$i(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.





$$\alpha = \frac{R}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = -9 \mp j4.359$$



$$V(t) = e^{-\alpha t}[A_1 \cos \omega_d t + A_2 \sin \omega_d t]$$
$$= e^{-9t}[A_1 \cos 4.359t + A_2 \sin 4.359t]$$

From $i(0+) = 1A$

$\Rightarrow -C\frac{dV(0+)}{dt} = 1A \Rightarrow \frac{dV(0+)}{dt} = -50 V/s.$

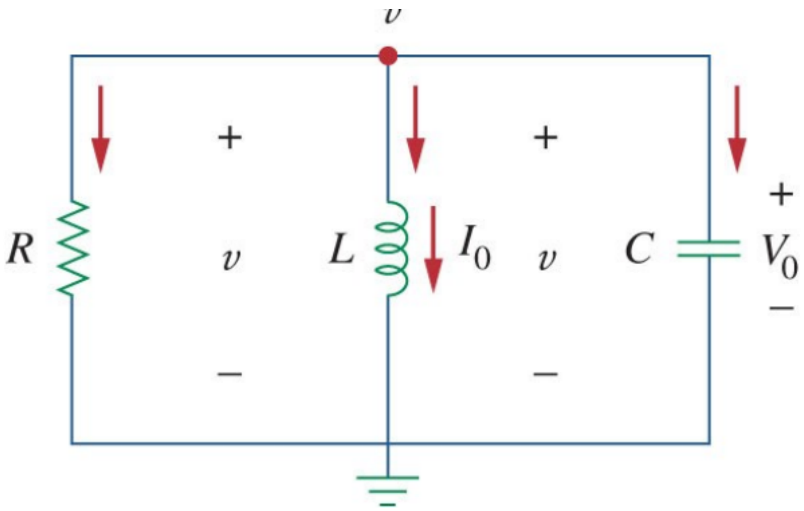
$\Rightarrow \begin{cases} \frac{dV(0+)}{dt} = -9A_1 + 4.359A_2 = -50 \\ V(0+) = A_1 = 6 \end{cases} \Rightarrow \begin{cases} A_1 = 6, \\ A_2 = 0.918. \end{cases}$

$\Rightarrow v(t) = e^{-9t}(6\cos 4.359t + 0.918\sin 4.359t)\ V$

More Notes

Case		$\alpha = \frac{R}{2L}, \omega_0 = \sqrt{\frac{1}{LC}}$	
1	overdamped	$\alpha > \omega \Leftrightarrow R > 2\sqrt{\frac{L}{C}}$	$V_C(t) = A_1e^{s_1t} + A_2e^{s_2t}$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ $\begin{cases} s_1A_1 + s_2A_2 = \frac{i(0_+)}{c} \\ A_1 + A_2 = v(0_+) \end{cases}$
2	critically damped	$\alpha = \omega \Leftrightarrow R = 2\sqrt{\frac{L}{C}}$	$V_C(t) = (A_1t + A_2)e^{st}$ $s = -\alpha$ $\begin{cases} A_1 + sA_2 = \frac{i(0_+)}{c} \\ A_2 = v(0_+) \end{cases}$
3	underdamped	$\alpha < \omega \Leftrightarrow R < 2\sqrt{\frac{L}{C}}$	$V_C(t) = e^{-\alpha t}[A_1cos\omega_dt + A_2sin\omega_dt]$ $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$ $\begin{cases} \omega_0A_2 - \alpha A_1 = \frac{i(0_+)}{c} \\ A_1 = v(0_+) \end{cases}$

parallel RLC circuit



$$\begin{cases} V_c(t) = L\frac{di_L(t)}{dt} \\ \frac{V_c}{R} + i_L + c\frac{dV_c}{dt} = 0 \end{cases} \Rightarrow \frac{L}{R}\frac{di_L(t)}{dt} + LC\frac{d^2 i_L(t)}{dt^2} + i_L(t) = 0$$

$$\Rightarrow \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = 0$$

1. ***overdamped*** 过阻尼 s have two different real solution

   where $(\frac{1}{2RC})^2 - \frac{1}{LC} > 0$, let $\alpha = \frac{1}{2RC}, \omega_0 = \frac{1}{\sqrt{LC}}$ so the equation about the circuit can be interpret into $s^2 + 2\alpha s + \omega_0^2 = 0$, so that $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$.

   Therefore $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$,where A_1, A_2 can be found using two initial conditions.

2. ***critically damped*** 临界阻尼 s have only one real solution

   where $(\frac{1}{2RC})^2 - \frac{1}{LC} = 0$, let $\alpha = \frac{1}{2RC}, \omega_0 = \frac{1}{\sqrt{LC}}, \alpha = \omega_0$ $s = s_1 = s_2 = -\alpha$

   Therefore $i_L(t) = (A_1 t + A_2)e^{st}$,where A_1, A_2 can be found using two initial conditions.

3. ***underdamped*** 欠阻尼 s_1, s_2 are complex

   where $(\frac{1}{2RC})^2 - \frac{1}{LC} < 0$, let $\alpha = \frac{1}{2RC}, \omega_0 = \frac{1}{\sqrt{LC}}$ so the equation about the circuit can be interpret into $s^2 + 2\alpha s + \omega_0^2 = 0$, so that $s_{1,2} = -\alpha \pm \sqrt{\omega_0^2 - \alpha^2 i}$, seeing that $V_C(t) \in \mathbb{R}$

   $$c_1 e^{s_1 t} + c_2 e^{s_2 t} = c_1 e^{-(\alpha + j\omega_d i)t} + c_2 e^{-(\alpha - j\omega_d i)t}$$
   $$(c_1 e^{s_1 t} + c_2 e^{s_2 t})^* = c_1^* e^{-(\alpha - j\omega_d i)t} + c_2^* e^{-(\alpha + j\omega_d i)t}$$

   so that $c_1 = c_2^*$, then

   $$i_L(t) = e^{-\alpha t}[A_1 cos\omega_d t + A_2 sin\omega_d t]$$

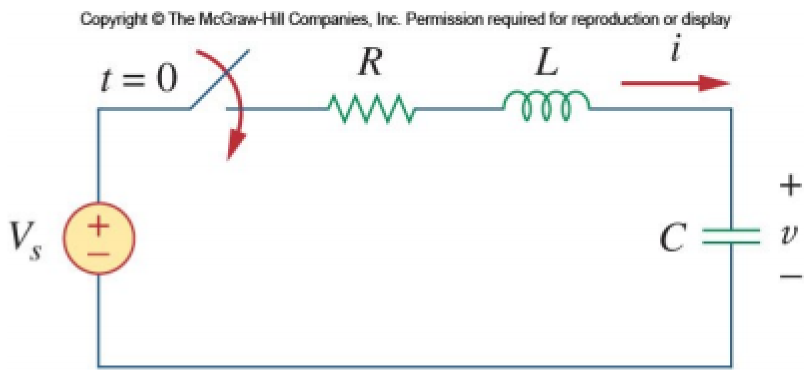
Case

$$\alpha = \frac{1}{2RC}, \omega_0 = \sqrt{\frac{1}{LC}}$$

Case		$\alpha = \frac{1}{2RC}, \omega_0 = \sqrt{\frac{1}{LC}}$	
1	overdamped	$\alpha > \omega \Leftrightarrow \frac{1}{R} > 2\sqrt{\frac{L}{C}}$	$V_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ $\begin{cases} s_1 A_1 + s_2 A_2 = \frac{i(0_+)}{c} \\ A_1 + A_2 = v(0_+) \end{cases}$
2	critically damped	$\alpha = \omega \Leftrightarrow \frac{1}{R} = 2\sqrt{\frac{L}{C}}$	$V_C(t) = (A_1 t + A_2)e^{st}$ $s = -\alpha$ $\begin{cases} A_1 + sA_2 = \frac{i(0_+)}{c} \\ A_2 = v(0_+) \end{cases}$
3	underdamped	$\alpha < \omega \Leftrightarrow \frac{1}{R} < 2\sqrt{\frac{L}{C}}$	$V_C(t) = e^{-\alpha t}[A_1 cos\omega_d t + A_2 sin\omega_d t]$ $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$ $\begin{cases} \omega_0 A_2 - \alpha A_1 = \frac{i(0_+)}{c} \\ A_1 = v(0_+) \end{cases}$

Step response 阶跃响应

series RLC circuit



$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{V}{LC} = \frac{V_s}{LC} \qquad (*)$$

Solution : $V(t) = \underbrace{V_t(t)}_{\text{transient response}} + \underbrace{V_{ss}(t)}_{\text{steady state response}}$

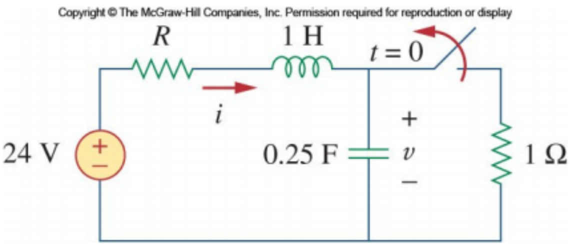
Statement: specifically, for step response, $V_t(t)$ is the same format as in natural response, $V_{ss} = V_s$, $v(t)$ is the solution to equation$(*)$

Case	$\alpha = \frac{R}{2L}, \omega_0 = \sqrt{\frac{1}{LC}}$

Case		$\alpha = \frac{R}{2L}, \omega_0 = \sqrt{\frac{1}{LC}}$	
1	overdamped	$\alpha > \omega \Leftrightarrow R > 2\sqrt{\frac{L}{C}}$	$V_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_s$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
2	critically damped	$\alpha = \omega \Leftrightarrow R = 2\sqrt{\frac{L}{C}}$	$V_C(t) = (A_1 t + A_2)e^{st} + V_s$ $s = -\alpha$
3	underdamped	$\alpha < \omega \Leftrightarrow R < 2\sqrt{\frac{L}{C}}$	$V_C(t) = e^{-\alpha t}[A_1 cos\omega_d t + A_2 sin\omega_d t] + V_s$ $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$

example

- Find $v(t)$ and $i(t)$ for $t > 0$.
  Consider three cases:
  - $R = 5\Omega$
  - $R = 4\Omega$
  - $R = 1\Omega$



When $R = 5\Omega$,

- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4A = C\frac{dv(0)}{dt}, \; v(0) = 4V, \; \frac{dv(0)}{dt} = 16$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2.5, \; \omega_0 = \frac{1}{\sqrt{LC}} = 2, \; s_{1,2} = -1, -4 \quad \text{Overdamped.}$$

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$

$\alpha = \frac{5\Omega}{2\times 1H} = 2.5\, rad/s$, $W_0 = \frac{1}{\sqrt{1\times 0.25}}\, rad/s = 2\, rad/s$. $\alpha > W_0$. overdamped.

$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - W_0^2} = -2.5 \pm \sqrt{2.5^2 - 2^2} = -1, -4$.

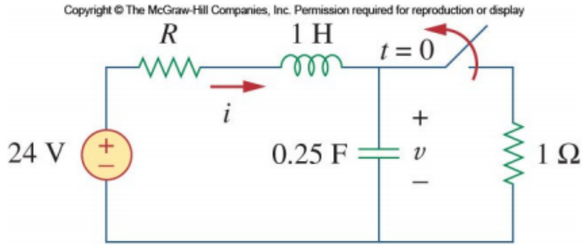
$V(t) = A_1 e^{-t} + A_2 e^{-4t} + 24V$.

From initial conditions:

$V(0+) = A_1 + A_2 + 24 = 4 \Rightarrow A_1 + A_2 = -20,$

$i(0+) = C\frac{dV(t)}{dt}\Big|_{t=0+} = C(-A_1 - 4A_2) = -\frac{1}{4}A_1 + A_2 = 4$ $\Bigg\} \Rightarrow$

$A_1 = -\frac{64}{3}$, $A_2 = \frac{4}{3}$. $V(t) = -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 24V$.



When $R = 4\Omega$,

- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4.8A = C\frac{dv(0)}{dt},\ v(0) = 4.8V,\ \frac{dv(0)}{dt} = 19.2$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2,\ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ s_{1,2} = -2 \quad \text{Critically damped}$$

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$

(b). $R=4\Omega$, $V(0+)=4.8V$, $i(0+)=4.8A$. Critically damped

$\alpha = \frac{4\Omega}{2\times 1H} = 2rad/s$, $\omega_0 = \frac{1}{\sqrt{1\times 0.25}} = 2rad/s$. $\alpha=\omega_0$.

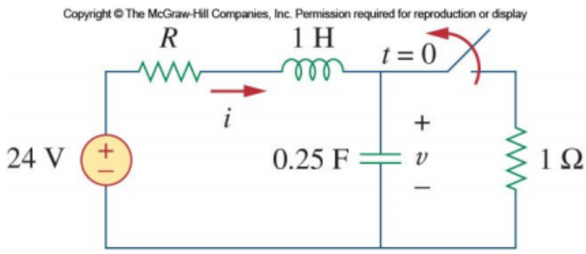
$\Rightarrow S_1 = S_2 = S = -\alpha = -2$.

$\Rightarrow V(t) = (A_1 t + A_2)e^{-2t} + 24V$.

$\Rightarrow V(t) = (-19.2t - 19.2)e^{-2t} + 24V$.

From Initial condition:

$$\left\{\begin{array}{l} V(0+) = A_2 + 24 = 4.8 \Rightarrow A_2 = -19.2 \\ i(0+) = C\frac{dV(t)}{dt}\Big|_{t=0+} = C(A_1 - 2A_2) = \frac{1}{4}A_1 - \frac{1}{2}A_2 = 4.8 \end{array}\right\} \Rightarrow A_1 = -19.2$$



When $R = 1\Omega$,

- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 12A = C\frac{dv(0)}{dt},\ v(0) = 12V,\ \frac{dv(0)}{dt} = 48$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 0.5,\ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ s_{1,2} = -0.5 \mp j1.936 \quad \text{Underdamped}$$

$$v(t) = v_{ss} + (A_1\cos 1.936t + A_2\sin 1.936t)e^{-0.5t}$$

$(c).\ R = 1\Omega,\ V(0+) = 12V,\ i(0+) = 12A.$

$\alpha = \frac{1\Omega}{2\times 1H} = 0.5\,rad/s.\quad \omega_0 = \frac{1}{\sqrt{1\times 0.25}} = 2\,rad/s.\quad \alpha < \omega_0.$ underdamped.

$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936.$

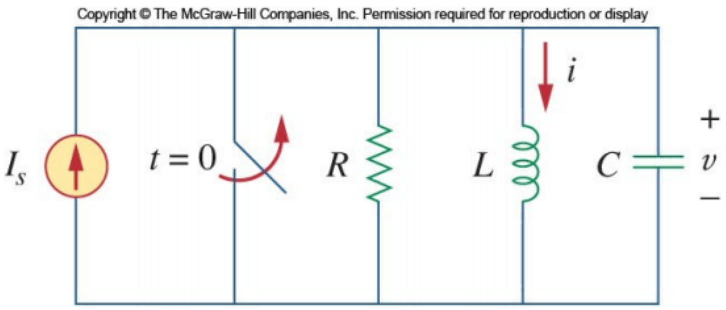
$V(t) = e^{-0.5t}(B_1 \cos 1.936t + B_2 \sin 1.936t) + 24V.$

From initial condition:

$$\left.\begin{array}{l} V(0+) = B_1 + 24 = 12 \Rightarrow B_1 = -12. \\ i(0+) = C \frac{dV(t)}{dt}\Big|_{t=0+} = \frac{1}{4}(-0.5B_1 + 1.936B_2) = 12 \end{array}\right\} \Rightarrow$$

$$\begin{cases} B_1 = -12, \\ B_2 = 21.694 \end{cases} \Rightarrow V(t) = e^{-0.5t}(-12\cos 1.936t + 21.694 \sin 1.936t) + 24V.$$

parallel RLC Circuit



Apply KCL,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

But

$$v = L\frac{di}{dt}$$

So we get

$$\frac{d^2i}{d^2t} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

$$\frac{d^2I}{dt^2} + \frac{1}{RC}\frac{dI}{dt} + \frac{I}{LC} = \frac{I_s}{LC} \qquad (*)$$

Solution : $I(t) = \underbrace{I_t(t)}_{\text{transient response}} + \underbrace{I_{ss}(t)}_{\text{steady state response}}$

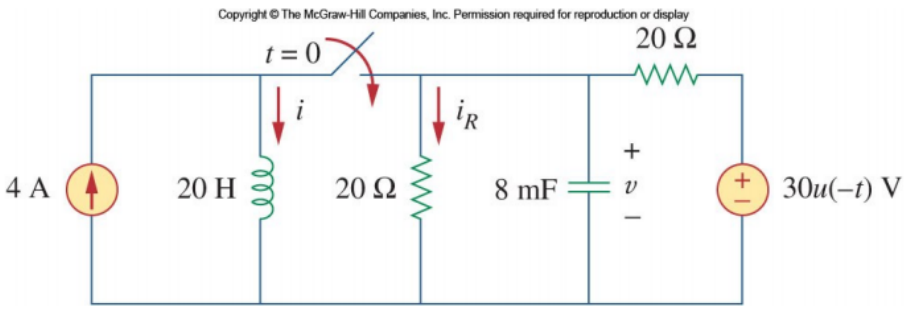
Statement: specifically, for step response, $I_t(t)$ is the same format as in natural response, $I_{ss} = I_s$, $v(t)$ is the solution to equation$(*)$

Case	$\alpha = \frac{R}{2L}, \omega_0 = \sqrt{\frac{1}{LC}}$

Case		$\alpha = \frac{R}{2L}, \omega_0 = \sqrt{\frac{1}{LC}}$	
1	overdamped	$\alpha > \omega \Leftrightarrow R > 2\sqrt{\frac{L}{C}}$	$I_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + I_s$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
2	critically damped	$\alpha = \omega \Leftrightarrow R = 2\sqrt{\frac{L}{C}}$	$I_L(t) = (A_1 t + A_2)e^{st} + I_s$ $s = -\alpha$
3	underdamped	$\alpha < \omega \Leftrightarrow R < 2\sqrt{\frac{L}{C}}$	$I_L(t) = e^{-\alpha t}[A_1 cos\omega_d t + A_2 sin\omega_d t] + I_s$ $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$

example

- Find $i(t)$ and $i_R(t)$ for $t > 0$.



Initial values ($t < 0$): $i(0) = 4A$, $v(0) = \frac{20}{20+20} \times 30V = 15V = L\frac{di(0)}{dt}$

For $t > 0$, $\alpha = \frac{1}{2RC} = \frac{1}{2\times10\times8\times10^{-3}} = 6.25$, $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times8\times10^{-3}}} = 2.5$

$$s_{1,2} = -6.25 \mp 5.7282$$

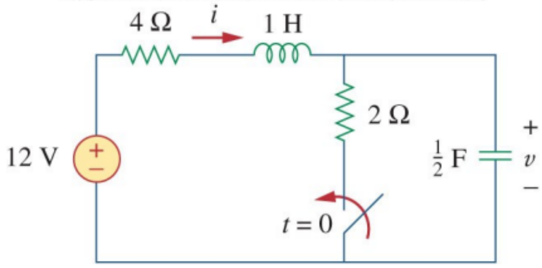
$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

General RLC Circuit

example

General RLC Circuits

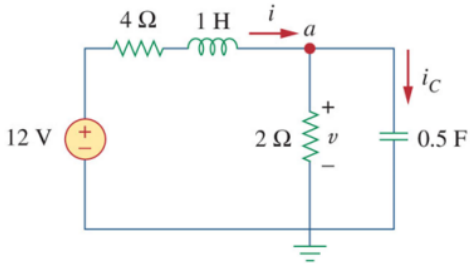
- Find the complete response v for $t > 0$ in the circuit.



1. Initial conditions

$$v(0^+) = v(0^-) = 12V,\ i(0^+) = i(0^-) = 0$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-12/2}{0.5} = -12\ V/s$$



2. Transient response

KCL at node *a*: $i = \frac{v}{2} + 0.5\frac{dv}{dt}$

KVL on left mesh: $4i + 1\frac{di}{dt} + v = 0$

$$\Rightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0 \Rightarrow v_t(t) = A_1e^{-2t} + A_2e^{-3t} = 12e^{-2t} - 4e^{-3t}$$

3. Steady-state response $v_{ss}(t) = 4V$

4. Put together

Summary

1. without excitation(Natural Response)自然响应

Case		$\frac{d^2y(t)}{dt} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = 0$	
1	overdamped	$\alpha > \omega$	$y(t) = A_1e^{s_1t} + A_2e^{s_2t}$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
2	critically damped	$\alpha = \omega$	$y(t) = (A_1t + A_2)e^{st}$ $s = -\alpha$
3	underdamped	$\alpha < \omega$	$y(t) = e^{-\alpha t}[A_1cos\omega_dt + A_2sin\omega_dt]$ $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$

2. with any excitation

$$\frac{d^2y(t)}{dt} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = f(t)$$
$$y(t) = y_h + y_p$$

y_h is the general solution to the corresponding homogeneous equation

y_p is a particular solution 特解

how to find y_p

1.

$$f(t)=e^{rt}P_m(t),P_m(t)\text{是关于t的m次多项式},r\in\mathbb{R}$$
$$\begin{cases} r\neq s_{1,2}, & y_p=Qm(t)e^{rt} \\ r=s_1 \text{ or } s_2(s_1\neq s_2), & y_p=tQ_m(t)e^{rt} \\ r=s_1=s_2, & y_p=t^2Q_m(t)e^{rt} \end{cases}$$

2.

$$f(t)=e^{rt}(Mcos\omega t+Nsin\omega t),r,M,N,\omega\in\mathbb{R},\omega\neq 0$$
$$\begin{cases} r\pm j\omega\neq s_{1,2}, & y_p=e^{rt}(B_1cos\omega t+B_2sin\omega t) \\ r\pm j\omega=s_{1,2}, & y_p=te^{rt}(B_1cos\omega t+B_2sin\omega t) \end{cases}$$

3.

$$

f(t) = f__1(t)+f__2(t)
y__p = y__{p__1} + y__{p__2}

$$

$$f(t)=f_1(t)+f_2(t)y_p=y_{p_1}+y_{p_2}$$