COMPUTATIONAL PRACTICUM

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1 Exact Solution

The differential equation is of the form:

$$y' = f(x, y) = \frac{4}{x^2} - \frac{y}{x} - y^2 \tag{1}$$

Also the initial values given are: $x_0 = 1, y_0 = 0, X = 7$ Due to division, x cannot be equal to zero.

We notice that equation 1 is a nonlinear nonhomogeneous differential equation which falls into Riccati equations which are of the format:

$$y' + a(x)y + b(x)y^2 = c(x) \tag{2}$$

where $a(x) = \frac{1}{x}, b(x) = 1, c(x) = \frac{4}{x^2}$

Therefore, to solve equation 1, we need firstly to find any particular solution y_1, then use the substitution $y = y_1(x) + z(x)$

To find y_1, we examine the original equation, it makes sense to guess that the particular solution is of the format $y_1(x) = \frac{C}{x}$, since both functions $a(x)$, $c(x)$ follow this format or its derivative.

Now to find C, we substitute y_1 in equation 1

$$y_1' = \frac{4}{x^2} - \frac{y_1}{x} - y_1^2 \tag{3}$$

$$-\frac{C}{x^2} = \frac{4}{x^2} - \frac{C}{x^2} - \frac{C^2}{x^2} \tag{4}$$

Multiplying by x^2, we get

$$C^2 = 4 \tag{5}$$

$$C = \pm 2 \tag{6}$$

Since we can take y_1 to be any particular solution, let's choose $C = -2$

Now we make the substitution $y = y_1 + z$ in equation 1:

$$(y_1 + z)' = \frac{4}{x^2} - \frac{(y_1 + z)}{x} - (y_1 + z)^2 \tag{7}$$

$$(\frac{-2}{x})' + z' = \frac{4}{x^2} - \frac{(\frac{-2}{x} + z)}{x} - (\frac{-2}{x} + z)^2 \tag{8}$$

$$\frac{2}{x^2} + z' = \frac{4}{x^2} + \frac{2}{x^2} - \frac{z}{x} - \frac{4}{x^2} - z^2 + \frac{4z}{x} \tag{9}$$

$$z' - \frac{3z}{x} = -z^2 \tag{10}$$

We can conclude that equation 10 is a nonlinear first order ordinary differential equation, and it has the format of Bernoulli equations which is:

$$z' + g(x)z = f(x)z^k \tag{11}$$

where $g(x) = \frac{-3}{x}, f(x) = -1, k = 2$

To solve equation 10, first we need to find any non-trivial solution z_c of the complementary equation:

$$z' + g(x)z = 0 \tag{12}$$

Then the solution of equation 10 would be $z = uz_c$, where $u(x)$ can be found using this equation:

$$\frac{\mathrm{d}u}{u^k} = f(x)z_c^{k-1}\mathrm{d}x \tag{13}$$

Now we proceed to find z_c:

$$z' - \frac{3z}{x} = 0 \tag{14}$$

$$z' = \frac{3z}{x} \tag{15}$$

$$\int \frac{\mathrm{d}z}{z} = \int \frac{3\mathrm{d}x}{x} \tag{16}$$

$$\ln|z| = 3\ln|x| + K \tag{17}$$

We can choose $K = 0$ since we need any non-trivial solution for z_c

$$z_c = x^3 \tag{18}$$

Now to find u, we substitute in equation 13:

$$\int \frac{\mathrm{d}u}{u^2} = \int -x^3\mathrm{d}x \tag{19}$$

$$-\frac{1}{u} = -\frac{x^4 + D}{4} \tag{20}$$

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$$u = \frac{4}{x^4 + D} \tag{21}$$

Now to find the solution of equation 10, we substitute in this equation:

$$z = u z_c \tag{22}$$

Substituting u and z_c

$$z = \frac{4x^3}{x^4 + D} \tag{23}$$

Now for the final step, we just need to write the solution of equation 1 using this equation:

$$y = y_1(x) + z(x) \tag{24}$$

Substituting y_1 and z, and renaming the constant D to C since it is the usual name for constants, we finally get:

$$y = \frac{-2}{x} + \frac{4x^3}{x^4 + C} \tag{25}$$

The constant C can be found substituting the initial values in an equation which results from solving equation 25 for C:

$$C = x_0^4 \left(\frac{2 - x_0 y_0}{x_0 y_0 + 2} \right) \tag{26}$$

So any initial conditions should satisfy $x_0 y_0 \neq -2$

We can also see that y is defined when $x \neq 0$, and when $x^4 + C \neq 0$, and when substituting the constant C:

$$x_0^4 + x_0^4 \left(\frac{2 - x_0 y_0}{x_0 y_0 + 2} \right) \neq 0 \tag{27}$$

$$x_0^4 + x_0^4 \left(\frac{2 - x_0 y_0}{x_0 y_0 + 2} \right) \neq 0 \tag{28}$$

dividing by x^4

$$\frac{4}{x_0 y_0 + 2} \neq 0 \tag{29}$$

Which is always true (assuming the previously mentioned conditions).

The final answer is:

$$ans = y = \frac{-2}{x} + \frac{4x^3}{x^4 + x_0^4 \left(\frac{2 - x_0 y_0}{x_0 y_0 + 2} \right)} \tag{30}$$

on $\{x | x \in R \ \& \ x_0 \in R \ \& \ y_0 \in R \ \& \ x \neq 0 \ \& \ x_0 y_0 \neq -2\}$

2 Application

The application finds the approximations, error, and error analysis graphs of the three methods: Euler, Improved Euler, and Runge Kutta.
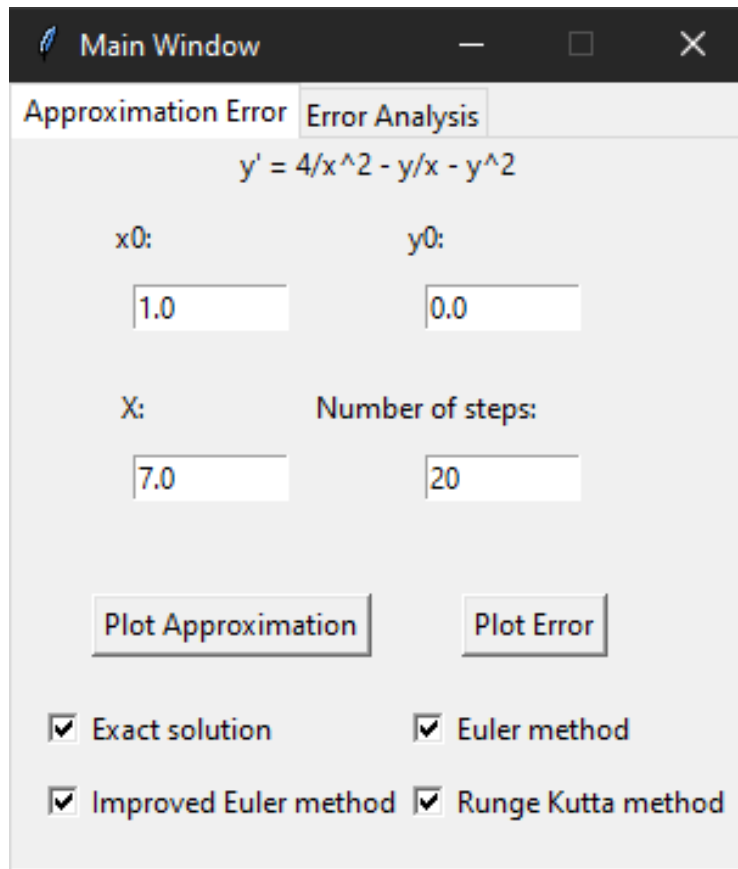
2.1 Used Libraries

I have used Python for this app, and these are the libraries I have used in the development: Matplotlib, numpy, tkinter.

2.2 User Interface

Application's main window has two tabs; one for finding approximations and errors, and the other one is for error analysis.

2.2.1 Approximations Error tab

Figure 1: Approximation Error tab

For the "Approximations Error" tab, a user enters the values for the initial values (x_0, y_0, X, and the number of steps), then they choose which graphs to plot using the check-boxes in the bottom. Finally, they click "Plot Approximation" button or "Plot Error" depending on what they want to see.

2.2.2 Error Analysis tab

Figure 2: Error Analysis tab

The "Error Analysis" tab's functionality is very similar to the previous tab's. First, the the user chooses values for N_0 and N, then they choose which methods they want tp plot. Finally, they click on the "Plot Error Analysis" button, to show the graph(s).

2.2.3 Graph Interface

Here are two figures showing the approximations and error graphs of the three methods (Euler, Improved Euler, Runge Kutta) when setting the initial values to be the same as in the PDF ($x_0 = 1, y_0 = 1, X = 7$), and the number of steps chosen is 20.
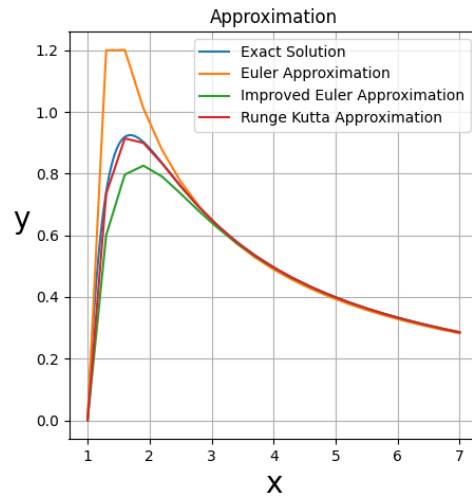

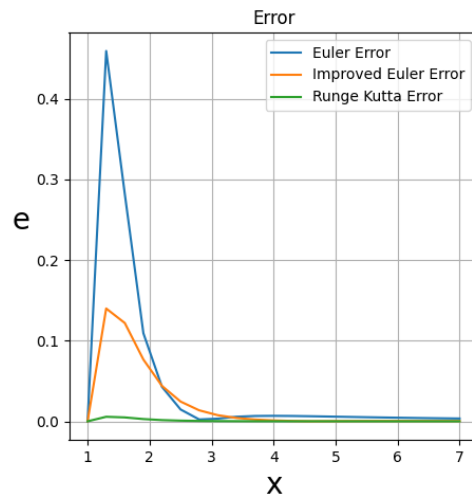Figure 3: An example of the approximation graphs plotting

Figure 4: An example of the error graphs plotting

Here is a figure showing the error analysis graph of the three methods (Euler, Improved Euler, Runge Kutta) when setting the initial values to be the same as in the PDF ($x_0 = 1, y_0 = 1, X = 7$), and choosing $N_0 = 20$, $N = 25$.
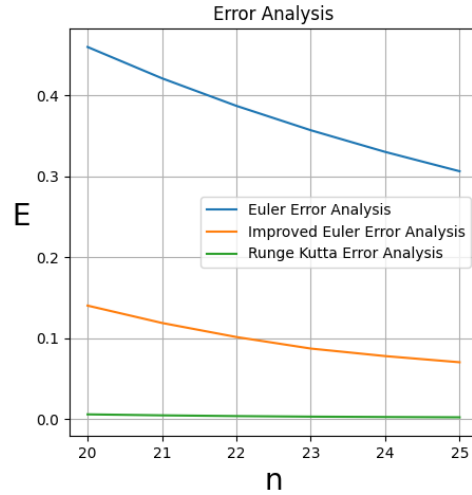

Figure 5: An example of the error analysis graphs plotting
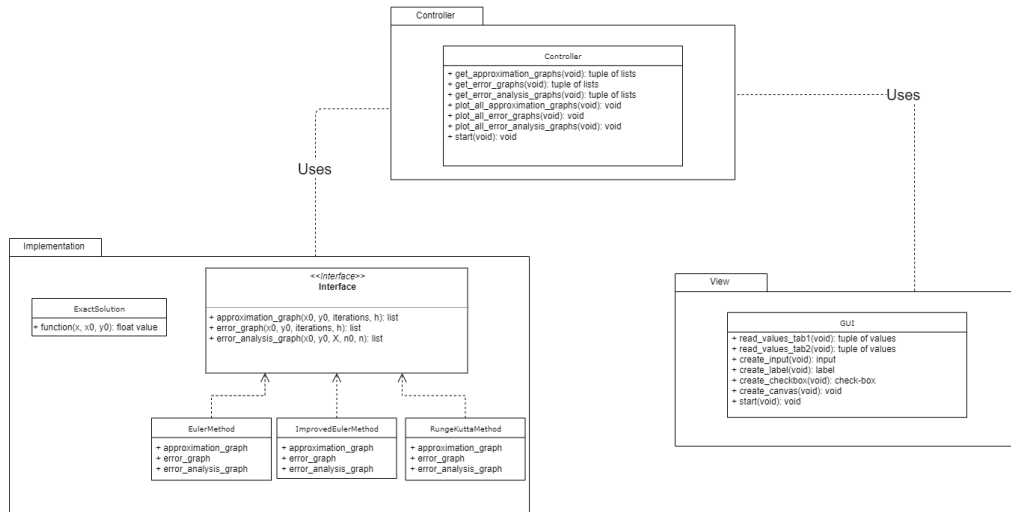
3   UML Diagram

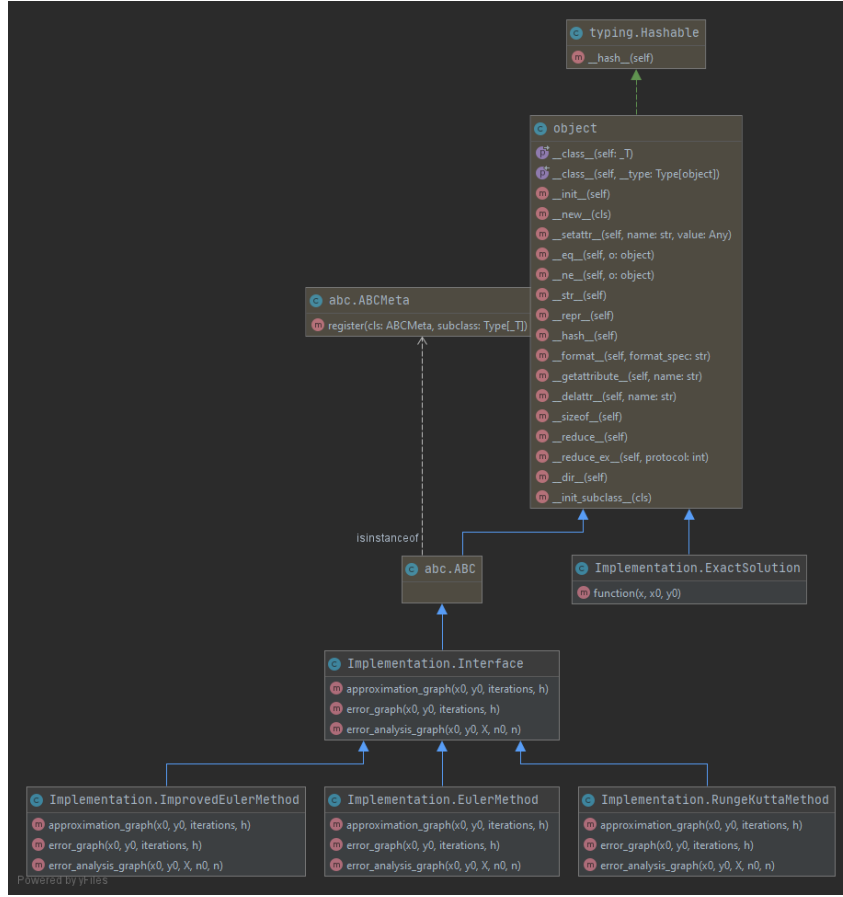Figure 6: The project diagram

Figure 7: A detailed diagram of the Implementation file

4   Code Snippets

There are three files in my project:

- Implementation: this file is for finding graphs of approximation, error, and and error analysis for each of the methods (Euler, Improved Euler, Runge Kutta).

- View: this file is for creating the user interface widgets.

- Controller: this file uses the other two files and controls the application as a whole.

4.1   Implementation

This file contains an interface class. The interface contains three static methods to be overridden. These methods are approximation graph, error graph, and error analysis.

Implementation file also contains three static classes implementing the interface (each class is for one approximation method). Below there is an example of the implementations of the Euler method class.
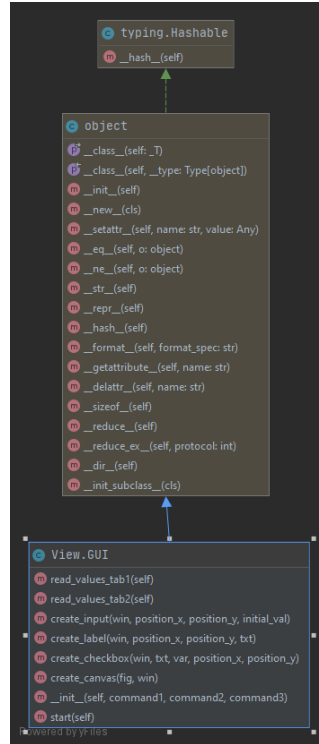
Figure 8: A detailed diagram of the View file

This file also contains a class for the exact solution. This class contains single static method for finding the value of the function given the initial conditions.

Finally, this file contains the function $f(x)$ from the original equation $y' = f(x)$.

4.2 View

This file contains mainly a single class, which is GUI. The class is responsible for creating the user interface, and making all its widgets (tabs, input boxes, etc). Below there is an example of a static method in GUI class, which is used to create inputs in a certain window.

4.3 Controller

This file basically manages the different parts of the application. It imports both Implementation and View files, and make the connections between them. Here are a list of the most important methods in it:

- get graphs: three methods of this type one for approximations graphs, the second is for error graphs, and the last one is for error analysis graphs. Each of these methods returns a tuple of 4 elements (please refer the figure below for further details).

- plot all graphs: three methods of this type one for approximations graphs, the second is for error graphs, and the last one is for error analysis graphs. This type of method
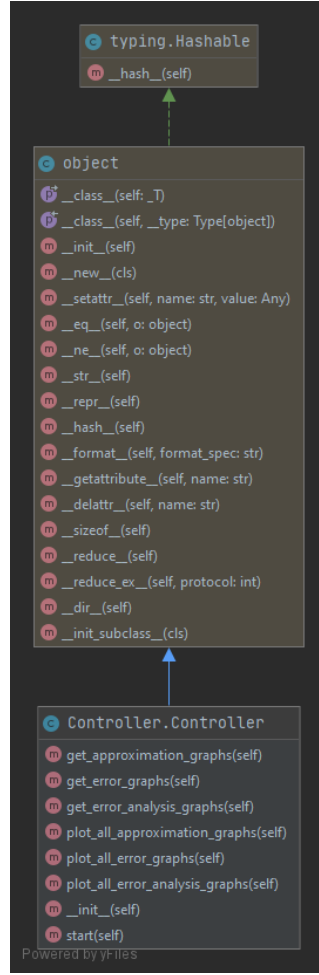
9

Figure 9: A detailed diagram of the Controller file

just creates window and plots the corresponding graphs on that window (please refer the figure below for further details).

5 Analysis

All of the graphs in the figures below are for these initial values: $(x_0 = 1, y_0 = 1, X = 7, Number of steps = 20, N_0 = 20, N = 25,)$.

5.1 Euler Method

As seen in the figures $[13 - 15]$, Euler approximation shows poor results with an error that can get to 0.48 at the beginning of the interval. This error is considered to be quite large taking into account that the number of steps is equal to 20.
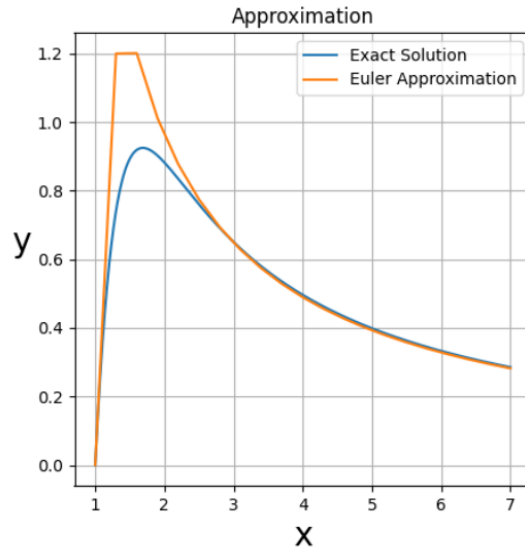
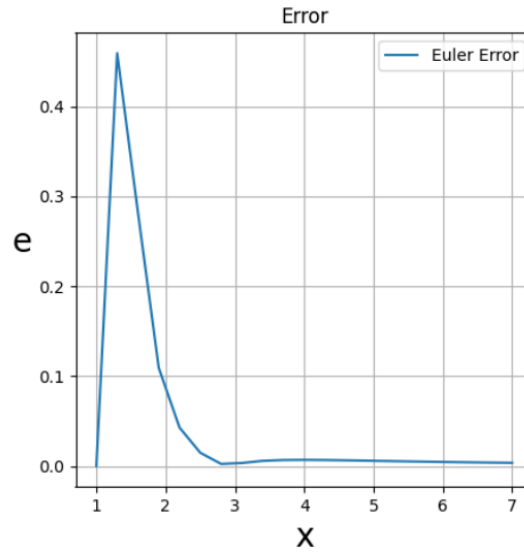Figure 10: Euler approximation graph with the exact solution

Figure 11: Euler approximation error graph

5.2 Improved Euler Method

As seen in the figures[16- 18], Improved Euler approximation shows better results than Euler with an error that can get to 0.14 at max at the beginning of the interval 1.3. This error is considered to be reasonable for an approximation (not too big, not too small).
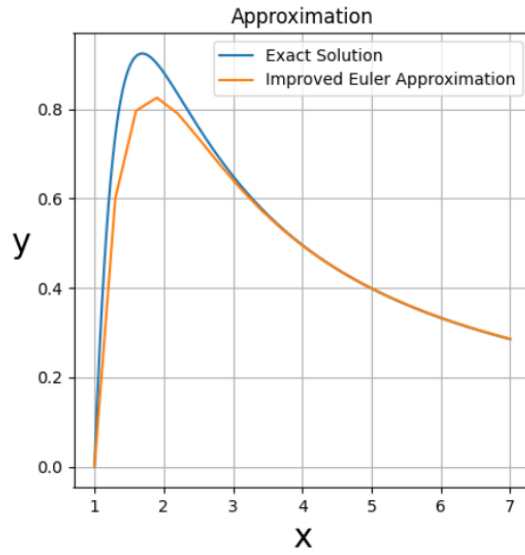
Figure 12: Improved Euler approximation graph with the exact solution
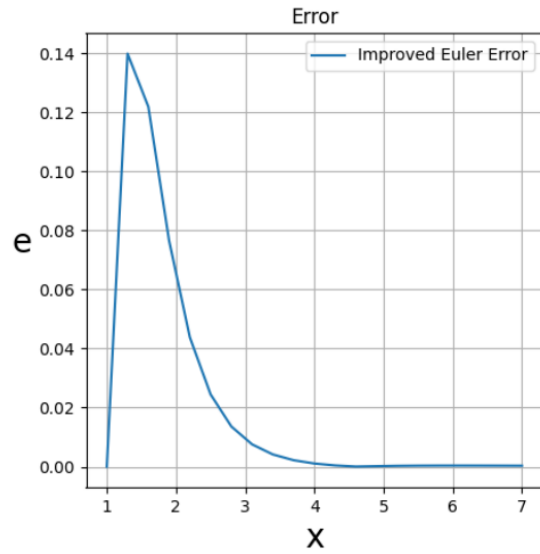

Figure 13: Improved Euler approximation error graph

5.3   Runge Kutta Method

As seen in the figures $[19 - 21]$, Runge Kutta approximation shows by far the best results among all the other approximation methods. The error is considered to be small as it gets at max to 0.0057.
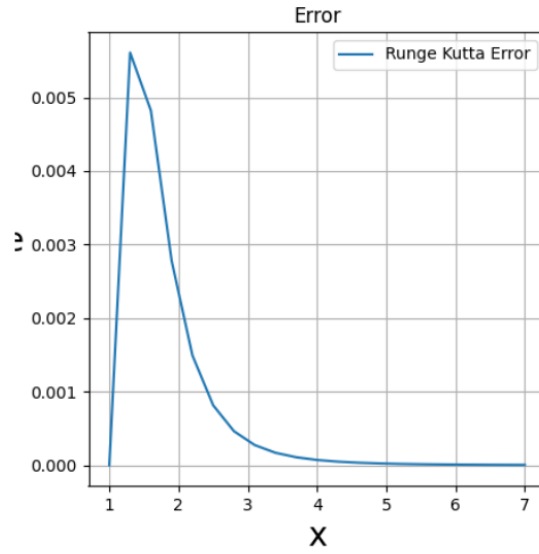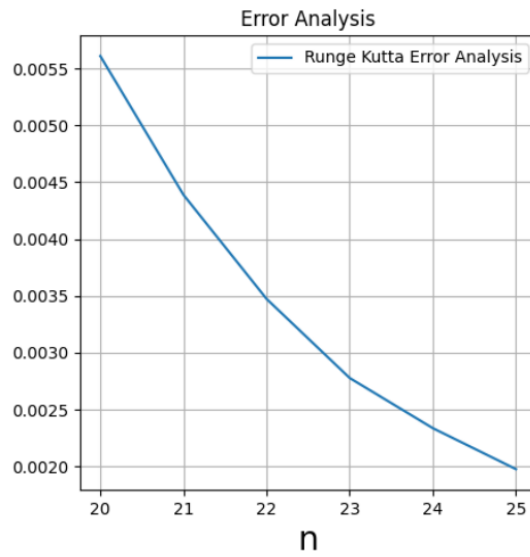
Figure 14: Runge Kutta approximation error graph

Figure 15: Runge Kutta error analysis graph

6   Conclusion

As discussed in the analysis section, the Runge Kutta approximation is the best method among the others, followed by Improved Euler, and finally Euler method to be the one with the maximum error.