Macroeconomics III - Problem Set 1

Murilo A. P. Pereira

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Question 1 - Stochastic Processes

(a) Consider the AR(1) process below,

$$y_t = \mu(1-\rho) + \rho y_{t-1} + \epsilon_t \quad \text{where} \;\; \epsilon_t \sim N(0, \sigma^2)$$

The unconditional standard deviation of y_t is $\sigma_y = \dfrac{\sigma}{\sqrt{1-\rho^2}}$. Now I will construct a discrete markov chain with the state space $\{y^i\}_{i=1}^N$ and the transition matrix $\Pi = \pi_{ij}$.

First I will specify the grid, defining the upper and lower bound and equally-spaced points.

$$y_1 = \mu - m\sqrt{\frac{\sigma^2}{1-\rho^2}} \text{and} y_N = \mu + m\sqrt{\frac{\sigma^2}{1-\rho^2}}$$

where m is a scaling parameter. Then, $z_2, z_3 ... z_{N-1}$ are defined such that,

$$\{y^i\}_{i=1}^N = \left\{ y_1, y_1 + \frac{y_N - y_1}{N-1}, ..., y_1 + \frac{N-2}{N-1}(y_N - y_1), y_N \right\}$$

Next we create the borders for each interval $[y_i, y_{i+1}]$.

$$m_i = \frac{y_{i+1} + y_i}{2} = y_1 + (2i-1)\frac{d}{2} \quad \text{where } d = \frac{y_N - y_1}{N-1}$$

$$y_i \in \begin{cases} (-\infty, m_1] & \text{if } i = 1 \\ (m_{i-1}, m_i] & \text{if } 1 < i < N \\ [m_N, \infty) & \text{if } i = N \end{cases}$$

If j=2,...,N-1

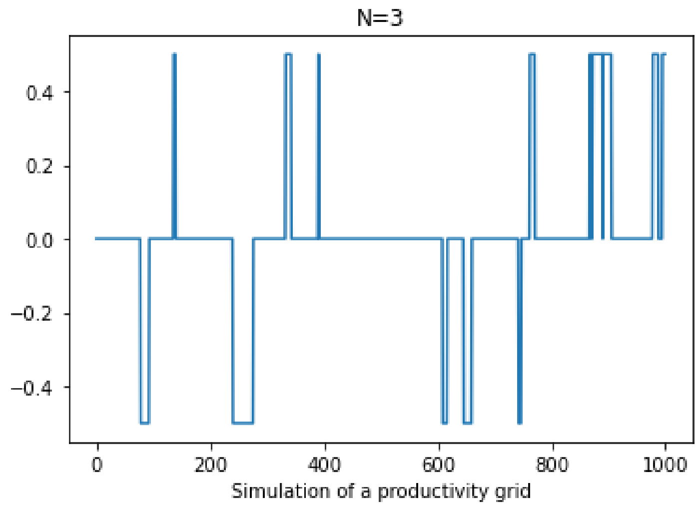
$$\begin{aligned} \pi_{ij} &= Pr(y_{t+1} = y_j | y_t = y_i) = Pr(\mu(1-\rho) + \rho y_i + \epsilon_{t+1} = z_j) \\ &\approx Pr(m_{j-1} \leq \mu(1-\rho) + \rho y_i + \epsilon_{t+1} \leq z_j) \\ &= \Phi\left\{ \frac{m_j - \rho y_i - \mu(1-\rho)}{\sigma} \right\} - \Phi\left\{ \frac{m_{j-1} - \rho y_i - \mu(1-\rho)}{\sigma} \right\} \end{aligned}$$

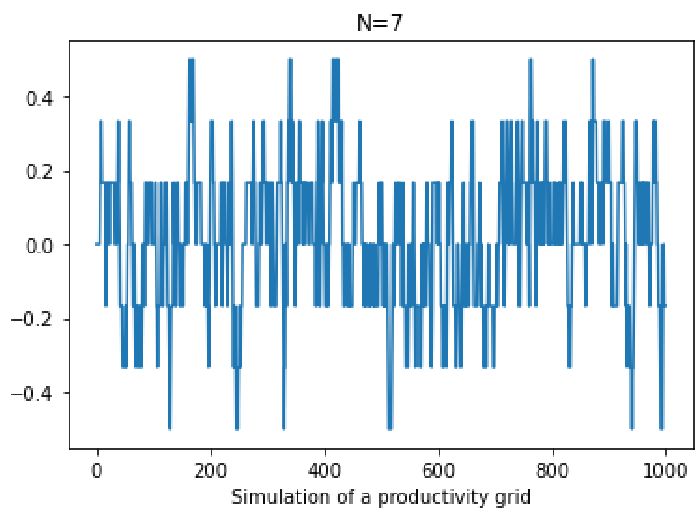
And following a similar method we can find that,

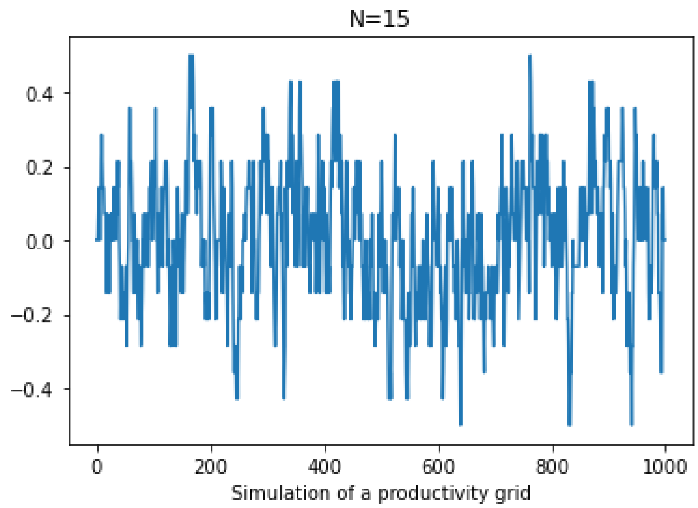
$$\pi_{i1} = \Phi\left\{ \frac{m_1 - \rho y_i - \mu(1-\rho)}{\sigma} \right\}$$

$$\pi_{iN} = 1 - \Phi\left\{ \frac{m_{N-1} - \rho y_i - \mu(1-\rho)}{\sigma} \right\}$$

(b) The code is the file attached. Considering N=3 we still have a big discrepancy with the data in continuous time. As we increase the grids, this difference will diminish. We can see from the graphs that the change from 7 to 15 is less significant in terms of approximation than the change from 3 to 7 grids. Tauchen makes the case that the approximation is adequate for most purposes when N = 9 and r = 3.






N=15
0.4
0.2
0.0
−0.2
−0.4
0
200
400
600
800
1000
Simulation of a productivity grid

Question 2 - RBC Model

(a) In order to write the decentralized equilibrium I start defining the problems of the firm. Firms owns only technology and produce using only capital and labor as inputs. The technology production function is $y_t = z_t f(k_t, h_t)$. Firms maximize profits, the problem is

$$Max_{k_t,h_t} z_t f(k_t, h_t) - r_t k_t - w_t h_t$$

and the first order conditions are

$$\begin{aligned} k: \quad & z f_k(k,h) - r = 0 \\ h: \quad & z f_h(k,h) - w = 0 \end{aligned}$$

Now consider the Household problem, recall that I am assuming households owns capital.

$$V(k,z) = Max_{c,k',n} = \{u(c, 1-h) + \beta E[V(k', z')]\}$$

subject to

$$\begin{aligned} & c + k' = rk + wh + (1-\delta)k \\ & c, k' \geq 0 \end{aligned}$$

The capital is supplied inelastically when interest rates are positive. Also, depreciation is paid by the the household.

The recursive competitive equilibrium is a list such that $V(k,z)$ is the value function for households. Variables c, h, k represents individuals and firms decisions and r, w represents factor prices. Holseholds maximize utility, firms maximiza profits and markets clear. The market clearing condition in goods markets is such that $C(K,z) + K' = zF(K,H) + (1-\delta)K$, where the upper case letters represents aggregate variables.

(b) The social planner solves the following problem

$$V(k_t, z_t) = Max_{c_t,k_{t+1},h_t} E_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\gamma}(1-h_t)^{1-\gamma})^{1-\mu}}{1-\mu}$$

subject to

$$\begin{aligned} k_{t+1} + c_t &= z_t k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t \\ ln(z_{t+1}) &= \rho ln(z_t) + \epsilon_{t+1} \quad \text{where} \quad \epsilon_{t+1} \sim N(0, \sigma^2) \end{aligned}$$

We can write c_t such that

$$c_t = z_t k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t - k_{t+1}$$

Now our problem can be written such that

$$Max_{c_t,k_{t+1},h_t} E_0 \sum_{t=0}^{\infty} \beta^t u(z_t f(k_t, h_t) + (1-\delta)k_t - k_{t+1})$$

subject to

$$\begin{aligned} & 0 \leq h_t < 1 \\ & k_{t+1} \geq 0 \end{aligned}$$

where

$$u(c_t,(1-h_t)) = \frac{(c_t^{\gamma}(1-h_t)^{1-\gamma})^{1-\mu}}{1-\mu}$$
$$f(k_t,h_t) = z_t k_t^{\alpha} h_t^{1-\alpha}$$

Consider $V(k_0,z_0)$ the value function given k_0 and z_0. Writing in a recursive way we have

$$V(k,z) = Max_{k'\geq 0, 0\leq h<1}\ u(zf(k,h)+(1-\delta)k-k',1-h)+\beta E_0[V(k',h')]$$

Note that the problem repeats every time and V is a time invariant function.

(c) The parameter μ is related to the intertemporal elasticity of substitution, $1/\mu$. It influences the speed that the economy converges to steady state. The higher the value of μ, the higher the speed of convergence. Considering the utility function given, setting $\mu = 2$, we are assuming that the agents are more risk averse than in the case $\mu = 1$, log utility function in this case. The parameter β set how agents discount the future and it can be seem as a function of the impacience rate. Assuming that the model period is a quarter, β is related to an intertemporal discount rate close to 5.3% and a depreciation rate, δ, close to 4.9%. In item (d) I will show the steady-state capital to labor ratio $\bar{k}/\bar{h}$, plugging in the equation that determines $\bar{r}$, we find the relation between β and $\bar{r}$. Finally, α represents the share of capital in the economy.

(d) Assuming that there is no uncertainty, I will consider the problem defined in item (b). Taking the first order conditions we have

$$\begin{aligned}
k': &\quad -u_c(c,1-h)+\beta V_k(k',z')=0 && (1)\\
&\quad [c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma}=\beta V_k(k',z') && (2)\\
h: &\quad -u_h(c,1-h)+u_c(c,1-h)zf_h(k,h)=0 && (3)\\
&\quad [c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}(1-\gamma)c^{\gamma}(1-h)^{-\gamma}=[c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma}zf_h(k,h) && (4)
\end{aligned}$$

where

$$c = zf(k,h)+(1-\delta)-k' \tag{5}$$

Using the Envelope Theorem condition

$$V_k(k,z) = u_c(c,1-h)(zf_k(k,h)+1-\delta) \tag{6}$$
$$V_k(k,z) = [c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma}(zk^{\alpha}h^{1-\alpha}+1-\delta) \tag{7}$$

Since envelope holds for any t

$$V_k(k',z') = u_c(c',1-h')(zf_k(k',h')+1-\delta) \tag{8}$$
$$V_k(k',z') = [c'^{\gamma}(1-h')^{1-\gamma}]^{-\mu}\gamma c'^{\gamma-1}(1-h')^{1-\gamma}(zk'^{\alpha}h'^{1-\alpha}+1-\delta) \tag{9}$$

Now we can substitute equation (8) in (1) to get the Euler equation

$$u_c(c,1-h) = \beta[u_c(c',1-h')(zf_k(k',h')+1-\delta)] \tag{10}$$

that is equivalent to

$$[c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma} = \beta[[c'^{\gamma}(1-h')^{1-\gamma}]^{-\mu}\gamma c'^{\gamma-1}(1-h')^{1-\gamma}(zk'^{\alpha}h'^{1-\alpha}+1-\delta)] \tag{11}$$

and the intratemporal equation is

$$u_h(c,1-h) = u_c(c,1-h)zf_h(k,h) \tag{12}$$

or

$$[c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}(1-\gamma)c^{\gamma}(1-h)^{-\gamma} = [c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma}zf_h(k,h) \tag{13}$$

Now, in order to calibrate γ I will consider all variables in the same period, steady steady, taking out the primes in variables $t+1$. Using equation (11), we can cancel out terms and find

$$1 = \beta[\alpha(\frac{k}{h})^{\alpha-1} + 1 - \delta] \tag{14}$$

isolating k we find

$$k_{kss} = (\frac{\beta\alpha}{1-\beta(1-\delta)})^{\frac{1}{1-\alpha}}h \tag{15}$$

Now, take the intratemporal equation

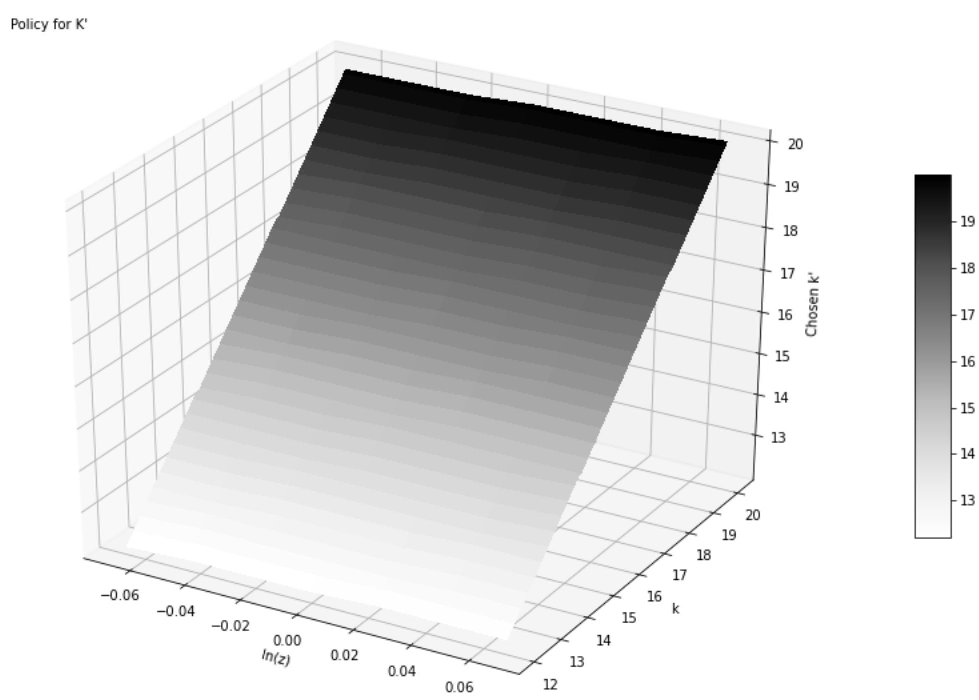
$$[c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}(1-\gamma)c^{\gamma}(1-h)^{-\gamma} = [c^{\gamma}(1-h)^{1-\gamma}]^{-\mu}\gamma c^{\gamma-1}(1-h)^{1-\gamma}zf_h(k,h) \tag{16}$$

Cancelling out on both sides of the equation we find

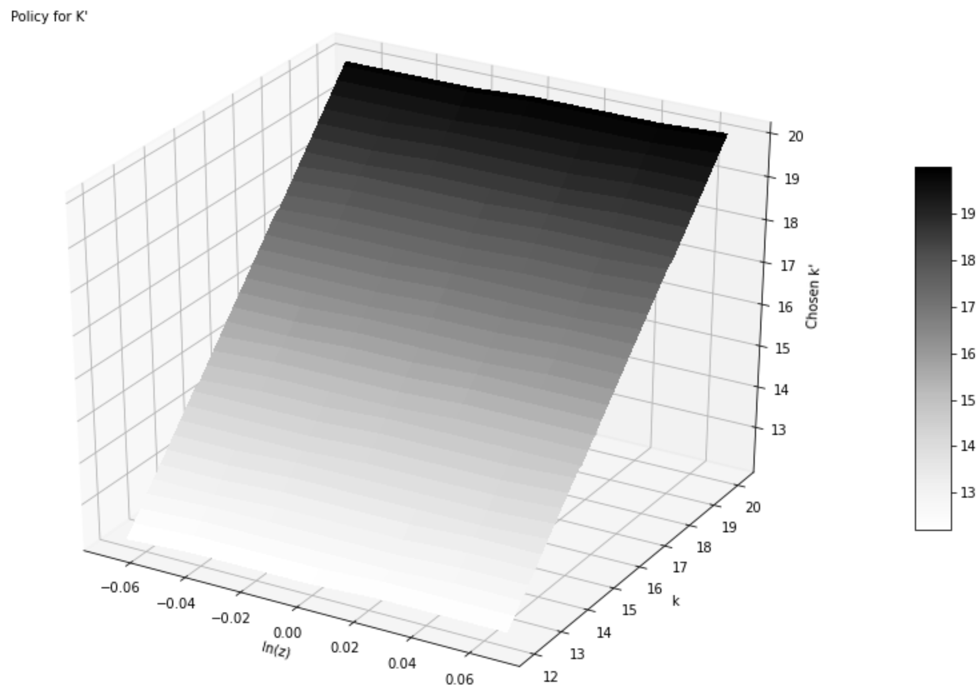
$$\frac{1-\gamma}{\gamma} = \frac{1}{c}(1-h)(1-\alpha)(\frac{k}{h})^{\alpha} \tag{17}$$

Finally, using fsolve on pyhton we can find that $\gamma \approx 0.3868$

(e) The number of iterations to convergence was 1429. The time necessary was 11min and 39s.



(f) Using now the Howard's improvement, iterating 20 times before updating the policy function we need 58 iterations so the error can be lower than the tolerance value considered. The time necessary for that was 1min and 1sec.



(g) Momenst of the data.

Consumption: Mean: 1.022818196085044 Variance: 0.0011533711413512866

Hours Worked: Mean: 8.00064352563388 Variance: 0.01814509617452932

Capital Mean: 16.1212532463534 Varaince: 0.5013516691850136

Investiment Mean: 0.19369310009954185 Variance: 0.0010000181849680562

Output Mean: 1.2165112961845856 Variance: 0.002679644765092331

(g) The question addresses a typical real business cycle problem. We have risk-averse consumers maximizing utility and firms maximizing profits. The problem is repeated every period, we can solve it iteratively. By the results that I have shown in the last item, we can see that agents smooth consumption, the data show a low variance for consumption. Also, agents works approximately 8 hours, in accordance to the steady state predetermined in the problem.

Question 3 - Occupational Choice

(a) In period t, individuals maximize their utilities subject to a constraint according to the the choice that they did between working for a wage w or to be an entrepreneur and produce a consumption good, y. Agents maximize

$$U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

where $u(c_t) = \dfrac{c^{1-\epsilon}}{1-\epsilon}$. After the agent know the productivity, if she decides to be a worker, her budget constraint will be

$$c_t = w_t + a_t(1+r) - a_{t+1} \tag{2}$$

and if the decision is to be an entrepreneur

$$c_t = zk_t^{\alpha} l^{\theta} - rk_t - wl_t + a_t(1+r) - a_{t+1} \tag{3}$$

Since the problem repeats every period, we can substitute and write the problem in a recursive way as a function of assets and productivity. Consider V_w the value function of the worker and V_e the value function of the entrepreneur, we have that

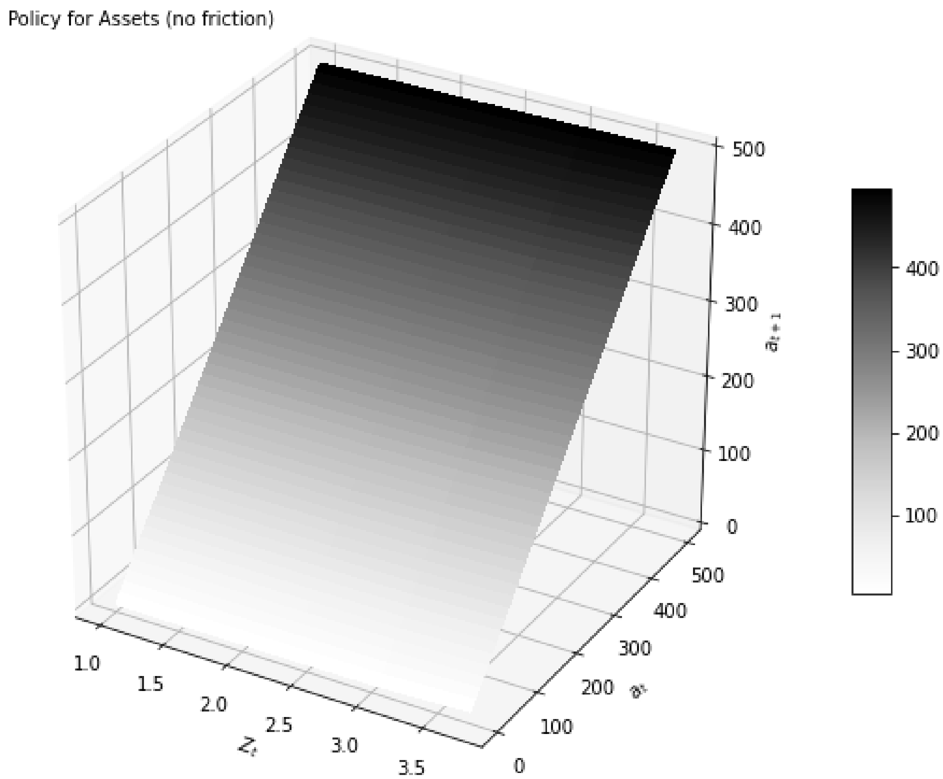
$$V_w(a,z) = u(w + a(1+r) - a') + \beta E_0 V(a', z') \tag{4}$$

$$V_e(a,z) = u(\pi + a(1+r) - a') + \beta E_0[\gamma V(a', z) + (1-\gamma)V(a', z')] \tag{5}$$

where $\pi = zk^{\alpha}l^{\theta} - rk - wl$

(b) Note that without financial friction, there is no relation between asset holding and the occupational choice of the agent. In this case the optimal occupational choice depend on productivity.

(c) The code is in the file attached.



(d) When there exist financial friction, the optimal choice for the agent depends on both productivity and the level of assets. We will see in the occupation map in the next item that there is a small change in the graphic, when the individual has a little ammount of assets today he is not able to contract optimal level of capital due to financial frictions and he might decide to be a worker.

In the case of financial friction the demand for capital might change if the friction is binding.

$$k = \lambda a \tag{6}$$

and the demand for labor is similar to he case in item a. Dividing the equation of w by the equation of r and substituting we can find l as a function of k. So we can write

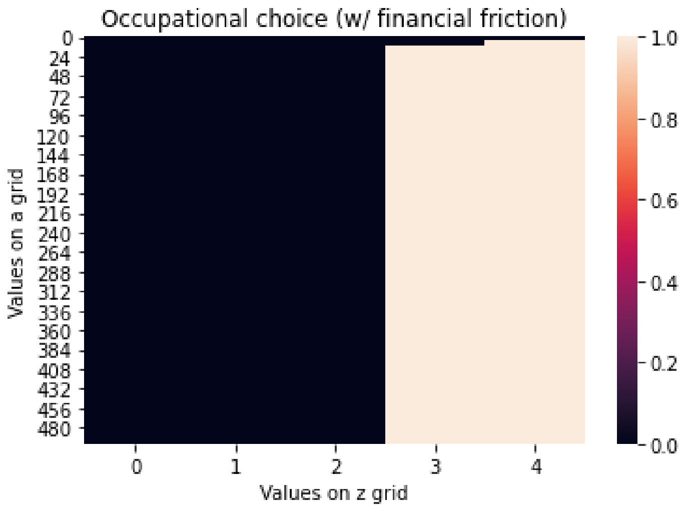
$$l = \left(\frac{w}{\theta z k^{\alpha}}\right)^{\dfrac{1}{\theta - 1}} \tag{7}$$

Considering the budget constraint of the entrepreneur presented before,

$$c_t = z k_t^{\alpha} l^{\theta} - r k_t - w l_t + a_t(1 + r) - a_{t+1} \tag{8}$$

in the case that the financial friction is binding we will have a demand for k that is not fully met. We can substitute k from equation (6) in the profit function to consider the new value function. Than we will have $V_e(a, z) = u(\pi + a(1 + r) - a') + \beta E_0[\gamma V(a', z) + (1 - \gamma)V(a', z')]$.

(e)



(f) The code is in the file attached. We can see in the figure below that the difference from the one in item (c) is subtle. From the heat map of occupational choice with friction we see that only for small values of assets there is some difference in the individual's choice due to the imposition of credit constraint, this fact is also reflected in the policy function.