Macro III: Problem Set 3
Deadline: Monday, 28/09/2020

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September 2020

1. **Aiyagari Model.** Time is discrete and indexed by $t = 0, 1, 2...$ Let $\beta \in (0,1)$ be the subjective discount factor, $c_t \geq 0$ be consumption at period t and l_t be labor supply at t. Agents are *ex-ante* identical and have the following preferences:

   Preferences:
   $$E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma_c}}{1-\sigma_c} + \gamma \frac{(1-l_t)^{1-\sigma_l}}{1-\sigma_l} \right) \right],$$
   where $\sigma_c, \ \sigma_l > 1, \gamma > 0$. Expectations are taken over an idiosyncratic shock, z_t, on labor productivity, where
   $$\ln(z_{t+1}) = \rho \ln(z_t) + \epsilon_{t+1}, \ \ \rho \in [0,1].$$
   Variable ϵ_{t+1} is an iid shock with zero mean and variance σ_ϵ^2. Markets are incomplete as in Huggett (1993) and Aiyagari (1994). There are no state contingent assets and agents trade a risk-free bond, a_{t+1}, which pays interest rate r_t at period t. In order to avoid a Ponzi game, we impose a natural borrowing limit.

   Technology: There is no aggregate uncertainty and the technology is represented by $Y_t = K_t^{\alpha} N_t^{1-\alpha}$. Let I_t be investment at period t. Capital evolves according to:
   $$K_{t+1} = (1-\delta)K_t + I_t.$$
   Let $\delta = 0.08$, $\beta = 0.96$, $\alpha = 0.4$, $\gamma = 0.75$ and $\sigma_c = \sigma_l = 2$.

   (a) Use a finite approximation for the autoregressive process
   $$\ln(z') = \rho \ln(z) + \epsilon.$$
   where ϵ' is normal iid with zero mean and variance σ_ϵ^2. Use a 7 state Markov process spanning 3 standard deviations of the log wage. Let ρ be equal to 0.98 and assume that $\sigma_z^2 = \frac{\sigma_\epsilon^2}{1-\rho^2} = 0.621$. Simulate this shock and report results.

(b) State the households' problem.

(c) State the representative firm's problem.

(d) Define the recursive competitive equilibrium for this economy.

(e) Write down a code to solve this problem. Find the policy functions for a', c, and l.

(f) Solve out for the equilibrium allocations and compute statistics for this economy. Report basic statistics about this economy, such as: the capital-to-output ratio, cumulative distribution of income (e.g., bottom 1%, 5%, 10%, 50%, top 1%, top 5%, top 10%), cumulative distribution of wealth (e.g., bottom 1%, 5%, 10%, 50%, top 1%, top 5%, top 10%).

2. **Hopenhayn model.** On paying a fixed operating cost $\kappa > 0$, an incumbent firm that hires n workers produces flow output $y = zn^{\alpha}$ with $0 < \alpha < 1$ where $z > 0$ is a firm-level productivity level. The productivity of an incumbent firm evolves according to an AR(1) in logs

$$\ln(z_{t+1}) = (1-\rho)\ln(\bar{z}) + \rho\ln(z_t) + \sigma\epsilon_{t+1},\ \rho \in (0,1),\ \sigma > 0$$

where $\epsilon_{t+1} \sim N(0,1)$. Firms discount flow profits according to a constant discount factor $0 < \beta < 1$. There is an unlimited number of potential entrants. On paying a sunk entry cost $\kappa_e > 0$, an entrant receives an initial productivity draw $z_0 > 0$ and then starts operating the next period as an incumbent firm. For simplicity, assume that initial productivity z_0 is drawn from the stationary productivity distribution implied by the AR(1) above.

Individual firms take the price p of their output as given. Industry-wide demand is given by the $D(p) = \bar{D}/p$ for some constant $\bar{D} > 0$. Let labor be the numeraire, so that the wage is $w = 1$. Let $\pi(z)$ and $v(z)$ denote respectively the profit function and value function of a firm with productivity z. Let v_e denote the corresponding expected value of an entering firm. Let $\mu(a)$ denote the (stationary) distribution of firms and let m denote the associated measure of entering firms.

(a) Derive an expression for the profit function.

(b) Set the parameter values $\alpha = 2/3$, $\beta = 0.8$, $\kappa = 20$, $\kappa_e = 40$, $\ln(\bar{z}) = 1.4$, $\sigma = 0.20$, $\rho = 0.9$ and $\bar{D} = 100$. Discretize the AR(1) process to a Markov chain on 33 nodes. Solve the model on this grid of productivity levels. Calculate the equilibrium price p^* and measure of entrants m^* . Let z^* denote the cutoff level of productivity below which a firm exits. Calculate the equilibrium z^* . Plot the stationary distribution of firms and the implied distribution of employment across firms. Explain how these compare to the stationary distribution of productivity levels implies by the AR(1).

(c) Now suppose the demand curve shifts, with $\bar{D}$ increasing to 120. How does this change the equilibrium price and measure of entrants? How does this change the stationary distributions of firms and employment? Give intuition for your results.

3. **Ramsey model in continuous time.** Consider the decentralised Ramsey model in continuous time. Households solve

$$\max_{c,l} \int_0^\infty e^{-\rho t} u(c,l) dt,$$

subject to

$$\dot{a} = w(1-l) + ra - c,$$

where c is consumption, a denotes assets and l is leisure. ρ is the subjective discount rate, r is the interest rate and w is the wage rate. Firms rent capital and labour from households to maximise

$$\max AK^\alpha N^{1-\alpha} - wN - (r+\delta)K,$$

where δ is the depreciation rate and A is a productivity factor.

(a) Write down the HJB associated with the households problem. Explain the steps to derive it.

From now on, assume that $u(c,l) = \log(c) + \eta \log(l)$. You can assume that $\rho = 0.04$ and $\eta = 0.75$

(b) For a given interest rate r and wage rate w, write down a code to solve the households problem.

(c) Write down the market clearing conditions.

Assume that $\delta = 0.06$, $A = 1$ and $\alpha = 0.33$.

(d) Write down the equations that describe the steady-state of the system and solve for the steady-state level of capital and labour supply.

(e) Write down a code to solve out the whole transition. Then simulate a permanent change in the TFP factor, such that A increases from $A = 1$ to $A = 1.2$. Plot the evolution of capital, labour and consumption.