Macroeconomics III - Problem Set 1

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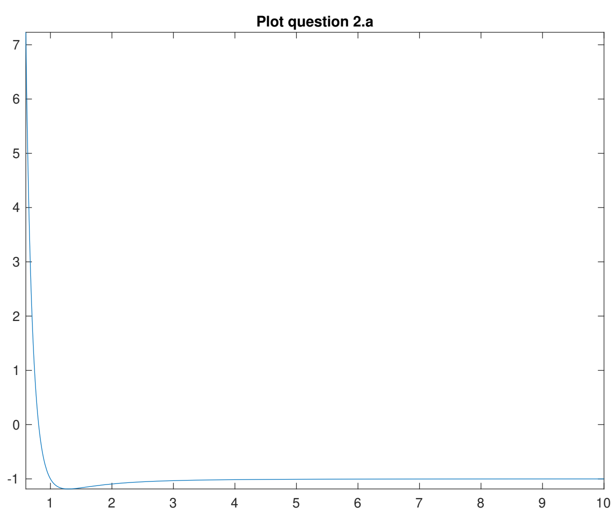
Question 1 - Bisection Method

This question is solved in the file **Question1.m** and the function bisection is in the file **bisection.m**. The answer is 2.2087.

Question 2 - Newton's Method

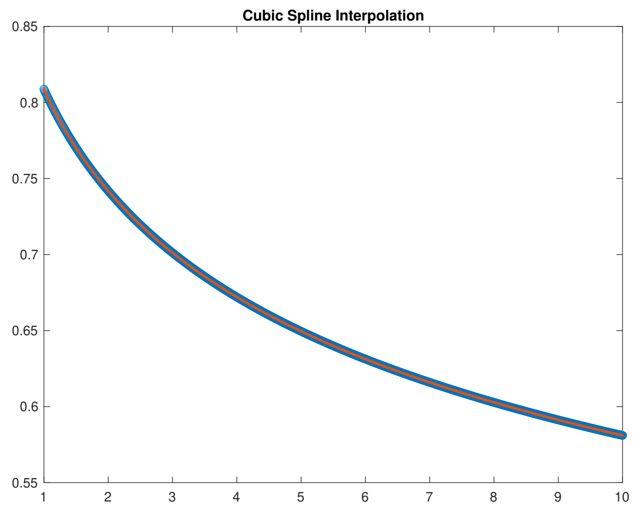
This question is solved in the file **Question2.m**, in order to solve the first three items I call **f2a.m** and to solve the last item **f2e.m**

(a) The answer to x^* such that $d(x^*) = 0$ is 0.8087.

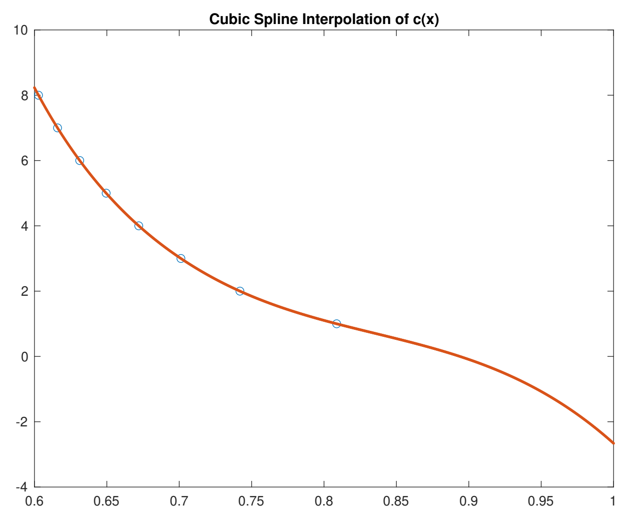


(b) Aswer contained in the file **Question2.m**.

(c) Solution using spline and 1000 nodes from 1 to 10.



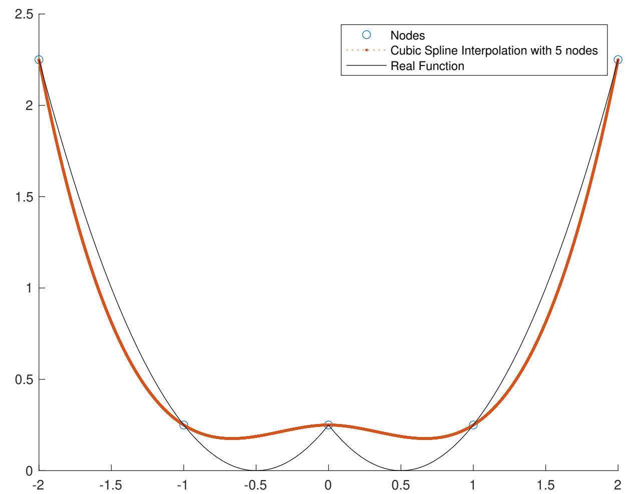
(d) Inverse function c(x).



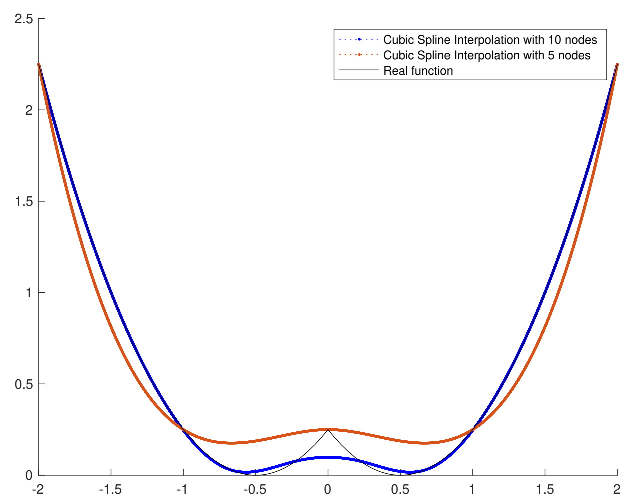
(e) For this item I also use the file **f2e.m**. The solution is 3.3173e-07.

Question 3 - Approximation Methods: Finite Element Methods

(a) The root of the mean square error aproximation is 0.1380.



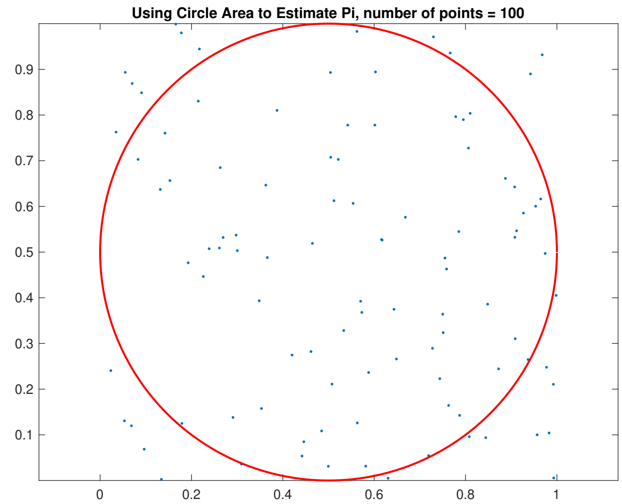
(b) The root of the mean square error aproximation is 0.0276. Since we have more nodes we get accuracy to construct our interpolation, then the mean square error in item b is lower than in item a.



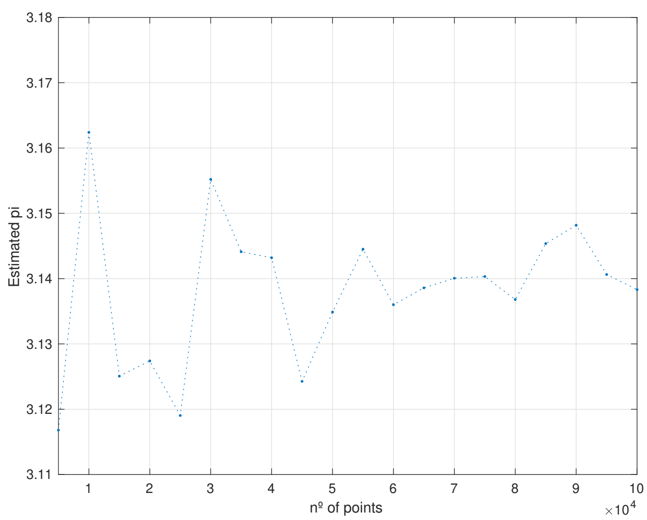
Question 4 - Estimating π

(a) This item is solved in file **Question4.m**.

(b) Randon draw of 100 points inside the square.



(c) Estimates for π using different amounts of random draws (n = 5000; 10000; 15000; ...; 100000).



Question 5 - Two Period Model

(a), (b) First consider the problem of the agent

$$\text{Max} \sum_{0}^{1} \beta^t (u(c_t) + \gamma v(l_t)), \quad \gamma > 0 \text{ and } \beta \in (0,1) \quad \text{subject to}$$

$$c_0 + a_1 \leq w_0 h_0 + \Pi_0$$

$$c_1 \leq w_1 h_1 + (1+r) a_1 + \Pi_1$$

Since the utility function is increasing the equality above at the budget constraint holds. From the two budget constraints we can calculate the intertemporal budget constraint,

$$c_0 + \frac{c_1}{1+r} = w_0 h_0 + \frac{w_1 h_1}{1+r} + \Pi_0 + \frac{\Pi_1}{1+r} \tag{1}$$

Considering that $l_t + h_t = 1$ we can find the first order condition of our problem taking the derivative with respect to c_0, c_1, h_0 and h_1 of the lagrange,

$$\sum_0^1 \beta^t (u(c_t) + \gamma v(l_t)) + \lambda [w_0 h_0 + \frac{w_1 h_1}{1+r} + \Pi_0 + \frac{\Pi_1}{1+r} - c_0 - \frac{c_1}{1+r}] \tag{2}$$

$$[c_0]: \ u'(c_0) = \lambda \tag{3}$$
$$[c_1]: \ \beta u'(c_1) = \frac{\lambda}{1+r} \tag{4}$$
$$[h_0]: \ \gamma v'(1-h_0) = \lambda w_0 \tag{5}$$
$$[h_1]: \ (1+r)\gamma\beta v'(1-h_1) = \lambda w_1 \tag{6}$$

Rearranging we get the Euler equations,

$$\left[\frac{c_1}{c_0}\right]^{\sigma} = \beta(1+r) \tag{7}$$
$$(1+r)\beta \left[\frac{1-h_1}{1-h_0}\right]^{-\sigma} = \frac{w_1}{w_0} \tag{8}$$

Now, consider the firms problem, it's given by

$$\Pi_t = Max_{L_t} \sum_{t=0}^{1} A_t (L_t)^{\alpha} - w_t H_t \tag{9}$$

The first order condition are

$$w_t = (1-\alpha) A_t L^{\alpha-1} \tag{10}$$

Then we rearrange and write the equtions and market clearing conditions in terms of w_0 and w_1 to solve the problem numerically.

$$L_0 = \left[\frac{w_0}{A_0 \alpha}\right]^{\frac{1}{\alpha-1}}, \qquad L_1 = \left[\frac{w_1}{A_1 \alpha}\right]^{\frac{1}{\alpha-1}} \tag{11}$$
$$y_0 = A_0 \left[\frac{w_0}{A_0 \alpha}\right]^{\frac{\alpha}{\alpha-1}}, \qquad y_1 = A_1 \left[\frac{w_1}{A_0 \alpha}\right]^{\frac{\alpha}{\alpha-1}} \tag{12}$$
$$L0 = (1-l_0), \qquad L1 = (1-l_1) \tag{13}$$
$$\tag{14}$$

Using the Euler equation (7), we can write

$$c_1 = [\beta(1+r)]^{\frac{1}{\sigma}} c_0 \tag{15}$$

Substituting the expression above in the intertemporal budget set (1) we have c_0 as a function of w_0 and w_1, that is

$$c_0 = \left[w_0 \left[\frac{w_0}{A_0 \alpha} \right]^{\frac{1}{\alpha - 1}} + \frac{w_1}{1+r} \left[\frac{w_1}{A_1 \alpha} \right]^{\frac{1}{1-\alpha}} \right] \frac{1+r}{1+r+[\beta(1+r)]^{\frac{1}{\sigma}}} \tag{16}$$

Now we can use the equation above, c_0 in (15) and substitute in

$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r} \tag{17}$$

Along with (8) we have two equation as a function of w_0 and w_1 to solve.

(c)-(f) I have explained in the last item the procedure to solve numerically. More explanation and the exercise solved is the file **Question5.m**, where items (c) to (f) are solved. The functions are in files **foc5.m** and **modelsolution.m**. First, considering A_0 lower, the model solution for w_0 implies lower wage, since now we have less productivity, the total output in time 0 is lower and also the aggregate output from both periods. Lower wages implies that agents will work less. Since agents work less, they will also consume less but it does not mean a higher a_1. Actually, the total assets is lower now and individual expend more in leisure.

Now considering A_0 at the initial level and A_1 lower, $A_1 = 0.9$. Since productivity is higher in period 0 than in period 1, individuals reallocate work to period 0. As a result of this reallocation, output and consumption rises in period zero. Also, the amount of asset now is higher, implying that agents save more to smooth consumption between periods.

When we have a lower consumption and labor elasticity, that is $\sigma = 1,5$, wages in period 0 and 1, in equilibrium are lower since agents offer work more due to the change in the utility function. When we do the analysis as the items before we see a similar behavior for the variables but now in a lower level of wages. Finally, consumption and aggregate output rises since agents offer more labor.

Question 6 - Growth Model

(a)

I will define a competitive equilibrium for this economy and write down the equations that describe it. First of all, consider the problem of the households.

$$Max \sum_{t=0}^{\infty} \beta^t N_t [ln(c_t) + \theta ln(1-h_t)] \qquad \text{subject to} \tag{1}$$

$$N_t c_t (1-\tau^c) + k_{t+1} = (1 - \delta + r_t(1-\tau^k)k_t + w_t h_t N_t (1-\tau^h) + \Pi \tag{2}$$

where Π represents profits of the firms. Also, we have that $N_{t+1} = (1+\eta)N_t$, $h_t + l_t = 1$ and initial capital stock k_0 is given. The evolution of capital is given by

$$K_{t+1} = (1-\delta)K_t + I, \qquad \delta \in (0,1) \tag{3}$$

Considering variables per capita we have

$$(1+\eta)k_{t+1} = (1-\delta)k_t + i_t \tag{4}$$

Hence the first order conditions are

$$[c_t]: \ \frac{1}{(1-\tau^c)c_t} = \lambda_t \tag{5}$$

$$[h_t]: \ \frac{\theta}{1-h_t} = w_t\lambda_t(1-\tau^h) \quad \Rightarrow \quad \lambda_t = \frac{\theta}{(1-h_t)w_t(1-\tau^h)} \tag{6}$$

$$[k_{t+1}]: \ \lambda_t = \beta\lambda_{t+1}[1-\delta+r_{t+1}(1-\tau^k)] \tag{7}$$

Now, consider the firms problem, it's given by

$$\Pi = Max_{K_t,H_t}\sum_{t=0}^{\infty} K_t^{\alpha}(A_tH_t)^{1-\alpha} - r_tK_t - w_tH_t \tag{8}$$

The first order conditions are

$$r_t = \alpha K_t^{\alpha-1}(A_tH_t)^{1-\alpha} \tag{9}$$

$$w_t = (1-\alpha)K_t^{\alpha}A_t^{1-\alpha}H^{-\alpha} \tag{10}$$

Also, every period the government balance it's budget

$$G_t = \tau^c N_t c_t + \tau^h w_t N_t h_t + \tau^k r_t k_t \tag{11}$$

The equilibrium is this economy is characterized by the sequence $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ and prices $\{\lambda_t, w_t, r_t\}$ such that households maximizes utility, firms maximize profits and markets clear. Then, we must have that

$$H_t = N_t h_t \tag{12}$$

$$K_{t+1} = (1-\delta)K_t + I \tag{13}$$

$$Y_t = N_t c_t + I_t + G_t \tag{14}$$

Manipulating the Euler equation and from the first order conditions we get that

$$\frac{1}{(1-\tau^c)c_t} = \frac{\theta}{(1-h_t)w_t(1-\tau^h)} \quad \Rightarrow \quad w_t = \frac{\theta(1-\tau^c)c_t}{(1-h_t)(1-\tau^h)} \tag{15}$$

$$\lambda_t = \beta\lambda_{t+1}[1-\delta+r_{t+1}(1-\tau^k)] \quad \Rightarrow \quad \frac{(1+\gamma)\tilde{c}_{t+1}}{\tilde{c}_t} = \beta[(1-\tau^k)\alpha\tilde{k}_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} + (1-\delta)] \tag{16}$$

where $\tilde{c}_t = \frac{C_t}{A_tN_t}$.

Now, from the equation for capital accumulation (4) we have that

$$(1+\gamma)(1+\eta)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{i}_t \tag{17}$$

Now substituting the government expending (11) in the resource constraint (14) and dividing everyting by A_tN_t yields

$$\tilde{k}_t^{\alpha}h_t^{1-\alpha} = \tilde{c}_t + \tilde{i}_t + \tau^c\tilde{c}_t + \tau^h\tilde{w}_t h_t + \tau^k r_t\tilde{k}_t \tag{18}$$

Considering (9), (10) and $\tilde{i}_t$ from (17), we have

$$(1+\gamma)(1+\eta)\tilde{k}_{t+1} + (1+\tau^c)\tilde{c}_t = (1-\delta)\tilde{k}_t + \tilde{k}_t^{\alpha}h_t^{1-\alpha}[1-\tau^h(1-\alpha)-\tau^k\alpha] \tag{19}$$

Therefore, from (15), (16) and (19) the following equations characterize the equilibrium in this economy

$$\frac{\theta(1+\tau^c)\tilde{c}_t}{1-h_t} = (1-\tau^h)(1-\alpha)\tilde{k}_t^{\alpha}h_t^{-\alpha} \tag{20}$$

$$\frac{(1+\gamma)\tilde{c}_{t+1}}{\tilde{c}_t} = \beta[(1-\tau^k)\alpha\tilde{k}_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} + (1-\delta)] \tag{21}$$

$$(1+\gamma)(1+\eta)\tilde{k}_{t+1} + (1+\tau^c)\tilde{c}_t = (1-\delta)\tilde{k}_t + \tilde{k}_t^{\alpha}h_t^{1-\alpha}[1-\tau^h(1-\alpha)-\tau^k\alpha] \tag{22}$$

(b)

Now I will find the variables growth rate along the balanced growth path equilibrium. The question state that $A_{t+1} = A_t(1+\gamma)$ and note that

$$\frac{c_{t+1}}{c_t} = g_c \qquad \frac{h_{t+1}}{h_t} = g_h \qquad \frac{k_{t+1}}{k_t} = g_k \qquad \frac{w_{t+1}}{w_t} = g_w \qquad \frac{r_{t+1}}{r_t} = g_r \tag{23}$$

From the Euler equation (16) we have

$$\frac{c_{t+1}}{c_t} = \beta(1-\delta) + (1-\tau^k)(\alpha K_{t+1}^{\alpha-1}(A_{t+1}N_{t+1}h_{t+1})^{1-\alpha})) \tag{24}$$

Since we know that in the BGP $\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = 1$, dividing the expression above by the same one in period t we have the following equation

$$\frac{K_{t+1}}{K_t} = (1+\gamma)(1+\eta)\frac{h_{t+1}}{h_t} \tag{25}$$

Since we must have that $g_h = \frac{h_{t+1}}{h_t} = 1$, because otherwise the marginal utility of leisure would go to infinity, then $g_K = (1+\gamma)(1+\eta)$ and $g_k = (1+\gamma)$

From (20) and (10), $g_c = 1+\gamma$. Finally, from (9) and (10) we have

$$g_w = \frac{w_{t+1}}{w_t} = \frac{K_{t+1}^{\alpha}A_{t+1}^{1-\alpha}H_{t+1}^{-\alpha}}{K_t^{\alpha}A_t^{1-\alpha}H_t^{-\alpha}} = 1+\gamma \tag{26}$$

$$g_r = \frac{r_{t+1}}{r_t} = \frac{K_{t+1}^{\alpha-1}(A_t+1H_t+1)^{1-\alpha}}{K_t^{\alpha-1}(A_tH_t)^{1-\alpha}} = 1 \tag{27}$$

(c)

In order to consider stationary variables I will work with "tilde" above the variable, as I have been doing in the previous items, i.e $\tilde{y}_t = \frac{Y_t}{A_tN_t}$. Also, I will omit the subscripts to consider the variable in steady state. I will work mainly with equation (20), (21) and (22), defined in item a. From (21) we have

$$\frac{(1+\gamma)\tilde{c}}{\tilde{c}} = \beta[\alpha(1-\tau^k)\tilde{k}^{\alpha-1}h^{1-\alpha} + 1 - \delta]$$

$$\tilde{h} = [\frac{1+\gamma-(1-\delta)\beta}{B\alpha(1-\tau^k)}]^{\frac{1}{1-\alpha}}\tilde{k} \quad \Rightarrow \quad h = \Upsilon\tilde{k} \tag{28}$$

where $\Upsilon = [\frac{1+\gamma-(1-\delta)\beta}{B\alpha(1-\tau^k)}]^{\frac{1}{1-\alpha}}$.

From (22) we can isolate $\tilde{c}$ to write

$$\tilde{c} = \frac{\tilde{k}^{\alpha}h^{1-\alpha}[1-\tau^h(1-\alpha)-\tau^k\alpha]-\tilde{k}\Delta}{(1+\tau^c)} \tag{29}$$

Substituting $\tilde{c}$ and h in (20) yields

$$\tilde{k} = \frac{(1-\alpha)(1-\tau^h)\Upsilon^{-\alpha}}{(1-\alpha)(1-\tau^h)\Upsilon^{1-\alpha}+\theta(\Upsilon^{1-\alpha}(1-\tau^h(1-\alpha)-\tau^k\alpha)-\Delta)} \tag{30}$$

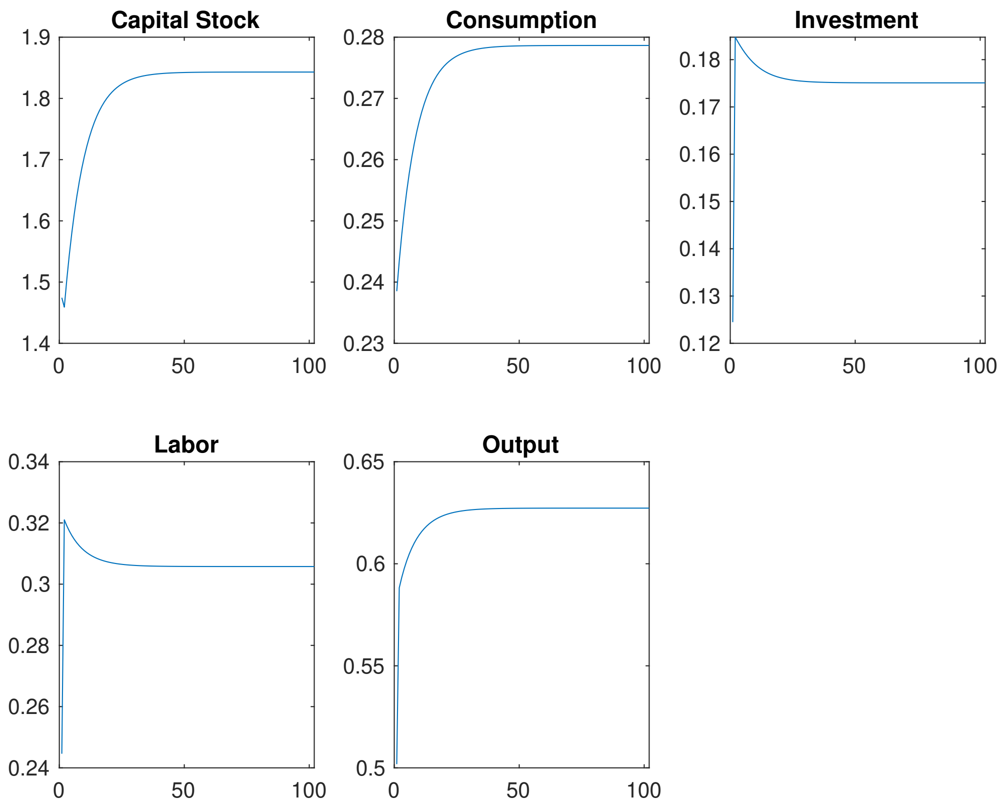
where $\Delta = \delta+\gamma+\eta+\eta\gamma$.

Finally, from the government equation (11) we have

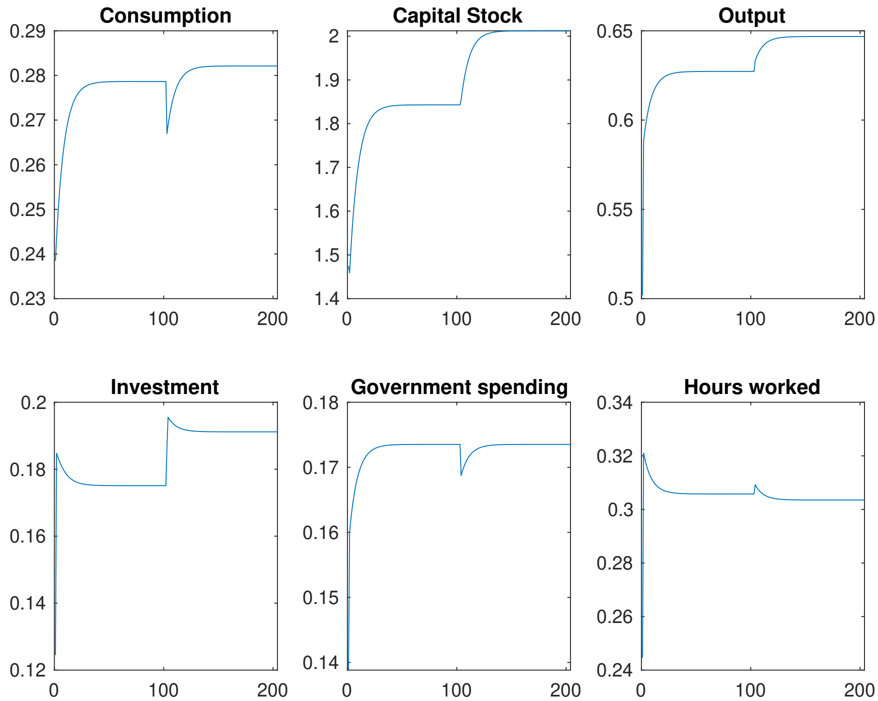
$$\tilde{g} = \tau^c\tilde{c}+\tau^h(1-\alpha)\tilde{k}^{\alpha}h^{1-\alpha}+\tau^k\alpha\tilde{k}^{\alpha}h^{1-\alpha} \tag{31}$$

(d) This item is solved in file **Question6.m**.

(e) Dynamics of capital, consumption, investment, labor and output per capita.



(f) The figures below show the dynamics of the economy after the tax on capital income is reduced from 0.15 to 0.10. As we could expect, it is clear that the new state steady state for capital is higher than the economy in item (d), since there is incentive to increase capital. At the moment of the change in tax we can see that consumption drops and investment increases, supporting the accumulation of capital and the new steady state. After some time, consumption stabilizes at a new and higher level, given the rise in output. As required in the question, the government spending remains the same in steady state and since the lower tax on capital is compensated with tax on income from labor, agents work less in the new steady state.



(g) The consumer must be compensated in 1.23%. From the figure below, as stated in the last exercise, the dynamics of the transition implies that the consumption drops and then stabilize in a higher level.