Linear Algebra

Week 1: System of linear equations

System of sentences

- **Complete**: there are as many equations as there are unknowns
  e.g., (x, y, z) would have 3 equations
- **Redundant**: some equations don't provide new information

$$\begin{aligned} 2x+y&=5 \\ 4x+2y&=10 \end{aligned} \longrightarrow \text{redundant}$$

- **Contradictory**: no solution, inconsistent equations

$$\begin{aligned} x+y&=5 \\ x+y&=6 \end{aligned} \rightarrow \text{no solution}$$

Singular vs. **Non-singular**

- **does not** have a unique solution
- **does** have a unique solution

System of equations

① $\begin{cases} x+y=10 \\ x+2y=12 \end{cases} \rightarrow \begin{aligned} \cancel{-x}-y&=-10 \\ \cancel{x}+2y&=12 \end{aligned} \rightarrow y=2 \rightarrow x+2=10 \rightarrow x=8$

complete
non-singular

② $\begin{cases} x+y=10 \\ 2x+2y=20 \end{cases} \longrightarrow$ there is not enough information (Redundant)
infinitely many solutions

redundant
singular

③ $\begin{cases} x+y=10 \\ 2x+2y=24 \end{cases} \rightarrow$ not possible to solve

contradictory
singular

What is a linear equation?

Linear

$a + b = 10$

$2a + 3b = 15$

$\boxed{3.4}a + \boxed{48.99}b + \boxed{2}c = \boxed{122.5}$

Numbers (scalars)

Non-linear

$a^2 + b^2 = 10$

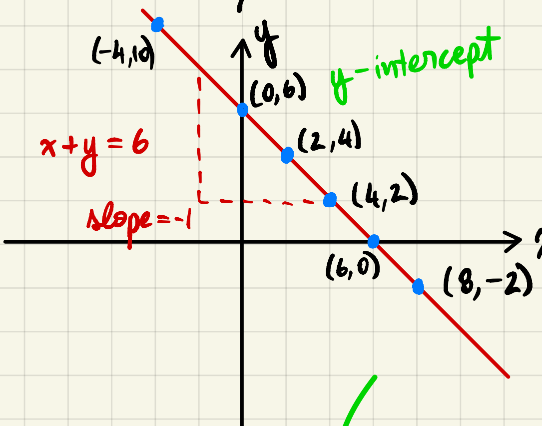
$\sin(a) + b^5 = 15$

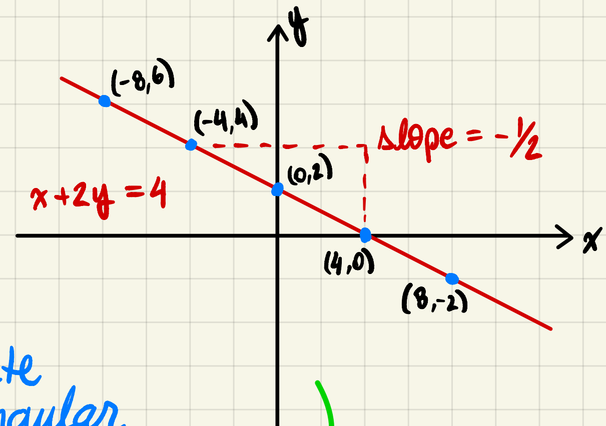
$2^a - 3^b = 0$

$\vdots$

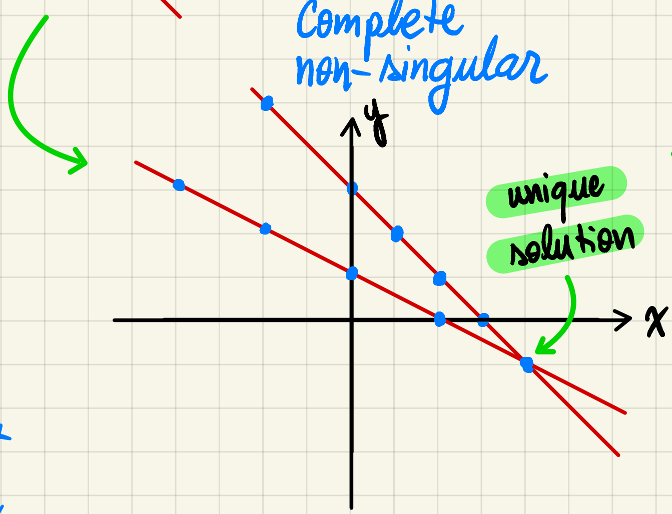
System of equations as lines

linear equation → line

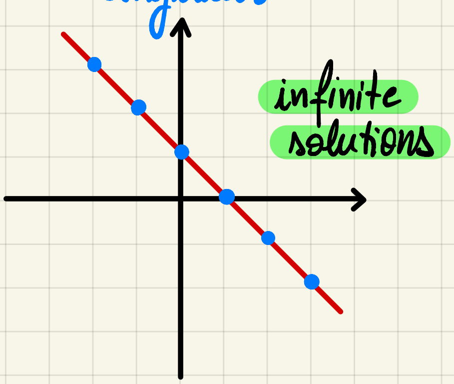




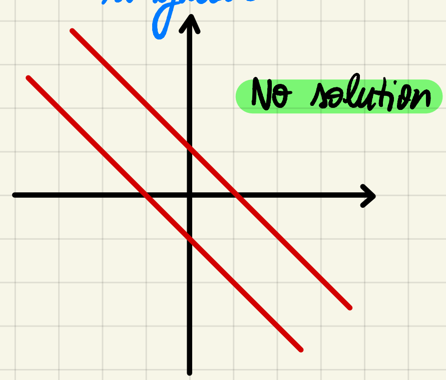
Complete
non-singular



Redundant
singular



Contradictory
singular

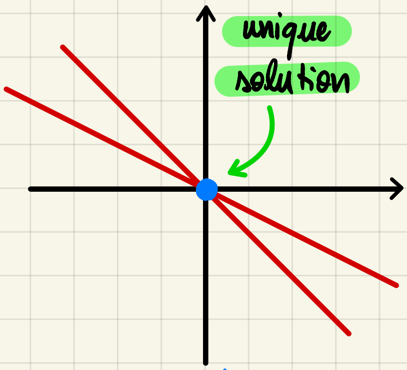


$$\begin{cases} 3a + 2b = 8 \\ 2a - b = 3 \end{cases} \rightarrow \begin{matrix} 3a + \cancel{2b} = 8 \\ 4a - \cancel{2b} = 6 \end{matrix} \rightarrow \begin{matrix} 7a = 14 \\ a = 2 \end{matrix} \qquad \begin{matrix} 3 \cdot 2 + 2b = 8 \\ 2b = 2 \end{matrix} \quad b = 1$$

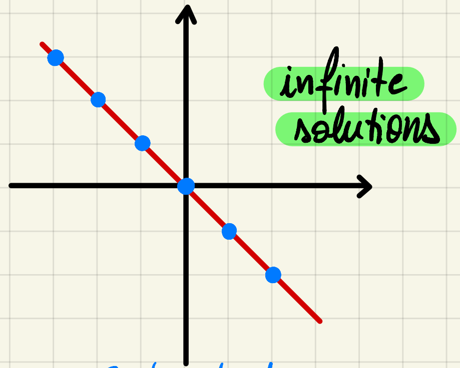
Complete, non-singular

Geometric notion of singularity

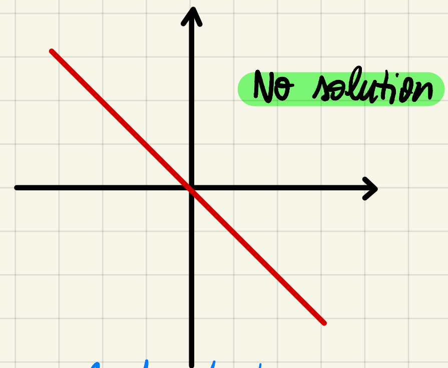
$$\begin{matrix} a + b = \boxed{0} \\ a + 2b = \boxed{0} \end{matrix} \qquad \begin{matrix} a + b = \boxed{0} \\ 2a + 2b = \boxed{0} \end{matrix} \qquad \begin{matrix} a + b = \boxed{0} \\ 2a + 2b = \boxed{0} \end{matrix}$$



Complete
non-singular



Redundant
singular



Contradictory
singular

- when the constants become zero the systems of linear equations pass through the origin
- constants don't determine singularity

System of equations as matrices

$$\begin{matrix} a + b \\ a + 2b \end{matrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{matrix} a + b \\ 2a + 2b \end{matrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Linear dependence and independence

$$\begin{matrix} a + b \\ 2a + 2b \end{matrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{matrix} a + b \\ a + 2b \end{matrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

rows are linearly dependent

rows are linearly independent

The determinant

A matrix is singular if:

① $[a \ \ b] \cdot K = [c \ \ d]$

② $a \cdot K = c$
$b \cdot K = d$

③ $\frac{a}{c} = \frac{b}{d} = K$

④ $ad = bc$

⑤ $ad - bc = 0$

Determinant

E.g.,
$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
$ad - bc = 0$
$1 \cdot 2 - 1 \cdot 1 = 1$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
$1 \cdot 2 - 2 \cdot 1$
$2 - 2 = 0$

Determinant

- non-zero when matrix is non-singular
- zero when matrix is singular

Find the determinant of the following matrices:

① $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$

$5 \cdot 3 - (-1 \cdot 1)$
$15 + 1 = \underline{\underline{16}}$

② $\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

$2 \cdot 3 - (-6 \cdot -1)$
$6 - 6 = \underline{\underline{0}}$

Practice Quiz

①

$2b + 3m = 15$
$2b + 4m = 16$

② $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 15 \\ 16 \end{bmatrix}$

③ Determinant of the matrix

$\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} = 2 \cdot 4 - 2 \cdot 3 = \underline{\underline{2}}$

non-singular

④ Linear independent

⑤ Solve the system of equations

$\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 15 \\ 16 \end{bmatrix}$

$2x + 3y = 15$
$2x + 4y = 16$

$\cancel{-2x} - 3y = -15$
$\cancel{2x} + 4y = 16$

$\underline{\underline{y = 1}}$

$2x + 3y = 15$

$2x + 3 = 15$
$2x = 12$
$\underline{\underline{x = 6}}$

System of equations 3×3

① $\begin{cases} a + b + c = 10 \\ a + 2b + c = 15 \\ a + b + 2c = 12 \end{cases}$ $a = 3$ $b = 5$ $c = 2$ Complete non-singular

② subtract eq.1 from eq.2

$$\begin{array}{r} -a - b - c = -10 \\ a + 2b + c = 15 \\ \hline 0 + b + 0 = 5 \\ b = 5 \end{array}$$

③ subtract eq.1 from eq.3

$$\begin{array}{r} -a - b - c = -10 \\ a + b + 2c = 12 \\ \hline 0 + 0 + c = 2 \\ c = 2 \end{array}$$

④ substitute b an c in the first eq. to solve for a

$$\begin{array}{r} a + b + c = 10 \\ a + 5 + 2 = 10 \\ a = 3 \end{array}$$

Solve the following systems:

① $\begin{cases} a + b + c = 10 \\ a + b + 2c = 15 \\ a + b + 3c = 20 \end{cases}$

$$\begin{array}{r} -\cancel{a} - \cancel{b} - c = -10 \\ \cancel{a} + \cancel{b} + 2c = 15 \\ \hline c = 5 \end{array}$$

$$\begin{array}{r} a + b + c = 10 \\ a + b + 5 = 10 \\ a + b = 5 \end{array}$$

Infinitely many solutions

Redundant singular

② $\begin{cases} a + b + c = 10 \\ a + b + 2c = 15 \\ a + b + 3c = 18 \end{cases}$

$$\begin{array}{r} -\cancel{a} - \cancel{b} - c = -10 \\ \cancel{a} + \cancel{b} + 2c = 15 \\ \hline c = 5 \end{array}$$

$$\begin{array}{r} a + b + 5 = 10 \\ a + b = 5 \end{array}$$

$$\begin{array}{r} a + b + 3c = 18 \\ a + b + 15 = 18 \\ a + b = 3 \end{array}$$

Contradictory singular

No solutions!

③ $\begin{cases} a + b + c = 10 \\ 2a + 2b + 2c = 20 \\ 3a + 3b + 3c = 30 \end{cases}$

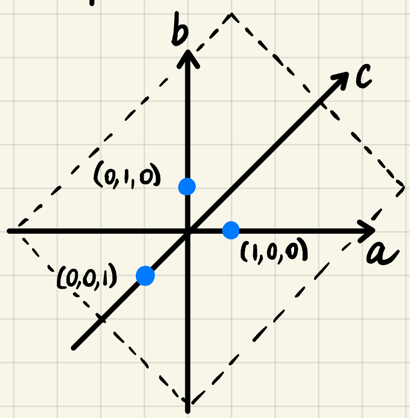
Infinite solutions

Redundant singular

⊛ constants don't matter for singularity

System of equations as planes

linear equation in 3 variables → plane



$a+b+c=1$
$1+0+0=1$
$0+1+0=1$
$0+0+1=1$

Linear dependence and independence

① $a=1$ → $a+0b+0c=1$
$b=2$ → $0a+b+0c=2$
$a+b=3$ → $a+b+0c=3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Row 1 + Row 2 = Row 3

Row 3 depends on Rows 1 and 2

Rows are linearly dependent

② $a+b+c=0$ → $a+b+c=0$
$a+b+2c=0$
$a+b+3c=0$ → $a+b+3c=0$

$2a+2b+4c=0$
$\downarrow \div 2$
$a+b+2c=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Average of Row 1 and 3 is Row 2

Row 2 depends on rows 1 and 3

Rows are linearly dependent

③ $a+b+c=0$
$a+2b+c=0$
$a+b+2c=0$

No relations between equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

No relation between rows

Rows are linearly independent

Quiz: determine if matrices are linearly dependent / independent

① $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix}$

$$a + 0b + c = 0$$
$$0a + b + 0c = 0$$
$$3a + 2b + 3c = 0$$

$$\begin{aligned} + \quad 3 \cdot a + 0 + 3c &= 0 \\ 0 + 2b + 0 &= 0 \\ \hline 3a + 2b + 3c &= 0 \end{aligned}$$

Dependent

② $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

$$a + b + c = 0$$
$$a + b + 2c = 0$$
$$0 + 0 - c = 0$$

$$\begin{aligned} \cancel{a} + \cancel{b} + c &= 0 \\ -\cancel{a} - \cancel{b} - 2c &= 0 \\ \hline -c &= 0 \end{aligned}$$

Dependent

③ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$$a + b + c = 0$$
$$0 + 2b + 2c = 0$$
$$0 + 0 + 3c = 0$$

Independent

④ $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 2 & 4 & 10 \end{bmatrix}$

$$a + 2b + 5c = 0$$
$$0 + 3b - 2c = 0$$
$$2a + 4b + 10c = 0$$

$$2(a + 2b + 5c) = 0$$
$$2a + 4b + 10c = 0$$

Dependent

Determinant 3x3

Diagonals in 3x3 matrix

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\begin{bmatrix} \boxed{1} & & \\ & \boxed{2} & \\ & & \boxed{2} \end{bmatrix}$ $\begin{bmatrix} & \boxed{1} & \\ & & \boxed{1} \\ \boxed{1} & & \end{bmatrix}$ $\begin{bmatrix} & & \boxed{1} \\ \boxed{1} & & \\ & \boxed{1} & \end{bmatrix}$

$+1 \cdot 2 \cdot 2$ $\quad +1 \cdot 1 \cdot 1$ $\quad +1 \cdot 1 \cdot 1$

$$Det = 4 + 1 + 1$$
$$-2 - 1 - 2$$

$Det = 1$

$\begin{bmatrix} & & \boxed{1} \\ & \boxed{2} & \\ \boxed{1} & & \end{bmatrix}$ $\begin{bmatrix} \boxed{1} & & \\ & & \boxed{1} \\ & \boxed{1} & \end{bmatrix}$ $\begin{bmatrix} & \boxed{1} & \\ \boxed{1} & & \\ & & \boxed{2} \end{bmatrix}$

$-1 \cdot 2 \cdot 1$ $\quad -1 \cdot 1 \cdot 1$ $\quad -1 \cdot 1 \cdot 2$

Find the determinant

① $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 3 & 3 \end{bmatrix} = \begin{cases} +1 \cdot 1 \cdot 3 + 0 \cdot 0 \cdot 3 + 0 \cdot 3 \cdot 1 = 3 \\ -3 \cdot 1 \cdot 1 - 1 \cdot 3 \cdot 0 - 0 \cdot 0 \cdot 3 = -3 \end{cases}$

$= 3 - 3 = 0$ singular

② $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{cases} -1 + 0 + 0 = -1 \\ 0 - 0 + 1 = 1 \end{cases}$

$= -1 + 1 = 0$ singular

③ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{cases} 6 + 0 + 0 \\ 0 - 0 - 0 \end{cases} = 6$ non-singular

triangular matrix

④ $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 2 & 4 & 10 \end{bmatrix} = \begin{cases} 30 - 8 + 0 \\ -30 + 8 - 0 \end{cases} = 0$ singular

Quiz

①

$$2b + m + 5c = 20$$
$$b + 2m + c = 10$$
$$2b + m + 3c = 15$$

②

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 15 \end{bmatrix}$$

③ Determinant = ?

$$\text{Det} = \begin{cases} 2\cdot2\cdot3 + 2\cdot1\cdot1 + 1\cdot1\cdot5 = 12+2+5 = 19 \\ -2\cdot2\cdot5 - 2\cdot1\cdot1 - 1\cdot1\cdot3 = -20-2-3 = -25 \end{cases}$$

$$\text{Det} = 19 - 25 = -6 \text{ (non-singular)}$$

④ linearly (in)dependent?

$$\left.\begin{aligned} 2b + m + 5c &= 0 \\ b + 2m + c &= 0 \\ 2b + m + 3c &= 0 \end{aligned}\right\} \text{linearly independent}$$

⑤ solve the system of equations

$$\begin{cases} 2b + m + 5c = 20 \\ b + 2m + c = 10 \\ 2b + m + 3c = 15 \end{cases}$$

$$\cancel{2b} + \cancel{m} + 5c = 20$$
$$-\cancel{2b} - \cancel{m} - 3c = -15$$
$$2c = 5$$
$$c = 2.5$$

$$\cancel{2b} + m + 5c = 20$$
$$-\cancel{2b} - 4m - 5 = -20$$
$$-3m + 7.5 = 0$$
$$-3m = -7.5$$
$$m = 2.5$$

$$2b + m + 5c = 20$$
$$2b + 2.5 + 12.5 = 20$$
$$2b = 5 \quad b = 2.5$$

⑥

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 2 & 1 \\ x & y & z \end{bmatrix} \quad \begin{aligned} x &= 3 \\ y &= 3 \\ z &= 6 \end{aligned}$$

⑦

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix} \quad \text{Det} = \begin{cases} +1\cdot2\cdot5 + 2\cdot2\cdot1 + 0\cdot4\cdot3 \\ -1\cdot2\cdot3 - 4\cdot2\cdot1 - 0\cdot2\cdot5 \end{cases}$$

$$\text{Det} = 10 + 4 + 0$$
$$-6 - 8 - 0$$
$$4 - 4 + 0$$

Det = 0, singular

Week 1 summary

- linear equation = line
- a system of linear equations may have
  - unique (one) solution
  - infinitely many solutions
  - no solution
- line passes through the origin when the constant is zero
- Determinant finds the singularity of a matrix
  - non-zero when matrix is non-singular
  - zero when matrix is singular
- Singular vs. Non-singular

Singular	Non-singular
does not have a unique solution	does have a unique solution

- System of equations
  - Complete: many equations = many unknowns
  - Redundant: no new information
  - Contradictory: no solution