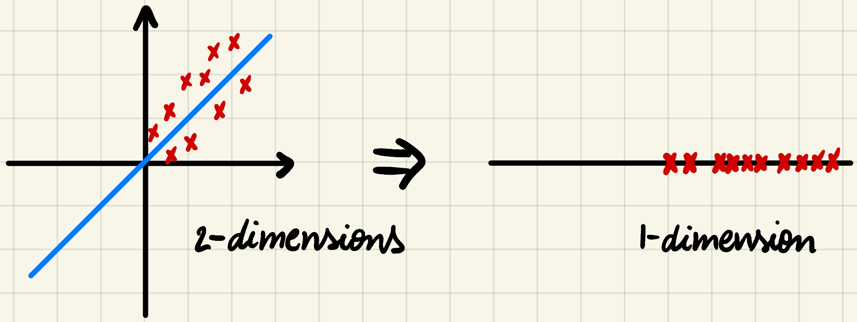
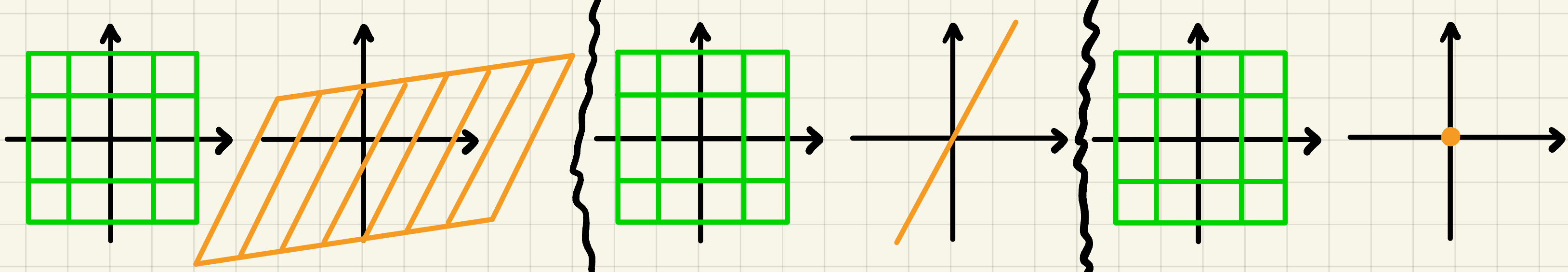
Determinants and Eigenvectors

Principal Component Analysis



Rank of linear transformations

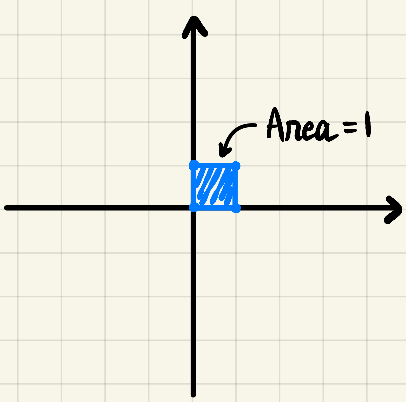


Rank 2: $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

Rank 1: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Rank 0: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

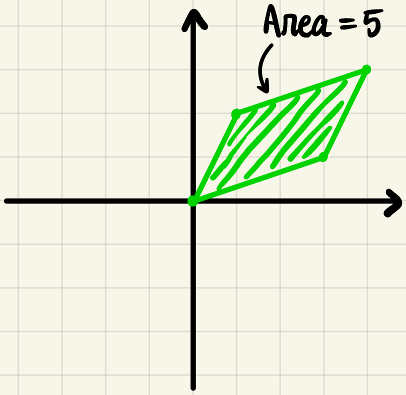
Determinant as an area



$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$Det = 3 \cdot 2 - 1 \cdot 1$

$Det = 5$



- The determinant of the matrix is the area of the image of the fundamental basis formed by the unit square on the left

Determinant of a product

$\det(AB) = \det(A) \cdot \det(B)$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

$\det = 5 \quad \det = 3 \quad \det = 15$

- The product of a singular and non-singular matrix is singular

Since $\det(AB) = \det(A) \cdot \det(B)$, if either A or B has $\det = 0$ (singular) the product will vanish, thus $\det(AB) = 0$, resulting in a singular matrix

Quiz: find the determinant

① $\begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$

$\det = 0.4 \cdot 0.6 - (-0.2 \cdot -0.2)$

$= 0.24 - 0.04$

$\det = 0.2$

② $\begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$

$\det = 0.25 \cdot 0.625 - (-0.125 \cdot -0.25)$

$\det = 0.125$

Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\det(AB) = \det(A) \cdot \det(B)$

$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$

$\det(\underbrace{I}_{1}) = \det(A) \cdot \det(\underbrace{A^{-1}}_{\frac{1}{\det(A)}})$

Quiz

① What's the determinant of matrix W?

$$W = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det = \begin{cases} 0 + 0 - 1 = -1 \\ 0 - 1 - 0 = -1 \end{cases}$$

$$\det = -2$$

② Calculate the Inverse matrix W^{-1}

$$W = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

* using Gaussian Elimination

augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

perform elementary row operations

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap rows

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$W^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

③ What's the inverse matrix W^{-1} multiplied with the identity matrix?

$$W \cdot W^{-1} = I$$

$$I \cdot W^{-1} = W^{-1}$$

④ Is the rank singular or non-singular?

→ Non-singular

⑤ $\vec{y} = \vec{b} \cdot W$, what's $\vec{y}$?

$$\vec{b} = [5 \;\; -2 \;\; 0]$$

$$\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + (-2 \cdot 2) + 0 \cdot -1 \\ 5 \cdot 1 + -2 \cdot 0 + 0 \cdot 1 \\ 5 \cdot 0 + (-2 \cdot 1) + 0 \cdot 0 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 5 - 4 + 0 \\ 5 - 0 + 0 \\ 0 - 2 + 0 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$ non-singular

⑥ T/F: the det. of a prod. of matrices is always the prod. of the det. of the matrices

$$\det(AB) = \det(A) \cdot \det(B)$$

True

⑦ Extract the 1st and 3rd column of matrix Z

$$Z = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 2 & 2 \\ -7 & 1 & 0 \end{bmatrix}$$

what's their dot product?

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{c} = \vec{a} \cdot \vec{b} \qquad \vec{c} = 3 \cdot 2 + 1 \cdot 2 - 7 \cdot 0$$

$$\vec{c} = 8$$

⑧ What's A·B ?

$$A = \begin{bmatrix} 5 & 2 & 3 \\ -1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & 0 \\ 8 & -1 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5\cdot1 + 2\cdot2 + 3\cdot8, & 5\cdot0 + 2\cdot1 + (3\cdot-1), & 5\cdot-4 + 2\cdot0 + 3\cdot0 \\ -1\cdot1 + (-3\cdot2) + 2\cdot8, & -1\cdot0 + (-3\cdot1) + 2\cdot-1, & (-1\cdot-4) + -3\cdot0 + 2\cdot0 \\ 0\cdot1 + 1\cdot2 + (-1\cdot8), & 0\cdot0 + 1\cdot1 + (-1\cdot-1), & 0\cdot-4 + 1\cdot0 + -1\cdot0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5+4+24, & 0+2-3, & -20+0+0 \\ -1-6+16, & 0-3-2, & 4+0+0 \\ 0+2-8, & 0+1+1, & 0+0+0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 33 & -1 & -20 \\ 9 & -5 & 4 \\ -6 & 2 & 0 \end{bmatrix}$$

⑨ Calculate the det. of the inverse matrix A·B

$$\det(A\cdot B)^{-1} = 1/\det(A\cdot B)$$

$$\det(A\cdot B) = \begin{cases} 0 + 24 - 360 = -336 \\ 600 - 264 + 0 = +336 \end{cases} = 0$$

$$\det(A\cdot B)^{-1} = 1/0$$ = cannot be computed

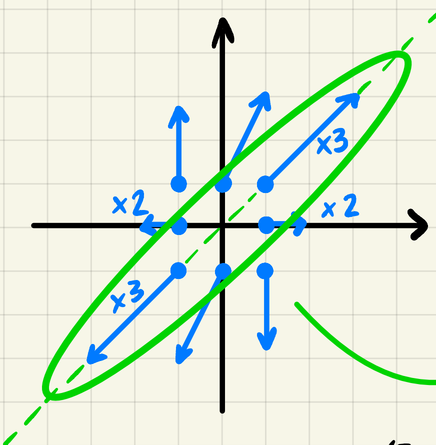
⑩ Select the correct statements:

- The determinant of an inverse matrix is the inverse of the determinant of the matrix

  ↳ $\det(A\cdot B)^{-1} = 1/\det(A\cdot B)$

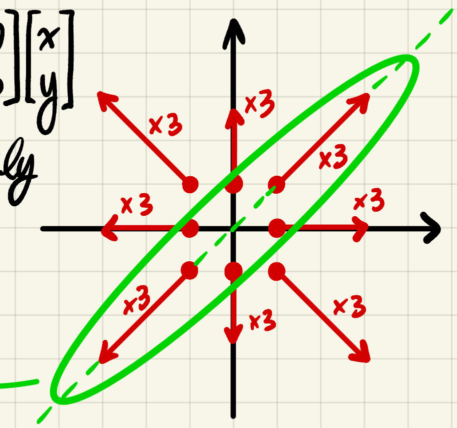
- Singular matrices are non-invertible

Eigenvalues and Eigenvectors



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

match for infinetely many points



$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ singular matrix

$$\det\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

Finding eigenvalues

- If λ is an eigenvalue: $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for infinitely many (x, y)

  $\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions

  $\det\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$

- Characteristic polynomial $(2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$ $\quad$ $\lambda = 2$, $\lambda = 3$

Finding Eigenvectors

$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 2 \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ $\begin{aligned} 2x + y &= 2x \\ 0x + 3y &= 2y \end{aligned}$ $\quad$ $\begin{aligned} x &= 1 \\ y &= 0 \end{aligned}$ $\quad$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3 \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ $\begin{aligned} 2x + y &= 3x \\ 0x + 3y &= 3y \end{aligned}$ $\quad$ $\begin{aligned} x &= 1 \\ y &= 1 \end{aligned}$ $\quad$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Quiz: ① find eigenvalues and eigenvectors of this matrix $A=\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}$

eingenvalues: $p_A(\lambda) = \det(A-\lambda I)$

$$= \det\left(\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \begin{vmatrix}9-\lambda & 4\\4 & 3-\lambda\end{vmatrix}$$

$$= (9-\lambda)(3-\lambda) - 4\cdot 4 = 27 - 9\lambda - 3\lambda + \lambda^2 - 16 = 11 - 12\lambda + \lambda^2$$

$$p_A(\lambda) = (11-\lambda)(1-\lambda) \qquad \lambda = 11,\ \lambda = 1$$

② Find the eigenvectors of the matrix

for $\lambda=11$, $\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 11\cdot\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \begin{cases}9x+4y = 11x\\4x+3y = 11y\end{cases} = \begin{matrix}-2x+4y=0\\4x-8y=0\end{matrix}$ ← same results

OR

for $\lambda=11$, $\begin{bmatrix}9-\lambda & 4\\4 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}9-11 & 4\\4 & 3-11\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2 & 4\\4 & -8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$ $\quad \begin{matrix}-2x+4y=0\\4x-8y=0\end{matrix}$

$x=2y$

$$\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}9\cdot 2+4\cdot 1\\4\cdot 2+3\cdot 1\end{bmatrix} = \begin{bmatrix}22\\11\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

for $\lambda=1$, $\begin{bmatrix}9-\lambda & 4\\4 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8 & 4\\4 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$ $\quad \begin{matrix}8x+4y=0\\4x+2y=0\end{matrix}$ $\quad y=-2x\ \begin{bmatrix}1\\-2\end{bmatrix}$

or

$-y=2x\ \begin{bmatrix}-1\\2\end{bmatrix}$

$$\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}9\cdot 1+4\cdot -2\\4\cdot 1+3\cdot -2\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-9+8\\-4+6\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

Quiz

① What's the characteristic polynomial for the matrix

$A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$,

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - (-3\cdot 1)$$

$$= 12 - 2\lambda - 6\lambda + \lambda^2 + 3$$

$$p_A(\lambda) = \lambda^2 - 8\lambda + 15 \qquad \lambda = 3 \text{ or } 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{64 - 60}}{2}$$

$$x_1 = \frac{8+2}{2} \qquad x_2 = \frac{8-2}{2}$$

$$x_1 = 5 \qquad x_2 = 3$$

② What are the eigenvectors?

for $\lambda = 3$ $\begin{bmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\begin{cases} -x + y = 0 \\ -3x + 3y = 0 \end{cases}$ $x = y \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 5$ $\begin{bmatrix} 2-5 & 1 \\ -3 & 6-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\begin{cases} -3x + y = 0 \\ -3x + y = 0 \end{cases}$ $y = 3x \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

③ What's the eigenvalue of the identity matrix?

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, eingenvalue = 1

④ Find the eigenvalues of $A \cdot B$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ → identity matrix

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda) - 2\cdot 0 = 4 - 5\lambda + \lambda^2$$

$$\lambda = 1 \text{ or } 4$$

⑤ What are the eigenvectors of the matrix?

$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, $\lambda = 1$ or 4

for $\lambda = 1$, $\begin{bmatrix} 1-1 & 2 \\ 0 & 4-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} 2y = 0 \\ 3y = 0 \end{cases}$ $\vec{v_1} = (2,3)$

WHY??

for $\lambda = 4$, $\begin{bmatrix} 1-4 & 2 \\ 0 & 4-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} -3x + 2y = 0 \\ 0 + 0 = 0 \end{cases}$ $\vec{v_2} = (1,0)$

⑥ Which of the vectors span the matrix $W = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 5 \\ 3 & -2 & -1 \end{bmatrix}$

$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$

⑦ Given matrix P, whats the eigenbasis?

$P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ eigenbasis $= \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$?

⑧ What's the characteristic polynomial for the matrix?

$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $p_A(\lambda) = \det(A - \lambda I)$

$$p_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 1 & -2 \\ 4 & 0-\lambda & 1 \\ 2 & 1 & -1-\lambda \end{bmatrix}$$

$$= \begin{cases} (3-\lambda)(-\lambda)(-1-\lambda) + 2 - 8 = (-3\lambda + \lambda^2)(-1-\lambda) - 6 = 3\lambda + 3\lambda^2 - \lambda^2 - \lambda^3 - 6 = -\lambda^3 + 2\lambda^2 + 3\lambda - 6 \\ -4\lambda - (3-\lambda) - (4 \cdot (-1-\lambda)) = -\cancel{4\lambda} - 3 + \lambda + 4 + \cancel{4\lambda} = \lambda + 1 \end{cases}$$

$$\begin{array}{r} -\lambda^3 + 2\lambda^2 + 3\lambda - 6 \\ \lambda + 1 \\ \hline -\lambda^3 + 2\lambda^2 + 4\lambda - 5 \end{array}$$

⑨ For a non-singular matrix A with real entries and eigenvalue i:

- $1/i$ is an eigenvalue of A^{-1}