Linear Algebra

Week 2: Solving system of linear equations

Solving non-singular system of linear equations

$$\begin{cases} 5a + b = 17 \\ 4a - 3b = 6 \end{cases}$$

$$\begin{aligned} 20a + 4b &= 68 \\ -\ 20a + 15b &= -30 \\ \hline 19b &= 38 \\ b &= 2 \end{aligned}$$

$$\begin{aligned} 5a + 2 &= 17 \\ 5a &= 15 \\ a &= 3 \end{aligned}$$

$$\begin{cases} 2a + 5b = 46 \\ 8a + b = 32 \end{cases}$$

$$\begin{aligned} 8a + 20b &= 184 \\ -\ 8a - b &= -32 \\ \hline 19b &= 152 \\ b &= 8 \end{aligned}$$

$$\begin{aligned} 8a + 8 &= 32 \\ 8a &= 24 \\ a &= 3 \end{aligned}$$

Solving singular system of linear equations

$$\begin{cases} 5a + b = 11 \\ 10a + 2b = 22 \end{cases}$$

Infinitely many solutions

Systems with more variables

$$\begin{cases} a + b + 2c = 12 \\ 3a - 3b - c = 3 \\ 2a - b + 6c = 24 \end{cases} \quad \begin{cases} a + b + 2c = 12 \\ 6b + 7c = 33 \\ -3b + 2c = 0 \end{cases} \quad \begin{cases} a + b + 2c = 12 \\ 6b + 7c = 33 \\ c = 3 \end{cases}$$

$a = 4$
$b = 2$
$c = 3$

$$\begin{aligned} -\cancel{3a} - 3b - 6c &= -36 \\ \cancel{3a} - 3b - c &= 3 \\ \hline -6b - 7c &= -33 \\ 6b + 7c &= 33 \end{aligned}$$

$$\begin{aligned} -\cancel{2a} - 2b - 4c &= -24 \\ \cancel{2a} - b + 6c &= 24 \\ \hline -3b + 2c &= 0 \end{aligned}$$

$$\begin{aligned} \cancel{6b} + 7c &= 33 \\ -\cancel{6b} + 4c &= 0 \\ \hline 11c &= 33 \\ c &= 3 \end{aligned}$$

$$\begin{aligned} 6b + 21 &= 33 \\ 6b &= 12 \\ b &= 2 \end{aligned}$$

$$\begin{aligned} a + 2 + 6 &= 12 \\ a &= 4 \end{aligned}$$

Matrix row reduction (Gaussian Elimination)

Original System	Intermediate System	Solved system
$\begin{cases} 5a + b = 17 \\ 4a - 3b = 6 \end{cases}$ $\rightarrow$	$\begin{aligned} a + 0.2b &= 3.4 \\ b &= 2 \end{aligned}$ $\rightarrow$	$\begin{aligned} 1a + 0b &= 3 \\ 0a + 1b &= 2 \end{aligned}$

Original matrix	Upper diagonal matrix	Diagonal matrix
$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$ $\rightarrow$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ $\rightarrow$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
	Row echelon form	Reduced row echelon form

Row echelon form

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 things may happen

$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ two 1's in the diagonal

$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ one 1 in the diagonal

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ no 1's in the diagonal

the zero's below diagonal

any number to the right of 1's

zero's to the right of zeros

Row operations that preserve singularity

$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix}$

Determinant $= 5 \cdot 3 - 1 \cdot 4 = 11$

$\begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$

Det $= 4 \cdot 1 - 3 \cdot 5 = -11$

Method of elimination Quiz

① $\begin{cases} x + y = 4 \\ -6x + 2y = 16 \end{cases}$

$$\begin{array}{r} -2x - 2y = -8 \\ -6x + 2y = 16 \\ \hline -8x = 8 \end{array}$$

$x = -1$

$x + y = 4$
$-1 + y = 4$
$y = 5$

② Calculate the determinant, singular or non-singular?

$$\begin{bmatrix} 4 & -3 \\ 7 & -8 \end{bmatrix}$$

Det $= -32 + 21 = -11$, non-singular

③

$$\begin{bmatrix} -3 & 8 & 1 \\ 2 & 2 & -1 \\ -5 & 6 & 2 \end{bmatrix}$$

Det $= \begin{cases} -12 + 40 + 12 = 40 \\ +10 - 18 - 32 = -40 \end{cases} = 0$, singular

④ Determine if matrix has linearly (in)dependent rows:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 2a-d & 2b-e & 2c-f \end{bmatrix}$$

Linearly dependent

⑤ which operations do not change the singularity of the matrix?

- Adding a row to another
- Switching rows
- Multiplying a row by nonzero scalar

⑥

$$\begin{bmatrix} a & a \\ b & c \end{bmatrix}$$

$c \neq a$ only if $a = b$
$c \neq b$

⑦

$$\begin{aligned} x + y + z &= 10{,}000 && (1) \\ x &= 2y && (2) \\ 0.02x + 0.03y + 0.04z &= 260 && (3) \end{aligned}$$

- (2) into (1):

$2y + y + z = 10{,}000$
$3y + z = 10{,}000$ (4)

- (4) into (3):

$0.02 \cdot 2y + 0.03y + 0.04z = 260$
$0.07y + 0.04z = 260$ (5)

- solve for (4) and (5):

$3y + z = 10{,}000$
$0.07y + 0.04z = 260$

$$\begin{array}{r} 260 \\ \times\ 25 \\ \hline 1300 \\ 5200 \\ \hline 6500 \end{array}$$

$$\begin{array}{r} 3y + z = 10{,}000 \\ -1.75y - z = -6{,}500 \\ \hline 1.25y = 3{,}500 \end{array}$$

$y = 2{,}800$

- substitute y in (4)

$3y + z = 10{,}000$

$8{,}400 + z = 10{,}000$
$z = 1{,}600$

Rank of a matrix

Rank = how much information a matrix holds

$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$ non-singular, 2 pieces of information, Rank = 2

$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ singular, 1 piece of information, Rank = 1

Rank is the largest number of linearly independent rows/columns in the matrix

Calculating the Row echelon form

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \xrightarrow[\text{leftmost coefficient}]{\text{divide each row by}} \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \xrightarrow[\text{the leftmost non-zero coefficient}]{\text{divide second row by}} \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{rr} 1 & -0.75 \\ -(1 & 0.2) \\ \hline 0 & -0.95 \end{array}$$

Row echelon form

Rank = 2

2 ones in the diagonal

non-singular

Singular matrices

Row echelon form

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

Rank = 1

1 one in the diagonal

singular

$$\begin{array}{rr} 1 & 0.2 \\ -(1 & 0.2) \\ \hline 0 & 0 \end{array}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank = 0

0 ones in the diagonal

singular

3x3 matrix

$$\begin{cases} a+b+2c=12 \\ 3a-3b-c=3 \\ 2a-b+6c=24 \end{cases} \longrightarrow \begin{aligned} a+b+2c&=12 \\ -6b+7c&=-33 \\ 6c&=18 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -3 & -1 \\ 2 & -1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

Matrix — Row echelon form matrix

Quiz

① $\begin{cases} 7f + 5a + 3c = 120 \\ 3f + 2a + 5c = 70 \\ f + 2a + c = 20 \end{cases}$

④

$$\begin{bmatrix} 7 & 5 & 3 \\ 3 & 2 & 5 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 120 \\ 70 \\ 20 \end{bmatrix}$$

⑤ solve the system:

$$\begin{cases} 7f + 5a + 3c = 120 \\ 3f + 2a + 5c = 70 \\ f + 2a + c = 20 \end{cases}$$

$$\begin{aligned} 21f + 15a + 9c &= 360 \\ -21f - 14a - 35c &= -490 \\ \hline a - 26c &= -130 \end{aligned}$$

$$\begin{aligned} 9a - 234c &= -1170 \\ -9a - 4c &= -20 \\ \hline -238c &= -1190 \end{aligned}$$

$c = 5$

$$\begin{aligned} 7f + 5a + 3c &= 120 \\ 7f + 0 + 15 &= 120 \\ 7f &= 105 \end{aligned}$$

$f = 15$

$$\begin{aligned} 7f + 5a + 3c &= 120 \\ a - 26c &= -130 \\ -9a - 4c &= -20 \end{aligned}$$

$$\begin{aligned} 7f + 5a + 3c &= 120 \\ -7f - 14a - 7c &= -140 \\ \hline -9a - 4c &= -20 \end{aligned}$$

$$\begin{aligned} a - 26.5 &= -130 \\ a - 130 &= -130 \end{aligned}$$

$a = 0$

⑥

$$\begin{bmatrix} 7 & 5 & 3 \\ 3 & 2 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

$\text{Det} = \begin{cases} 14 + 18 + 25 = 57 \\ -6 - 70 - 15 = -91 \end{cases} = -34$

non-singular

⑦ $\begin{cases} 7f + 5a + 3c = 120 \\ a - 26c = -130 \\ -238c = -1190 \end{cases}$

Rank = 3

Week 2 Summary

- Determinant finds the singularity of a matrix
  - non-zero when matrix is non-singular
  - zero when matrix is singular

- Singular vs. Non-singular

  does not have a unique solution — does have a unique solution

- Linearly (in)dependent

  ↳ linear combination of vectors

- independent: systems of linear equations cannot be derived from operations such a +, −, ×, ÷

  each equation provides unique info.

- dependent: one or more equations in the system can be derived from the others. This implies redundancy in the system

- Rank = how much info. a matrix holds

Rank is the largest number of linearly independent rows/columns in the matrix