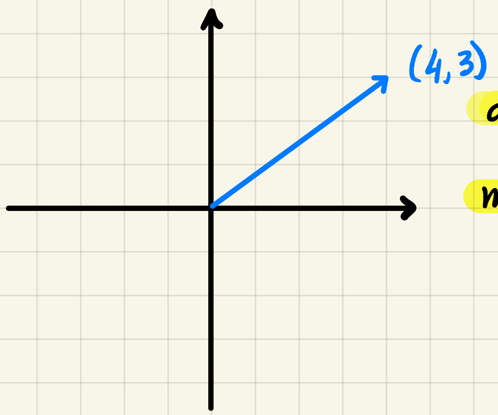
Linear Algebra

Week 3: Vectors and Matrices

Vectors and their properties

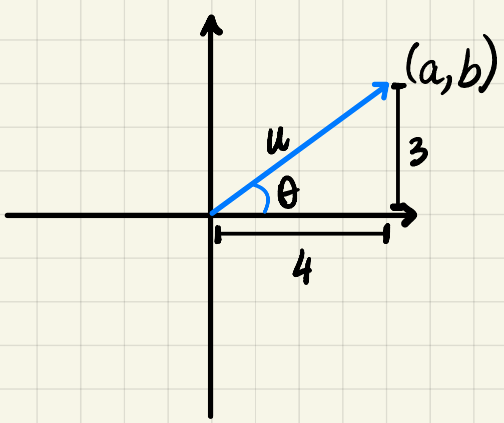
Vector = objects with magnitude (length) and direction
usually represented by an ordered list of numbers



distance = 4+3 = 7

magnitude = $\sqrt{4^2+3^2} = 5$

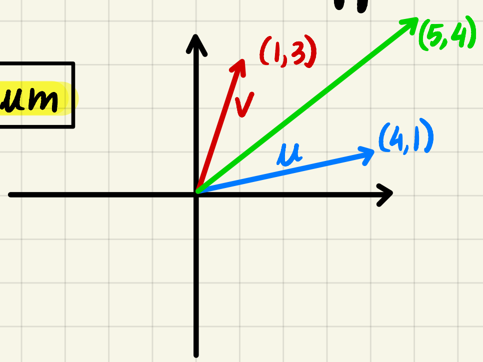
Norms



- L1-norm = $|(a,b)|_1 = |a| + |b|$
- L2-norm = $|(a,b)|_2 = \sqrt{a^2+b^2}$
- $\tan(\theta) = \frac{3}{4}$
- $\theta = \arctan(3/4) = 0.64$

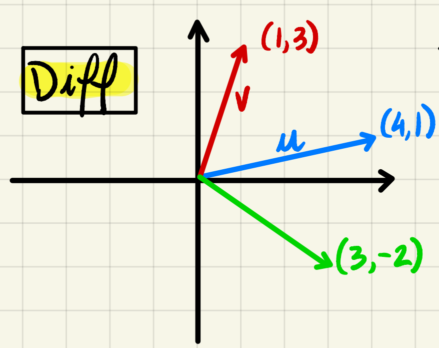
Sum and Difference

Sum



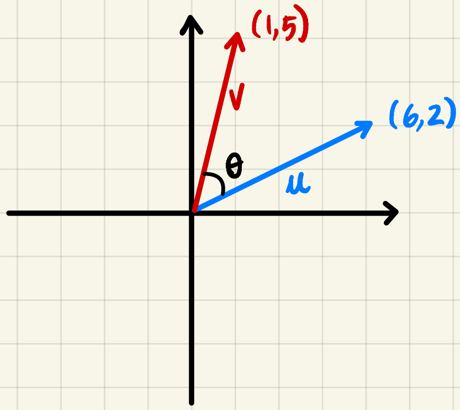
$u + v = (4+1, 1+3)$
$= (5,4)$

Diff



$u - v = (4-1, 1-3)$
$= (3,-2)$

Distances

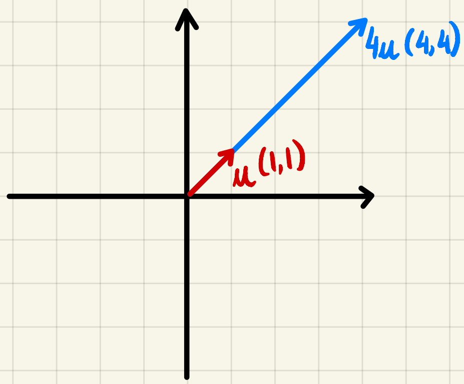


L1-distance $= |u - v|_1 = |5| + |-3| = 8$

L2-distance $= |u - v|_2 = \sqrt{5^2 + 3^2} = 5.83$

cosine distance $= \cos\theta$

Multiplying a vector by a scalar



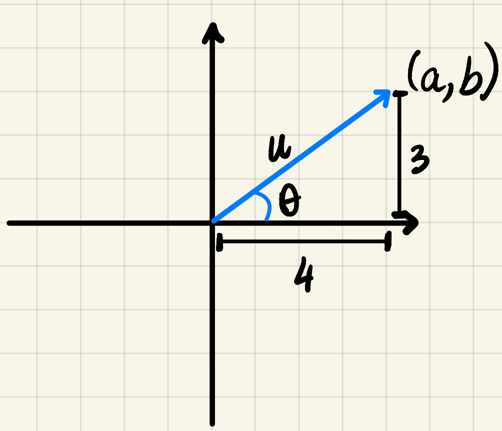
$u = (1,1)$
$\lambda = 4$
$\lambda u = (4,4)$

Dot product

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

$$2 \cdot 3 + 4 \cdot 5 + 1 \cdot 2 = 28$$

Norm of a vector using dot product



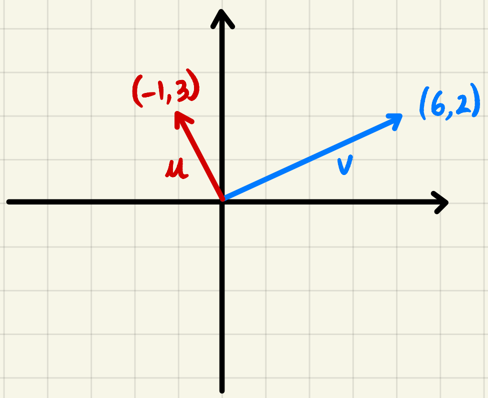
$$\sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 25$$

L2-norm $= \sqrt{\text{dot product}(u,u)}$

$$|u|_2 = \sqrt{\langle u, u \rangle}$$

Orthogonal vectors have dot product 0

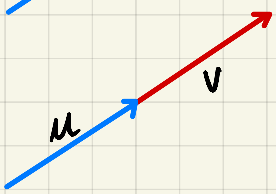


$$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$$

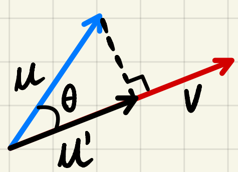
Orthogonal = perpendicular

Dot product (continued)

$$\langle u, u \rangle = |u|^2 = |u| \cdot |u|$$

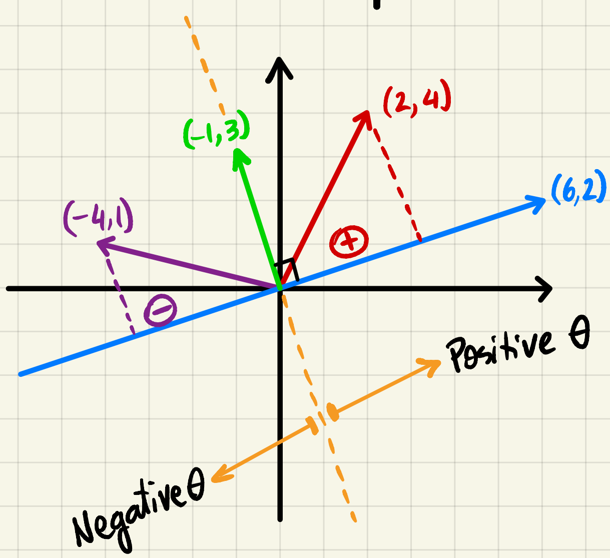


$$\langle u, v \rangle = |u| \cdot |v|$$



$$\langle u, v \rangle = |u'| \cdot |v|$$
$$= |u||v| \cos\theta$$

Geometric dot product



$$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20 \quad \text{Positive}$$

$$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -22 \quad \text{Negative}$$

Equations as dot product

System of equations vs Matrix product

$$a + b + c = 10$$
$$a + 2b + c = 15$$
$$a + b + 2c = 12$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

Vector Quiz

① Vectors have magnitude and direction

② $\vec{u} = (1,3)$
$\vec{v} = (6,2)$

$\vec{u} + \vec{v} = (7,5)$

③ $\vec{u} - \vec{v} = (-5,1)$

④ Dot product

$\vec{a} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ 6 \\ -4 \end{bmatrix}$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= -1 \cdot -3 + 5 \cdot 6 + 2 \cdot -4 \\ &= +3 + 30 - 8 \\ &= 25 \end{aligned}$$

⑤ if $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

$\vec{a} = 0$, $\vec{b}$ = any vector

⑥

$$\begin{bmatrix} 3 & 5 & 1 \\ 7 & -2 & 4 \\ -6 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 15 \end{bmatrix}$$

Dot product terminology

algebraic operation that takes two vectors $x = [x_1 \ x_2 \ \dots \ x_n]^T \in \mathbb{R}^n$ and $y = [y_1 \ y_2 \ \dots \ y_n]^T \in \mathbb{R}^n$ and returns a single scalar.

It's defined as:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Matrix Multiplication

$$\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}\cdot\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}2 & -1\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} & \begin{bmatrix}2 & -1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix} \\ \begin{bmatrix}0 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} & \begin{bmatrix}0 & 2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}5 & 0\\ 2 & 4\end{bmatrix}$$

Identity Matrix

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}=\begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}$$

Matrix inverse

$$\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Identity matrix

$$\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}2/5 & -1/5\\ -1/5 & 3/5\end{bmatrix}$$

$$\begin{bmatrix}3 & 1\end{bmatrix}\cdot\begin{bmatrix}a\\c\end{bmatrix} = 1 \qquad 3a + c = 1$$

$$\begin{bmatrix}3 & 1\end{bmatrix}\cdot\begin{bmatrix}b\\d\end{bmatrix} = 0 \qquad 3b + d = 0$$

$$\begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}a\\c\end{bmatrix} = 0 \qquad a + 2c = 0$$

$$\begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}b\\d\end{bmatrix} = 1 \qquad b + 2d = 1$$

$a = 2/5$

$b = -1/5$

$c = -1/5$

$d = 3/5$

Find the inverse of the matrix

$$\begin{bmatrix}5 & 2\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow \begin{aligned}5a + 2c &= 1\\ 5b + 2d &= 0\\ a + 2c &= 0\\ b + 2d &= 1\end{aligned}$$

$$\begin{aligned}5a + 2c &= 1\\ -a - 2c &= 0\\ \hline 4a &= 1\\ a &= 1/4\end{aligned}$$

$$\begin{aligned}1/4 + 2c &= 0\\ 2c &= -1/4\\ c &= -1/8\end{aligned}$$

$$\begin{aligned}5b + 2d &= 0\\ -b - 2d &= -1\\ \hline 4b &= -1\\ b &= -1/4\end{aligned}$$

$$\begin{aligned}-1/4 + 2d &= 1\\ 2d &= 5/4\\ d &= \frac{5}{8}\end{aligned}$$

Find the inverse of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{matrix} a + c = 1 \\ b + d = 0 \\ 2a + 2c = 0 \\ 2c + 2d = 1 \end{matrix}$$

$$\left.\begin{matrix} a + c = 1 \\ 2a + 2c = 0 \\ \\ b + d = 0 \\ 2b + 2d = 1 \end{matrix}\right\} \text{impossible to solve}$$

Which matrices have an inverse?

$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$

non-singular matrix

Invertible

$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$

non-singular matrix

Invertible

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$

singular matrix

Non-invertible

Det = 5
$5^{-1} = 0.2$

Det = 8
$8^{-1} = 0.125$

Det = 0

Non-zero determinants

Zero determinant

Neural Networks and matrices

AND operator

AND	x	y
No	0	0
No	1	0
No	0	1
Yes	1	1

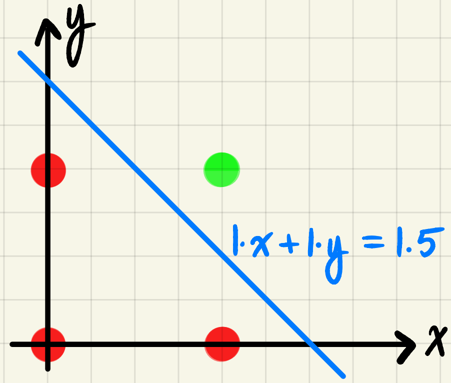
Model
1
1

=

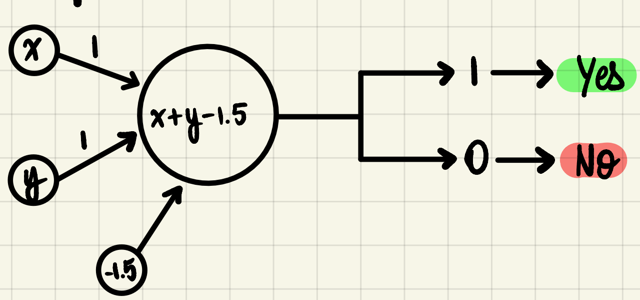
Dot prod
0
1
1
2

check: > 1.5 ? →

Check
No
No
No
Yes



Perceptron



Quiz

① Find the distance between the vectors $d(\vec{v}, \vec{w})$
$\vec{v} = (1,0,7)$
$\vec{w} = (0,-1,2)$

$d = |\vec{v} - \vec{w}|$

$d = \sqrt{1^2 + 1^2 + 5^2}$

$d = \sqrt{27}$

② What's the magnitude of the vector from P to Q?
$P: (1,0,-3)$
$Q: (-1,0,-3)$

$\vec{v} = P - Q$
$\vec{v} = [1+1,\ 0-0,\ -3+3]$
$\vec{v} = [2\ \ 0\ \ 0]$

$L2\text{-norm} = \sqrt{2^2 + 0^2 + 0^2}$
$= \sqrt{4}$
$= 2$

③ Dot product:

- dot product of two vectors is always a scalar
- dot product of orthogonal vector is always 0

④ Calculate the norm $\|v\|$ of vector $\vec{v} = (1,-5,2,0,-3)$

$\|v\| = \sqrt{\langle v, v \rangle}$

$= \sqrt{1^2 + (-5)^2 + 2^2 + 0^2 + (-3)^2}$

$= \sqrt{1 + 25 + 4 + 0 + 9}$

$\|v\| = \sqrt{39}$

⑤ Vector with greatest norm

$[2\ 5] = \sqrt{29}$

$[1\ 0\ -2\ 0\ -1] = \sqrt{6}$

$[1\ 2\ -3] = \sqrt{14}$

$[2\ 2\ 2\ 2] = \sqrt{16}$

$[0\ 0\ 0\ 0] = 0$

⑥ dot product of $\vec{a} \cdot \vec{b}$

$\vec{a} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -3 \\ 6 \\ -4 \end{bmatrix}$

$\vec{a} \cdot \vec{b} = [-1\ 5\ 2] \begin{bmatrix} -3 \\ 6 \\ -4 \end{bmatrix}$

$= 3 + 30 - 8$
$= 25$

⑦ Multiplication $M_1 \cdot M_2$

$M_1 = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix}$

$M_1 \cdot M_2 = \begin{bmatrix} 2\cdot5 + (-1\cdot0) & 2\cdot-2 + (-1\cdot1) \\ 3\cdot5 + (-3\cdot0) & 3\cdot-2 + (-3\cdot1) \end{bmatrix}$

$= \begin{bmatrix} 10 & -5 \\ 15 & -9 \end{bmatrix}$

⑧ dot product

$\vec{w} = \begin{bmatrix} -9 \\ -1 \end{bmatrix}, \vec{z} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$

$\vec{w} \cdot \vec{z} = [-9\ -1] \begin{bmatrix} -3 \\ -5 \end{bmatrix}$

$= 27 + 5$

$= 32$

Summary

- **Vector** = objects with magnitude (length) and direction
  usually represented by an ordered list of numbers
- **L1-norm** = distance (Manhattan)

  $\|x\|_1 := \sum_{i=1}^{n} |x_i|$ sum of abs. values of the vectors elements
- **L2-norm** = distance (Euclidean)

  $$\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$

- **Dot product terminology**

algebraic operation that takes two vectors $x = [x_1 \ x_2 \ \dots \ x_n]^T \in \mathbb{R}^n$ and $y = [y_1 \ y_2 \ \dots \ y_n]^T \in \mathbb{R}^n$ and returns a single scalar.

It's defined as:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Matrix Multiplication Terminology

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, the matrix product $C = AB$ is defined to be the $m \times p$ matrix, such that

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where a_{ik} are the elements of matrix A and b_{kj} are the elements of matrix B, and $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, p$

In other words c_{ij} is the dot product of the i-th row of A and the j-th column of B