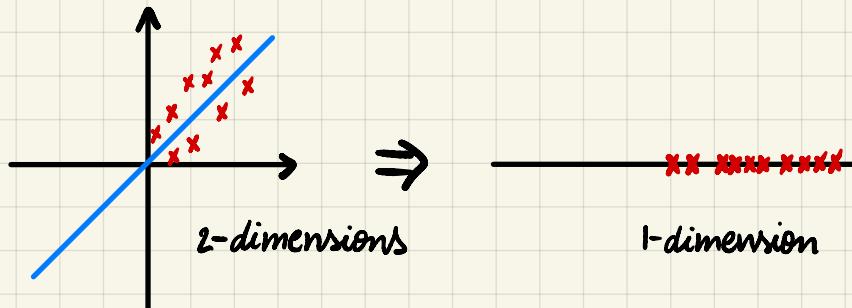


Determinants and Eigenvectors

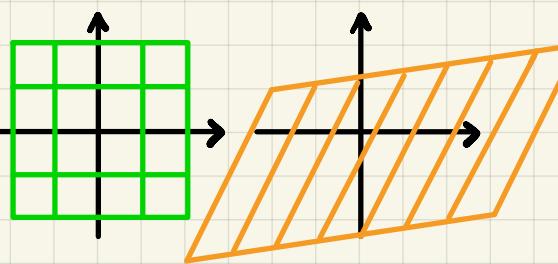
Principal Component Analysis



Rank of linear transformations

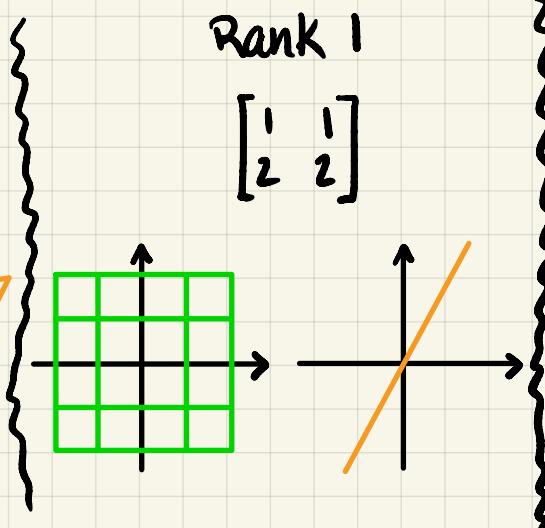
Rank 2

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$



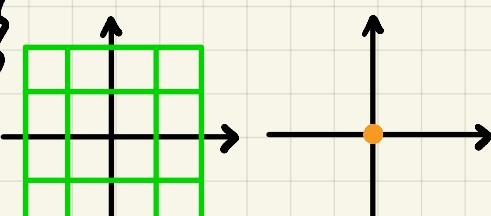
Rank 1

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

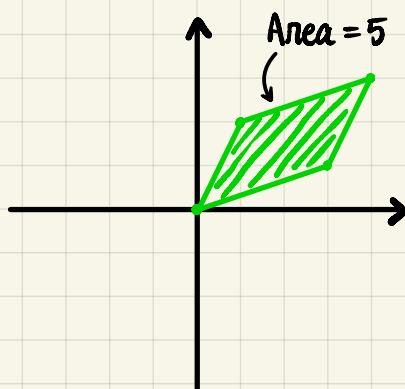
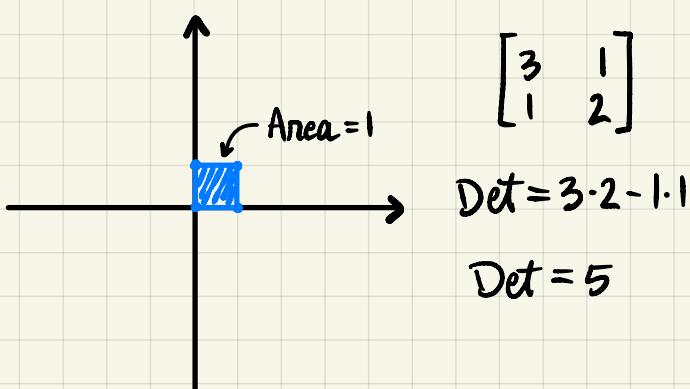


Rank 0

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



Determinant as an area



- The determinant of the matrix is the area of the image of the fundamental basis formed by the unit square on the left

Determinant of a product

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

$\det = 5$ $\det = 3$ $\det = 15$

- The product of a singular and non-singular matrix is singular. Since $\det(AB) = \det(A) \cdot \det(B)$, if either A or B has $\det=0$ (singular), the product will vanish, thus $\det(AB)=0$, resulting in a singular matrix.

Quiz: find the determinant

$$① \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\begin{aligned} \det &= 0.4 \cdot 0.6 - (-0.2 \cdot -0.2) \\ &= 0.24 - 0.04 \end{aligned}$$

$$\det = 0.2$$

$$② \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

$$\det = 0.25 \cdot 0.625 - (-0.125 \cdot -0.25)$$

$$\det = 0.125$$

Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$$

$$\det(I) = \det(A) \cdot \det(A^{-1})$$

\downarrow \uparrow
 $\frac{1}{\det(A)}$

Quiz

① What's the determinant of matrix W ?

$$W = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det = \begin{cases} 0 + 0 - 1 = -1 \\ 0 - 1 - 0 = -1 \end{cases}$$

$$\det = -2$$

② Calculate the Inverse matrix W^{-1}

$$W = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

* using Gaussian Elimination

augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

perform elementary row operations

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

$$\text{swap rows} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

$$W^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

③ What's the inverse matrix W^{-1} multiplied with the identity matrix?

$$W \cdot W^{-1} = I$$

$$I \cdot W^{-1} = W^{-1}$$

④ Is the rank singular or non-singular?

→ Non-singular

⑤ $\vec{y} = \vec{b} \cdot W$, what's \vec{y} ?

$$\vec{b} = [5 \ 2 \ 0]$$

$$\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + (-2 \cdot 2) + 0 \cdot -1 \\ 5 \cdot 1 + -2 \cdot 0 + 0 \cdot 1 \\ 5 \cdot 0 + (-2 \cdot 1) + 0 \cdot 0 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 5 - 4 + 0 \\ 5 - 0 + 0 \\ 0 - 2 + 0 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \text{ non-singular}$$

⑥ T/F: the det. of a prod. of matrices is always the prod. of the det. of the matrices

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

True

⑦ Extract the 1st and 3rd column of matrix Z

$$Z = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 2 & 2 \\ -7 & 1 & 0 \end{bmatrix}$$

what's their dot product?

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{c} = \vec{a} \cdot \vec{b}$$

$$\vec{c} = 3 \cdot 2 + 1 \cdot 2 - 7 \cdot 0$$

$$\vec{c} = 8$$

⑧ What's $A \cdot B$?

$$A = \begin{bmatrix} 5 & 2 & 3 \\ -1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & 0 \\ 8 & -1 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5 \cdot 1 + 2 \cdot 2 + 3 \cdot 8, & 5 \cdot 0 + 2 \cdot 1 + (3 \cdot -1), & 5 \cdot -4 + 2 \cdot 0 + 3 \cdot 0 \\ -1 \cdot 1 + (-3) \cdot 2 + 2 \cdot 8, & -1 \cdot 0 + (-3) \cdot 1 + 2 \cdot -1, & (-1) \cdot -4 + -3 \cdot 0 + 2 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 2 + (-1) \cdot 8, & 0 \cdot 0 + 1 \cdot 1 + (-1) \cdot -1, & 0 \cdot -4 + 1 \cdot 0 + -1 \cdot 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5+4+24, & 0+2-3, & -20+0+0 \\ -1-6+16, & 0-3-2, & 4+0+0 \\ 0+2-8, & 0+1+1, & 0+0+0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 33 & -1 & -20 \\ 9 & -5 & 4 \\ -6 & 2 & 0 \end{bmatrix}$$

⑨ Calculate the det. of the inverse matrix $A \cdot B$

$$\det(A \cdot B)^{-1} = \frac{1}{\det(A \cdot B)}$$

$$\det(A \cdot B) = \begin{cases} 0 + 24 - 360 = -336 \\ 600 - 264 + 0 = +336 \end{cases} = 0$$

$$\det(A \cdot B)^{-1} = \frac{1}{0} = \text{cannot be computed}$$

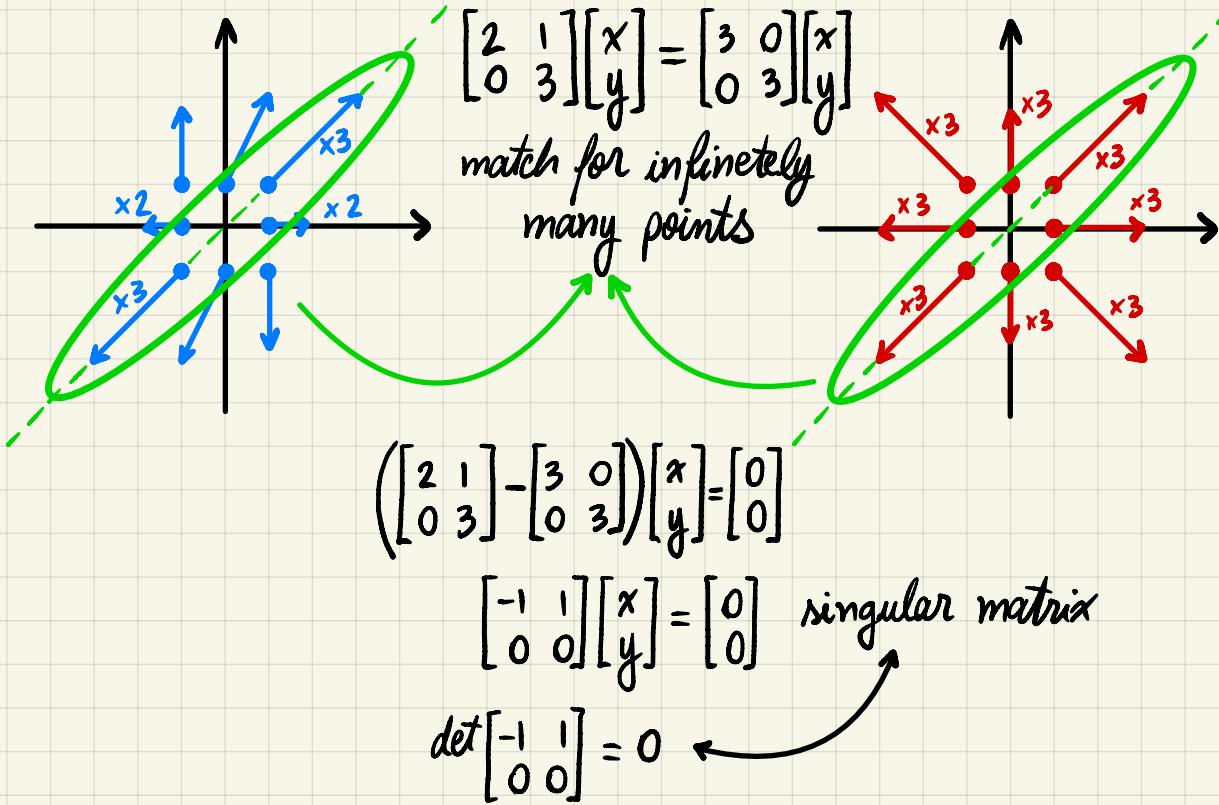
⑩ Select the correct statements :

- The determinant of an inverse matrix is the inverse of the determinant of the matrix

$$\hookrightarrow \det(A \cdot B)^{-1} = \frac{1}{\det(A \cdot B)}$$

- Singular matrices are non-invertible

Eigenvalues and Eigenvectors



Finding eigenvalues

- If λ is an eigenvalue: $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ for infinitely many (x, y)
 $\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions
 $\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$
- Characteristic polynomial $(2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$ $\lambda = 2$
 $\lambda = 3$

Finding Eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 2 \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} 2x + y &= 2x \\ 0x + 3y &= 2y \end{aligned}$$

$$\begin{aligned} x &= 1 \\ y &= 0 \end{aligned} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3 \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} 2x + y &= 3x \\ 0x + 3y &= 3y \end{aligned}$$

$$\begin{aligned} x &= 1 \\ y &= 1 \end{aligned} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quiz: ① find eigenvalues and eigenvectors of this matrix $A = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$

eigenvalues: $p_A(\lambda) = \det(A - \lambda I)$

$$= \det \left(\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \begin{vmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix}$$

$$= (9-\lambda)(3-\lambda) - 4 \cdot 4 = 27 - 9\lambda - 3\lambda + \lambda^2 - 16 = 11 - 12\lambda + \lambda^2$$

$$p_A(\lambda) = (11-\lambda)(1-\lambda) \quad \lambda = 11, \lambda = 1$$

② Find the eigenvectors of the matrix

for $\lambda = 11$, $\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 11 \cdot \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} 9x + 4y = 11x \\ 4x + 3y = 11y \end{cases} = \begin{cases} -2x + 4y = 0 \\ 4x - 8y = 0 \end{cases}$

OR

same results

for $\lambda = 11$, $\begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9-11 & 4 \\ 4 & 3-11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \begin{array}{l} -2x + 4y = 0 \\ 4x - 8y = 0 \end{array}$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \cdot 2 + 4 \cdot 1 \\ 4 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

for $\lambda = 1$, $\begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \begin{array}{l} 8x + 4y = 0 \\ 4x + 2y = 0 \end{array} \quad y = -2x \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \cdot 1 + 4 \cdot -2 \\ 4 \cdot 1 + 3 \cdot -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

or

$$-y = 2x \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9+8 \\ -4+6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Quiz

① What's the characteristic polynomial for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix},$$

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det \left(\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - (-3 \cdot 1)$$

$$= 12 - 2\lambda - 6\lambda + \lambda^2 + 3$$

$$p_A(\lambda) = \lambda^2 - 8\lambda + 15 \quad \lambda = 3 \text{ or } 5$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$x = \frac{8 \pm \sqrt{64 - 60}}{2}$$

$$x_1 = \frac{8+2}{2} \quad x_2 = \frac{8-2}{2}$$

$$x_1 = 5 \quad x_2 = 3$$

② What are the eigenvectors?

$$\text{for } \lambda = 3 \quad \begin{bmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \begin{cases} -x+y=0 \\ -3x+3y=0 \end{cases} \quad x=y \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{for } \lambda = 5 \quad \begin{bmatrix} 2-5 & 1 \\ -3 & 6-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \begin{cases} -3x+y=0 \\ -3x+y=0 \end{cases} \quad y=3x \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

③ What's the eigenvalue of the identity matrix?

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ eigenvalue} = 1$$

④ Find the eigenvalues of $A \cdot B$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{identity matrix}}$$

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det \left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det \left(\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix} \right) = (1-\lambda)(4-\lambda) - 2 \cdot 0 = 4 - 5\lambda + \lambda^2$$

$$\lambda = 1 \text{ or } 4$$

⑤ What are the eigenvectors of the matrix?

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \lambda = 1 \text{ or } 4$$

$$\text{for } \lambda=1, \begin{bmatrix} 1-1 & 2 \\ 0 & 4-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} 2y=0 \\ 3y=0 \end{cases}$$

$$y=0, x=1$$

$$\vec{v}_2 = (1, 0)$$

$$\text{for } \lambda=4, \begin{bmatrix} 1-4 & 2 \\ 0 & 4-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} -3x+2y=0 \\ 0+0=0 \end{cases}$$

$$\vec{v}_1 = (2, 3)$$

$$3x=2y \Rightarrow x=2, y=3$$

⑥ Which of the vectors span the matrix $W = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 5 \\ 3 & -2 & -1 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$$

⑦ Given matrix P, what's the eigenbasis?

$$P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{eigenbasis} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

⑧ What's the characteristic polynomial for the matrix?

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}, p_A(\lambda) = \det(A - \lambda I)$$

$$p_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 1 & -2 \\ 4 & 0-\lambda & 1 \\ 2 & 1 & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} &= (3-\lambda)(-\lambda)(-1-\lambda) + 2(-8) = (-3\lambda + \lambda^2)(-1-\lambda) - 6 = 3\lambda + 3\lambda^2 - \lambda^2 - \lambda^3 - 6 = -\lambda^3 + 2\lambda^2 + 3\lambda - 6 \\ &\{ -4\lambda - (3-\lambda) - (4 \cdot (-1-\lambda)) = -4\lambda - 3 + \lambda + 4 + 4\lambda = \lambda + 1 \end{aligned}$$

$$\frac{\lambda + 1}{-\lambda^3 + 2\lambda^2 + 4\lambda - 5}$$

⑨ For a non-singular matrix A with real entries and eigenvalue i:

- $\frac{1}{i}$ is an eigenvalue of A^{-1}