§5.11 Exercícios

1. Calcule as seguintes integrais triplas:

a) $\displaystyle\iiint_W z\,dxdydz$, onde W é a região no primeiro octante limitada pelos planos $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$ e o cilindro $y^2 + z^2 = 4$.

b) $\displaystyle\iiint_W xy^2z^3\,dxdydz$, onde W é a região no primeiro octante limitada pela superfície $z = xy$ e os planos $y = x$, $x = 1$ e $z = 0$.

c) $\displaystyle\iiint_W z\,dxdydz$, onde W é a região limitada pelas superfícies $z = \sqrt{x^2 + y^2}$, $y = x^2$, $z = 0$ e $y = 1$.

d) $\displaystyle\iiint_W y\cos(x + z)\,dxdydz$, onde W é a região limitada pelo cilindro $x = y^2$ e os planos $x + z = \dfrac{\pi}{2}$ e $z = 0$.

2. Calcule o volume dos sólidos W descritos abaixo.

a) W é limitado pelo cone $z = \sqrt{x^2 + y^2}$ e o parabolóide $z = x^2 + y^2$.

b) W é limitado pelas superfícies $z = 8 - x^2 - y^2$ e $z = x^2 + 3y^2$.

c) W é limitado pelas superfícies $z = 4 - x^2 - y^2$ e $z = y$, está situado no interior do cilindro $x^2 + y^2 = 1$, e $z \geq 0$.

d) W é limitado pelo cone $z = \sqrt{x^2 + y^2}$, pelo cilindro $x^2+y^2-\sqrt{x^2 + y^2} = x$ e pelo plano $z = 0$.

e) $W = \{(x, y, z) \in I\!R^3 \mid z \geq 1, x + y + z \leq 7, x \geq y^2\}$.

f) $W = \{(x, y, z) \in I\!R^3 \mid x^2 + y^2 + z^2 \leq a^2 \text{ e } z^2 \leq x^2 + y^2\}$.

g) $W = \{(x, y, z) \in I\!R^3 \mid x^2 + y^2 + z^2 \leq 4, x^2 + y^2 \leq 2y\}$.

h) $W = \{(x, y, z) \in I\!R^3 \mid x^2 + y^2 + z^2 \geq 4, x^2 + y^2 + (z - \sqrt{2})^2 \leq 2\,, z \leq \sqrt{3(x^2 + y^2)}\}$.

3. Calcule $\displaystyle\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2 + y^2}\,dz$.

4. Calcule as integrais triplas abaixo, usando uma mudança de variáveis conveniente.

a) $\displaystyle\iiint\limits_W z\,dxdydz$, onde
$W = \{(x,y,z) \in I\!R^3 \mid x^2+y^2+z^2 \leq 1, z \geq 0, x^2+y^2 \geq \frac{1}{4}\}$.

b) $\displaystyle\iiint\limits_W \frac{dxdydz}{z^2}$, onde W é o sólido limitado pelas superfícies $z = \sqrt{x^2+y^2}$, $z = \sqrt{1-x^2-y^2}$ e $z = \sqrt{4-x^2-y^2}$.

c) $\displaystyle\iiint\limits_W z\,dxdydz$, onde W é o sólido limitado pelas superfícies $z = \sqrt{x^2+y^2}$, $z = \sqrt{3(x^2+y^2)}$ e $x^2+y^2+z^2 = 4$.

d) $\displaystyle\iiint\limits_W xyz\,dxdydz$, onde
$W = \{(x,y,z) \in I\!R^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

e) $\displaystyle\iiint\limits_W (x^2+y^2+z^2)^{\frac{1}{2}}\,dxdydz$, onde W é a região dada por:
i) $W = \{(x,y,z) \in I\!R^3 \mid x^2+y^2+z^2 \leq a^2\}$.
ii) $W = \{(x,y,z) \in I\!R^3 \mid x^2+y^2+z^2 \leq x\}$.

f) $\displaystyle\iiint\limits_W \frac{dxdydz}{x^2+y^2+z^2}$, onde W é o sólido definido por
$W = \{(x,y,z) \in I\!R^3 \mid x^2+y^2+z^2 \leq 2y, z \leq \sqrt{x^2+y^2}, y \geq x \text{ e } x \geq 0\}$.

g) $\displaystyle\iiint\limits_W x\,dxdydz$,onde
$W = \{(x,y,z) \in I\!R^3 \mid 4 \leq x^2+(y-1)^2+z^2 \leq 9, x \geq 0, z \geq 0\}$.

5. Sabendo que a densidade em cada ponto de um sólido W é dada por $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$, determine a massa de W quando

$$W = \{(x,y,z) \in I\!R^3 \,|\, x^2+y^2+z^2 \leq 9 \text{ e } x^2+y^2+z^2 \geq 2y\}.$$

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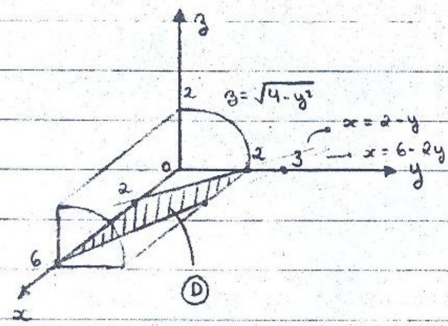
① a) $\displaystyle\iiint_W z\,dx\,dy\,dz$ 1º octante

$y=0$, $z=0$, $x+y=2$, $2y+x=6$

$y^2+z^2=4$



$0\le y\le 2$

$2-y\le x\le 6-2y$

$0\le z\le \sqrt{4-y^2}$

$$\iiint_W z\,dx\,dy\,dz = \int_0^2\int_{2-y}^{6-2y}\int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy =$$

$$= \int_0^2\int_{2-y}^{6-2y}\left[\frac{z^2}{2}\right]_0^{\sqrt{4-y^2}} dx\,dy = \int_0^2\int_{2-y}^{6-2y} 2-\frac{y^2}{2}\,dx\,dy =$$

$$= \int_0^2\left[2x - x\frac{y^2}{2}\right]_{2-y}^{6-2y} dy = \int_0^2 2(6-2y)-(6-2y)\frac{y^2}{2}-2(2-y)+(2-y)y^2/2\;dy =$$

$$= \int_0^2 12-4y-3y^2+y^3-4+2y+y^2-\frac{y^3}{2}\;dy =$$

$$= \int_0^2 \frac{y^3}{2}-2y^2-2y+8\;dy = \left[\frac{y^4}{8}-\frac{2y^3}{3}-y^2+8y\right]_0^2 =$$

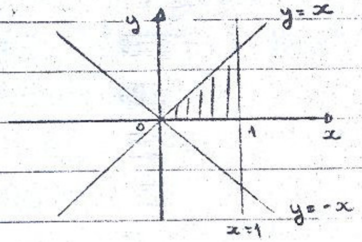
$$= 2-\frac{16}{3}-4+16 = \frac{14}{1_{/3}}-\frac{16}{3_{/1}} = \frac{42-16}{3} = \boxed{\frac{26}{3}}$$

b) $\iiint_W xy^2z^3 \, dx\,dy\,dz$

1° octante

$z = xy \quad y = x \quad x = 1$

$z = 0$



$z = cte = xy$

$z = 1 \therefore \quad xy = 1$

$y = 1/x$

$0 \le x \le 1 \qquad 0 \le z \le x \cdot y$

$0 \le y \le x$

$$\iiint_W xy^2z^3 \, dx\,dy\,dz = \int_0^1 \int_0^x \int_0^{xy} xy^2z^3 \, dz\,dy\,dx =$$

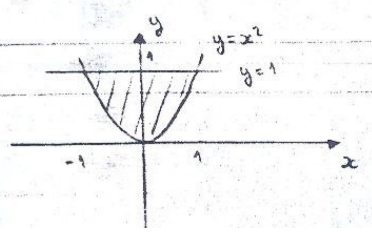
$$= \int_0^1 \int_0^x \left[xy^2 \frac{z^4}{4} \right]_0^{xy} dy\,dx = \int_0^1 \int_0^x \frac{x^5y^6}{4} \, dy\,dx =$$

$$= \frac{1}{4} \int_0^1 \left[\frac{x^5 y^7}{7} \right]_0^x dy = \frac{1}{28} \int_0^1 x^{12} \, dx =$$

$$= \frac{1}{28} \cdot \left[\frac{x^{13}}{13} \right]_0^1 = \boxed{\frac{1}{364}}$$

c) $\iiint_W z \, dx\,dy\,dz$

$z = \sqrt{x^2+y^2} \qquad y = x^2 \qquad z = 0 \quad y = 1$



TIPO I

$-1 \le x \le 1$

$x^2 \le y \le 1$

$0 \le z \le \sqrt{x^2+y^2}$

$$\iiint_W z\,dx\,dy\,dz = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx =$$

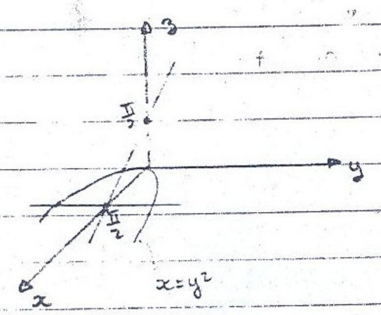
$$= \frac{1}{2}\int_{-1}^{1}\int_{x^2}^{1}\left[z^2\right]_0^{\sqrt{x^2+y^2}} dy\,dx = \frac{1}{2}\int_{-1}^{1}\int_{x^2}^{1} x^2+y^2\,dy\,dx =$$

$$= \frac{1}{2}\int_{-1}^{1}\left[x^2 . y + \frac{y^3}{3}\right]_{x^2}^{1} dx = \frac{1}{2}\int_{-1}^{1} x^2+\frac{1}{3}-x^4-\frac{x^6}{3}\,dx =$$

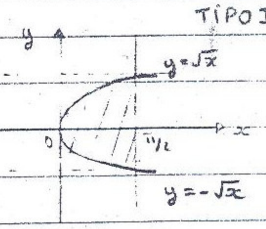
$$= \frac{1}{2}\left[\frac{x^3}{3}+\frac{x}{3}-\frac{x^5}{5}-\frac{x^7}{21}\right]_{-1}^{1} = \frac{1}{\cancel{2}}\times\cancel{2}\left[\frac{x^3}{3}+\frac{x}{3}-\frac{x^5}{5}-\frac{x^7}{21}\right]_0^1 =$$

$$= \frac{1}{3}+\frac{1}{3}-\frac{1}{5}-\frac{1}{21} = \frac{70-21-5}{105} = \boxed{\frac{44}{105}}\,//$$

d) $\iiint_W y . \cos(x+z)\,dx\,dy\,dz$ $\quad x=y^2 \quad x+z=\frac{\pi}{2} \quad z=0$



TIPO I



$0 \le x \le \pi/2$

$-\sqrt{x} \le y \le \sqrt{x}$

$0 \le z \le \pi/2 - x$

$$\iiint_W y . \cos(x+z)\,dx\,dy\,dz = \int_0^{\pi/2}\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{\pi/2-x} y . \cos(x+z)\,dz\,dy\,dx =$$

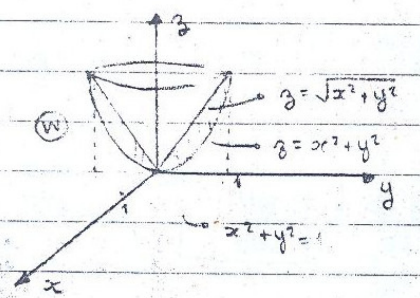
$$= \int_0^{\pi/2}\int_{-\sqrt{x}}^{\sqrt{x}}\left[y\ \text{sen}(x+z)\right]_0^{\pi/2-x} dy\,dx = \int_0^{\pi/2}\int_{-\sqrt{x}}^{\sqrt{x}} y . \text{sen}(\cancel{x}+\pi/2-\cancel{x}) - y . \text{sen}$$

dy dx

$$= \int_0^{\pi/2}\int_{-\sqrt{x}}^{\sqrt{x}} y - y \sin x \, dy\, dx = \int_0^{\pi/2} \left[\frac{y^2}{2} - \frac{y^2}{2} \sin x \right]_{-\sqrt{x}}^{\sqrt{x}} dx =$$

$$= \int_0^{\pi/2} \cancel{\frac{x}{2}} - \cancel{\frac{x}{2} \sin x} - \cancel{\frac{x}{2}} + \cancel{\frac{x}{2} \sin x} \, dx = \int_0^{\pi/2} 0 \, dx = \boxed{0}$$

(2) a) $z = \sqrt{x^2+y^2}$ cone $\qquad z = x^2+y^2$ parabolóide



intersecção:

$$\begin{cases} z = \sqrt{x^2+y^2} \\ z = x^2+y^2 \end{cases}$$

$(x^2+y^2)^2 = (\sqrt{x^2+y^2})^2$

$(x^2+y^2)^2 = (x^2+y^2)$

$x^2+y^2=1$

$x^2+y^2 \le z \le \sqrt{x^2+y^2}$

$$\begin{cases} x = r\cos\theta \\ y = r \sin\theta \\ z = z \end{cases}$$

$(r^2\cos^2\theta + r^2 \sin^2\theta) = 1$

$r^2 = 1 \quad \therefore \quad r = 1 \qquad 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$r^2 \le z \le r$

$$\text{vol}(W) = \iiint_W dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r \, dz\, dr\, d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \left[z \cdot r \right]_{r^2}^{r} dr\, d\theta = \int_0^{2\pi}\int_0^1 r^2 - r^3 \, dr\, d\theta =$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3} - \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} - \frac{1}{4} \, d\theta = \left[\frac{\theta}{12} \right]_0^{2\pi} = \boxed{\frac{\pi}{6}}$$

(/ /)

b) $z = 8 - (x^2+y^2)$ $\qquad z = x^2 + 3y^2$

interseção:

$$\begin{cases} z = 8-(x^2+y^2) \\ z = x^2+3y^2 \end{cases}$$

$8 - x^2 - y^2 = x^2 + 3y^2$

$2x^2 + 4y^2 = 8$

$\frac{x^2}{4} + \frac{y^2}{2} = 1$ (elipse)

$$\begin{cases} \frac{x}{2} = r\cos\theta \therefore x = 2r\cos\theta \\ \frac{y}{\sqrt{2}} = r\,\text{sen}\,\theta \therefore y = \sqrt{2}\,r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$x^2 + 3y^2 \leq z \leq 8-(x^2+y^2)$

- $z = 8-(x^2+y^2) = 8 - (4r^2\cos^2\theta + 2r^2\,\text{sen}^2\theta) =$

$= 8 - 4r^2\cos^2\theta - 2r^2\,\text{sen}^2\theta = 8 - 4r^2\cos^2\theta - 2r^2 + 2r^2\cos^2\theta$

$= 8 - 2r^2 - 2r^2\cos^2\theta$

- $z = x^2 + 3y^2 = 4r^2\cos^2\theta + 6r^2\,\text{sen}^2\theta =$

$= 4r^2\cos^2\theta + 6r^2 - 6r^2\cos^2\theta = 6r^2 - 2r^2\cos^2\theta$

- $$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} 2\cos\theta & -2r\,\text{sen}\,\theta & 0 \\ \sqrt{2}\,\text{sen}\,\theta & \sqrt{2}r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$= 2\sqrt{2}\,r\cos^2\theta + 2\sqrt{2}\,r\,\text{sen}^2\theta = 2\sqrt{2}\,r$

- $$\text{vol.}(W) = \iiint_W dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_{6r^2-2r^2\cos^2\theta}^{8-2r^2-2r^2\cos^2\theta} 2\sqrt{2}\,r\,dz\,dr\,d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi}\int_0^1 r\,(8-2r^2-\cancel{2r^2\cos^2\theta}-6r^2+\cancel{2r^2\cos^2\theta})\,dr\,d\theta =$$

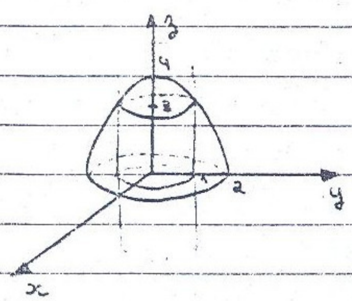
$$= 2\sqrt{2}\int_0^{2\pi}\int_0^1 r\,(8-8r^2)\,dr\,d\theta = 16\sqrt{2}\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta =$$

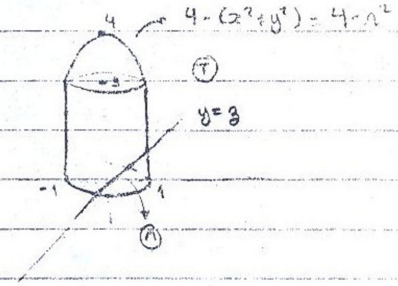
$$16\sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta = 16\sqrt{2}\int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{4}\right)d\theta =$$

$$= 16\sqrt{2}\cdot\left[\frac{\theta}{4}\right]_0^{2\pi} = 16\sqrt{2}\cdot\frac{2\pi}{4} = \boxed{8\sqrt{2}\,\pi}$$

c) $z = 4 - x^2 - y^2$ $\quad z = y$ $\quad$ no interior do cilindro:

$z = 4 - (x^2+y^2)$ $\quad\quad x^2+y^2 = 1 \quad z \geq 0$





$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

vol. (W) = volume total − volume de A

Ⓐ $\begin{cases} 0 \leq \theta \leq \pi \\ 0 \leq r \leq 1 \\ 0 \leq z \leq y = r\,\text{sen}\,\theta \end{cases}$

$$\int_0^{\pi}\int_0^1\int_0^{r\,\text{sen}\theta} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^1 \left[r\cdot z\right]_0^{r\,\text{sen}\theta} dr\,d\theta =$$

$$= \int_0^{\pi}\int_0^1 r^2\,\text{sen}\,\theta\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^3}{3}\,\text{sen}\,\theta\right]_0^1 d\theta =$$

$= \frac{1}{3}\int_0^{\pi} \operatorname{sen}\theta \, d\theta = \frac{1}{3}\left[-\cos\theta\right]_0^{\pi} = \frac{1}{3}(1+1) = \frac{2}{3} = V_A$

$$\text{(T)} \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z \le 3 \end{cases} \cup \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 3 \le z \le 4-r^2 \end{cases}$$

1º   2º

1º $\int_0^{2\pi}\int_0^{1}\int_0^{3} r \, dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} 3r \, dr\,d\theta = \frac{3}{2}\int_0^{2\pi} 1 \, d\theta = \frac{3}{\cancel{2}} \cdot \cancel{2}\pi = 3\pi$

2º $\int_0^{2\pi}\int_0^{1}\int_3^{4-r^2} r \, dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} r(4-r^2-3) \, dr\,d\theta =$

$= \int_0^{2\pi}\int_0^{1} r - r^3 \, dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{2\pi}{4_2} = \frac{\pi}{2}$

$V_T = 3\pi + \pi/2 = 7\pi/2$

$\Rightarrow V_W = V_T - V_A = \frac{7\pi}{2_{/3}} - \frac{2}{3_{/2}} = \boxed{\frac{21\pi - 4}{6}}$,,

d) $z = \sqrt{x^2+y^2}$ $x^2+y^2 - \sqrt{x^2+y^2} = x$ $z = 0$

cone

$$\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \\ z = z \end{cases}$$

$x^2+y^2 - \sqrt{x^2+y^2} = x$

$r^2 - r = r\cos\theta$

$\cancel{r}(r-1) = \cancel{r}\cos\theta$

$r - 1 = \cos\theta$

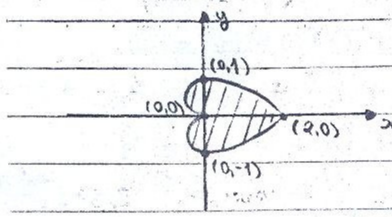
$r = \cos\theta + 1$

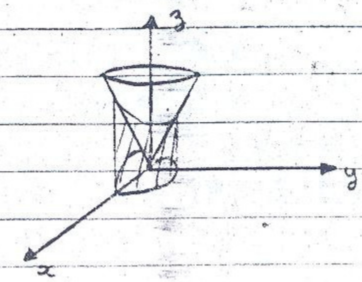
$0 \le \theta \le 2\pi$ CARDIÓIDE

$0 \le r \le 1 + \cos\theta$

$0 \le \theta \le 2\pi$

$0 \le z \le r$ $(\sqrt{x^2+y^2})$





$$vol(W) = \iiint_W dx\,dy\,dz = \int_0^{2\pi}\int_0^{1+\cos\theta}\int_0^{r} r\; dz\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\; dr\,d\theta = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1+\cos\theta} d\theta =$$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^3}{3}\, d\theta = \frac{1}{3}\int_0^{2\pi} 1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta \; d\theta =$$

$$= \frac{1}{3}\left[\theta + 3\sin\theta + \underbrace{\int_0^{2\pi} 3\cos^2\theta\, d\theta}_{⊛_1} + \underbrace{\int_0^{2\pi} \cos^3\theta\, d\theta}_{⊛_2}\right] =$$

$⊛_1$ $$\int \cos^2\theta\, d\theta = \frac{1}{2}\int 1 + \cos 2\theta\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

$⊛_2$ $$\int \cos^3\theta\, d\theta = \int \cos\theta\,(1-\sin^2\theta)\,d\theta = \int \cos\theta\, d\theta - \int \cos\theta \sin^2\theta\, d\theta =$$

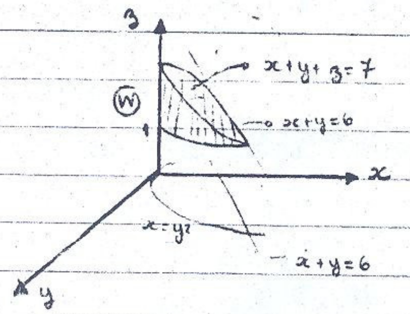
$$= \sin\theta - \frac{\sin^3\theta}{3}$$

$$= \frac{1}{3}\left[\theta + 3\sin\theta + \frac{3\theta}{2} + \frac{3\sin 2\theta}{4} + \sin\theta - \frac{\sin^3\theta}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3}\left[\frac{5\theta}{2}\right]_0^{2\pi} = \frac{10\pi}{6} = \boxed{\frac{5\pi}{3}}\;,,$$

1) $W = \{ (x,y,z) \in \mathbb{R}^3 / \ z \geq 1 \ , x+y+z \leq 7 \ , x \geq y^2 \}$

$z = 1 \qquad x+y+z = 7 \qquad x = y^2$

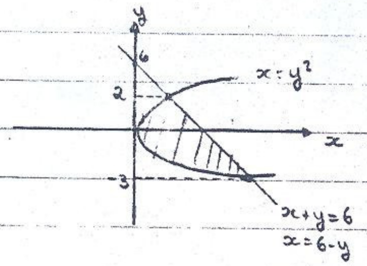


intersecção:

$$\begin{cases} z = 7 - x - y \\ z = 1 \end{cases}$$

$7 - x - y = 1$

$x + y = 6$



$$\left.\begin{matrix} x+y=6 \\ x=y^2 \end{matrix}\right\} \quad y^2 + y - 6 = 0$$

$$y = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \mp 5}{2}$$

$y = 2$ ou $y = -3$

$$\text{vol}(W) = \iiint_W dx\,dy\,dz \qquad \begin{matrix} y^2 \leq x \leq 6-y \\ -3 \leq y \leq 2 \end{matrix} \qquad 1 \leq z \leq 7-x-y$$

$$= \int_{-3}^{2}\int_{y^2}^{6-y}\int_{1}^{7-x-y} dz\,dx\,dy = \int_{-3}^{2}\int_{y^2}^{6-y} (7-x-y-1)\,dx\,dy =$$

$$= \int_{-3}^{2}\int_{y^2}^{6-y} 6-x-y\,dx\,dy = \int_{-3}^{2} \left[(6-y).x - \frac{x^2}{2} \right]_{y^2}^{6-y} dy =$$

$$= \int_{-3}^{2} (6-y)^2 - \frac{(6-y)^2}{2} - (6-y)y^2 + \frac{y^4}{2}\,dy =$$

$$= \int_{-3}^{2} \frac{(6-y)^2}{2} - (6-y)y^2 + \frac{y^4}{2}\,dy =$$

$$= \int_{-3}^{2} \frac{36-12y+y^2}{2} - 6y^2 + y^3 + \frac{y^4}{2}\, dy =$$

$$= \frac{1}{2}\int_{-3}^{2} 36-12y+y^2-12y^2+2y^3+y^4 \, dy =$$

$$= \frac{1}{2}\int_{-3}^{2} + y^4 + 2y^3 - 11y^2 - 12y + 36 \, dy =$$

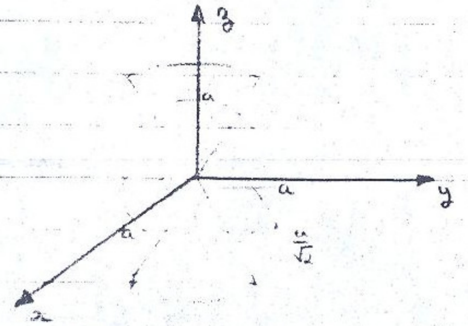
$$= \frac{1}{2}\left[+\frac{y^5}{5} + \frac{y^4}{2} - \frac{11y^3}{3} - 6y^2 + 36y \right]_{-3}^{2} =$$

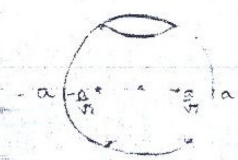
$$= \frac{1}{2}\left(+\frac{32}{5} + \frac{16}{2} - \frac{88}{3} - 24 + 72 + \frac{243}{5} - \frac{81}{2} - \frac{297}{3} + 54 + 108 \right) =$$

$$= \frac{1}{2}\left(+\frac{275}{5_{/6}} - \frac{65}{2_{/15}} - \frac{385}{3_{/10}} + \frac{210}{1_{/30}} \right) = \frac{1}{2} \cdot \frac{\overset{625}{3125}}{30} = \boxed{\frac{625}{12}}$$

f) $W = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2 \ \text{e} \ z^2 \leq x^2+y^2 \}$

$x^2+y^2+z^2 = a^2$ — esfera $\qquad x^2+y^2-z^2=0$ — cone





intersecção:

$x^2+y^2+z^2=a^2$ ∴ $x^2+y^2-z^2=0$

$z^2=a^2-x^2-y^2$ $z^2=x^2+y^2$

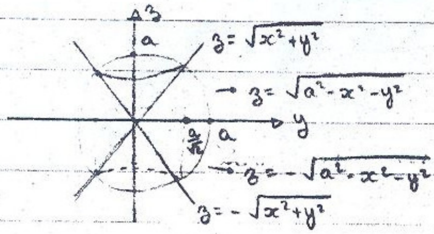
$z=\pm\sqrt{a^2-x^2-y^2}$ $z=\pm\sqrt{x^2+y^2}$

$$\left(\sqrt{a^2-x^2-y^2}\right)^2=\left(\sqrt{x^2+y^2}\right)^2$$

$$a^2-x^2-y^2=x^2+y^2$$

$$2x^2+2y^2=a^2$$

$$x^2+y^2=\frac{a^2}{2}$$



$$\begin{cases} x=r\cos\theta \\ y=r\,\mathrm{sen}\,\theta \\ z=z \end{cases}$$

$W_1=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le a^2/2,\ -\sqrt{x^2+y^2}\le z\le\sqrt{x^2+y^2}\}$

$\cup$

$W_2=\{(x,y,z)\in\mathbb{R}^3 \mid a^2/2\le x^2+y^2\le a^2,\ -\sqrt{a^2-x^2-y^2}\le z\le\sqrt{a^2-x^2-y^2}\}$

$W=W_1+W_2$

$Q=Q_1+Q_2$

$Q_1=\{(r,\theta,z)\in\mathbb{R}^3 \mid 0\le r\le a/\sqrt{2},\ 0\le\theta\le 2\pi,\ -r\le z\le r\}$

$\cup$

$Q_2=\{(r,\theta,z)\in\mathbb{R}^3 \mid a/\sqrt{2}\le r\le a,\ 0\le\theta\le 2\pi,\ -\sqrt{a^2-r^2}\le z\le\sqrt{a^2-r^2}\}$

$$\iiint_W dx\,dy\,dz=\iiint_Q r\,dr\,d\theta\,dz=$$

$$2\int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_0^{r} r\,dz\,dr\,d\theta+2\int_0^{2\pi}\int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta=$$

$$= 2\int_0^{2\pi}\int_0^{a/\sqrt{2}} r^2\,dr\,d\theta + 2\int_0^{2\pi}\int_{a/\sqrt{2}}^{a} r\,(a^2-r^2)^{1/2}\,dr\,d\theta = \qquad u = a^2 - r^2, \quad du = -2r\,dr$$

$$= \frac{2}{3}\int_0^{2\pi}\Big[r^3\Big]_0^{a/\sqrt{2}}\,d\theta + 2\cdot\left(-\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{2}{3}(a^2-r^2)^{3/2}\right]_{a/\sqrt{2}}^{a} d\theta =$$

$$= \frac{2}{3}\int_0^{2\pi}\frac{a^3}{2\sqrt{2}}\,d\theta - \frac{2}{3}\int_0^{2\pi} -\frac{a^3}{2\sqrt{2}}\,d\theta = \frac{4}{3}\cdot\frac{1}{\cancel{2}\sqrt{2}}\Big[a^3\cdot\theta\Big]_0^{2\pi} =$$

$$= \frac{2}{3\sqrt{2}}\cdot a^3\cdot 2\pi\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{4}\pi\sqrt{2}\cdot a^3}{3\cdot\cancel{2}} = \boxed{\frac{2\pi a^3\cdot\sqrt{2}}{3}} \;//$$

Usando mudança de variáveis esféricas:

$$\begin{cases} x = \rho\,\text{sen}\,\varphi\cos\theta \\ y = \rho\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = \rho\cos\varphi \end{cases} \qquad \begin{aligned} &x^2+y^2+z^2 = a^2 \\ &\rho^2 = a^2 \\ &\rho = a \;\therefore\; 0 \le \rho \le a \end{aligned}$$

$\pi/4 \le \varphi \le \pi/2$ por simetria: $0 \le \theta \le \pi$

[illegible]

$$vol(W) = 4\int_0^{\pi}\int_{\pi/4}^{\pi/2}\int_0^{a}\rho^2\,\text{sen}\,\varphi\;d\rho\,d\varphi\,d\theta =$$

$$= \frac{4}{3}\int_0^{\pi}\int_{\pi/2}^{\pi/4}\Big[\rho^3\,\text{sen}\,\varphi\Big]_0^a\,d\varphi\,d\theta = \frac{4}{3}\int_0^{\pi}\int_{\pi/2}^{\pi/4} a^3\,\text{sen}\,\varphi\;d\varphi\,d\theta =$$

$$= \frac{4}{3}\int_0^{\pi}\Big[-a^3\cos\varphi\Big]_{\pi/4}^{\pi/2}\,d\theta = \frac{4}{3}\int_0^{\pi} -a^3\left(-\frac{\sqrt{2}}{2}\right)d\theta =$$

$$= \frac{4}{3}\cdot\frac{\sqrt{2}}{2}\Big[a^3\,\theta\Big]_0^{\pi} = \boxed{2\pi a^3\sqrt{2}} \;//$$

g) $W = \{(x,y,z) \in \mathbb{R}^3 / \ x^2+y^2+z^2 \leq 4, \ x^2+y^2 \leq 2y\}$

$x^2+y^2+z^2 = 4$

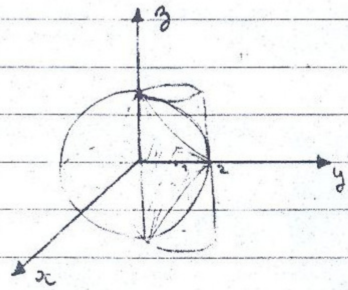
esfera

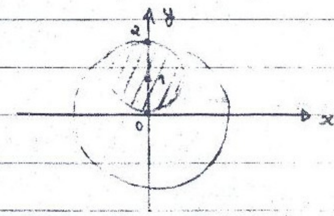
$x^2+y^2-2y=0$

$x^2+y^2-2y+1=1$

$x^2+(y-1)^2=1$

cilindro





$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$x^2+y^2=2y$

$r^2\cos^2\theta + r^2\,\text{sen}^2\theta = 2r\,\text{sen}\,\theta$

$r^2 = 2r\,\text{sen}\,\theta$

$r=0 \quad r = 2\,\text{sen}\,\theta$

$0 \leq \theta \leq \pi$

$0 \leq r \leq 2\,\text{sen}\,\theta$

$-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$

$x^2+y^2+z^2=4$

$r^2\cos^2\theta + r^2\,\text{sen}^2\theta + z^2 = 4$

$z^2 = 4-r^2 \therefore z = \pm\sqrt{4-r^2}$

$$\iiint_{W=g(Q)} dx\,dy\,dz = \iiint_{Q} r\,dr\,d\theta\,dz$$

$$= 2.\int_0^{\pi}\int_0^{2\,\text{sen}\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{2\,\text{sen}\theta} r.\sqrt{4-r^2}\,dr\,d\theta =$$

$u = 4-r^2$

$du = -2r\,dr$

$$= 2.\left(-\frac{1}{2}\right)\int_0^{\pi}\left[\frac{2}{3}(4-r^2)^{3/2}\right]_0^{2\,\text{sen}\theta} d\theta = -\frac{2}{3}\int_0^{\pi} (4-4\,\text{sen}^2\theta)^{3/2} - (4)^{3/2}\,d\theta =$$

$1-\text{sen}^2 = \cos^2$

$$= -\frac{2}{3}\int_0^{\pi} 8.\underbrace{(1-\text{sen}^2\theta)^{3/2}}_{=(\cos^2\theta)^{3/2} = (\cos^2\theta)^1.(\cos^2\theta)^{1/2}} - 8\,d\theta = -\frac{2}{3}\int_0^{\pi} 8\cos^2\theta.|\cos\theta| - 8\,d\theta =$$

$\cos^2\theta \quad |\cos\theta|$

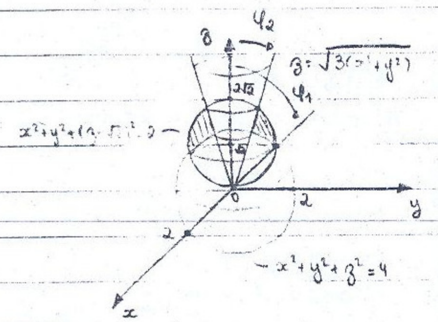
$$= -\frac{2}{3} \cdot 8 \int_0^{\pi} \cos^2\theta \cdot |\cos\theta| - 1 \, d\theta = -\frac{16}{3} \int_0^{\pi/2} \underset{(*)}{\cos^3\theta \, d\theta} - \int_{\pi/2}^{\pi} \underset{(*)}{\cos^3\theta \, d\theta} - \int_0^{\pi} d\theta =$$

$$= -\frac{16}{3} \left\{ \left[\operatorname{sen}\theta - \frac{\operatorname{sen}^3\theta}{3} \right]_0^{\pi/2} - \left[\operatorname{sen}\theta - \frac{\operatorname{sen}^3\theta}{3} \right]_{\pi/2}^{\pi} - [\theta]_0^{\pi} \right\} = \boxed{\frac{16\pi}{3} - \frac{64}{9}}$$

(*) $\int \cos^3\theta \, d\theta = \int \cos\theta (1 - \operatorname{sen}^2\theta) d\theta = \int \cos\theta \, d\theta - \int \operatorname{sen}^2\theta \cos\theta \, d\theta =$

$= \operatorname{sen}\theta - \operatorname{sen}^3\theta / 3$

h) $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \geq 4, \; x^2 + y^2 + (z - \sqrt{2})^2 \leq 2,$
$z \leq \sqrt{3(x^2 + y^2)} \}$



$x^2 + y^2 + z^2 - 2\sqrt{2}z + 2 = 2$
$x^2 + y^2 + z^2 = 2\sqrt{2}z$

coordenadas cartesianas

- $x^2 + y^2 + z^2 = 4$
- $x^2 + y^2 + z^2 = 2\sqrt{2}z$
- $z = \sqrt{3(x^2 + y^2)}$

coordenadas esféricas:

$\rho^2 = 4 \therefore \rho = 2 \qquad 0 \leq \varphi \leq \pi$

$\rho^2 = 2\sqrt{2}\rho\cos\varphi$

$\rho = 0 \quad \rho = 2\sqrt{2}\cos\varphi$

$0 \leq \varphi \leq \pi/2$

$\rho\cos\varphi = \sqrt{3\rho^2 \operatorname{sen}^2\varphi}$

$\rho\cos\varphi = \rho \operatorname{sen}\varphi \sqrt{3}$

$\operatorname{tg}\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\varphi = \pi/6 = \varphi_1$

(/ /)

- interseção das esferas:

$$\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2+z^2=2\sqrt{2}\,z \end{cases} \xRightarrow[\text{esféricas}]{\text{coord.}} \begin{cases} \rho=2 \\ \rho=2\sqrt{2}\cos\varphi \end{cases}$$

$$2=2\sqrt{2}\cos\varphi$$

$$\cos\varphi=\frac{\cancel{2}}{\cancel{2}\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2} \quad \therefore \varphi=\frac{\pi}{4}=\varphi_1$$

• traçando um raio saindo da origem e passando entre $\pi/6$ e $\pi/4$, encontramos 1º a esfera de raio 2, e depois a de raio $2\sqrt{2}$.

$$Q=\{(r,\theta,\varphi)\mid 0\le\theta\le 2\pi,\ \pi/6\le\varphi\le\pi/4,\ 2\le\rho\le 2\sqrt{2}\cos\varphi\}$$

$$\text{vol}(W)=\iiint_{W=g(Q)} dx\,dy\,dz=\iiint_Q \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=$$

$$=\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_2^{2\sqrt{2}\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=\frac{1}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\Big[\rho^3\sin\varphi\Big]_2^{2\sqrt{2}\cos\varphi}d\varphi\,d\theta=$$

$$=\frac{1}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\sin\varphi\,(16\sqrt{2}\cos^3\varphi-8)\,d\varphi\,d\theta=$$

$$=\frac{8}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/4} 2\sqrt{2}\sin\varphi\cos^3\varphi-\sin\varphi\;d\varphi\,d\theta= \qquad d(\cos\varphi)=-\sin\varphi$$

$$=\frac{8}{3}\int_0^{2\pi}\left(-\int_{\pi/6}^{\pi/4}2\sqrt{2}\cos^3\varphi\,d(\cos\varphi)\,d\varphi\,d\theta+\Big[\cos\varphi\Big]_{\pi/6}^{\pi/4}\right)d\theta=$$

$$=\frac{8}{3}\int_0^{2\pi}\left(-\frac{\cancel{2}\sqrt{2}\cos^4\varphi}{\cancel{4}\,2}\right)_{\pi/6}^{\pi/4}+\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\right)\;d\theta=$$

$$= \frac{8}{3}\int_0^{2\pi}\left(-\frac{\sqrt{2}}{2}\left[\frac{4}{16}-\frac{9}{16}\right]+\frac{(\sqrt{2}-\sqrt{3})}{2}\right)d\theta =$$

$$= \frac{\cancel{8}}{3}\cdot \cancel{2}\pi \left[\frac{5\sqrt{2}}{\cancel{32}_2}+\frac{16\sqrt{2}}{\cancel{32}_2}-\frac{16\sqrt{3}}{\cancel{32}_2}\right] =$$

$$= \boxed{\frac{\pi}{6}\cdot\left(21\sqrt{2}-16\sqrt{3}\right)_{\shortparallel}}$$

(3) $$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz = \qquad \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

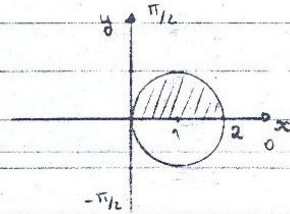
$\sqrt{2x-x^2} = y$

$2x - x^2 = y^2$

$x^2+y^2 = 2x \longrightarrow (x-1)^2+y^2 = 1$

$r^2 = 2r\cos\theta$

$r = 0 \quad r = 2\cos\theta$



$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq z \leq a$

$$= \int_0^a\int_0^{\pi/2}\int_0^{2\cos\theta} z\cdot r\cdot r \; dr\,d\theta\,dz$$

$$= \int_0^a\int_0^{\pi/2}\left[\frac{r^3}{3}\cdot z\right]_0^{2\cos\theta} d\theta\,dz = \frac{8}{3}\int_0^a\int_0^{\pi/2}\cos^3\theta\cdot z\,d\theta\,dz$$

$$\circledast \quad \int \cos^3\theta\,d\theta = \int\cos^2\theta\cdot\cos\theta\,d\theta = \int\cos^2\theta\,d(\text{sen}\,\theta)\,d\theta =$$

$$= \int(1-\text{sen}^2\theta)\,d(\text{sen}\,\theta)\,d\theta = \left[\text{sen}\,\theta - \frac{\text{sen}^3\theta}{3}\right]$$

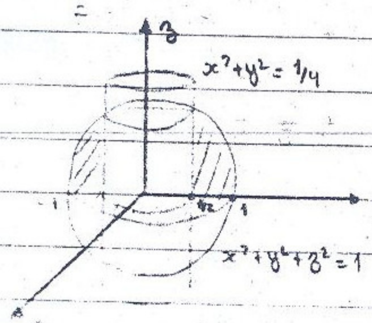
$$= \frac{8}{3} \int_0^a z \cdot \left[\operatorname{sen}\theta - \frac{\operatorname{sen}^3\theta}{3} \right]_0^{\pi/2} dz =$$

$$= \frac{8}{3} \int_0^a z \left(\frac{1}{1} - \frac{1}{3} \right) dz = \frac{16}{9} \int_0^a z \, dz =$$

$$= \frac{16}{9} \left[\frac{z^2}{2} \right]_0^a = \boxed{\frac{8a^2}{9}}$$

④ a) $\displaystyle\iiint_W z \, dx\, dy\, dz$

$W = \{ (x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 \leq 1 \,,\, z \geq 0 \,,\, x^2 + y^2 \geq 1/4 \}$



$$\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \\ z = z \end{cases}$$

$1/2 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq \sqrt{1 - r^2}$

$x^2 + y^2 + z^2 = 1$

$z^2 = 1 - x^2 - y^2$

$z^2 = 1 - r^2 \;\therefore\; z = \sqrt{1 - r^2}$

$$\text{vol}(W) = \iiint_{W = g(Q)} z \, dx\, dy\, dz = \iiint_Q z \cdot r \, dr\, d\theta\, dz$$

$$\int_0^{2\pi}\int_{1/2}^{1}\int_0^{\sqrt{1-r^2}} z r \, dz\, dr\, d\theta = \int_0^{2\pi}\int_{1/2}^{1} \left[r \cdot \frac{z^2}{2} \right]_0^{\sqrt{1-r^2}} dr\, d\theta =$$

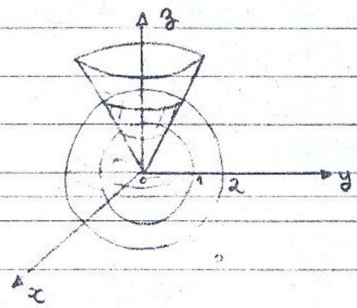
$$\int_0^{2\pi}\int_{1/2}^{1} \frac{r.(1-r^2)}{2}\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_{1/2}^{1} r - r^3 \, dr\, d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{1/2}^{1} d\theta = \frac{1}{2}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{64} \right) d\theta$$

$$= \frac{1}{2}\left[\frac{9 .\theta}{64} \right]_0^{2\pi} = \frac{1}{\cancel{2}} \cdot \frac{9}{64} \cdot \cancel{2}\pi = \boxed{\frac{9\pi}{64}}$$

b) $\displaystyle\iiint_W \frac{dx\,dy\,dz}{z^2}$ $\qquad z = \sqrt{x^2+y^2}$ $\qquad z = \sqrt{1-x^2-y^2}$

$z = \sqrt{4-x^2-y^2}$



$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$z = \sqrt{1-x^2-y^2}$ $\qquad z = \sqrt{4-x^2-y^2}$

$x^2+y^2+z^2 = 1$ $\qquad x^2+y^2+z^2 = 4$

$\rho^2 = 1$ $\qquad \rho^2 = 4$

$\rho = 1$ $\qquad \rho = 2$

$1 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$0 \leq \varphi \leq \frac{\pi}{4}$ (pq o cos e sen reta $x=y$)

$$\iiint_W dx\,dy\,dz = \iiint_Q \frac{1}{\cancel{\rho^2}\cos^2\varphi} \cancel{\rho^2}\operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_1^{2} \operatorname{sen}\varphi . (\cos\varphi)^{-2} \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \operatorname{sen}\varphi \cdot (\cos\varphi)^{-2} \, d\varphi\, d\theta =$$

$u = \cos\varphi$

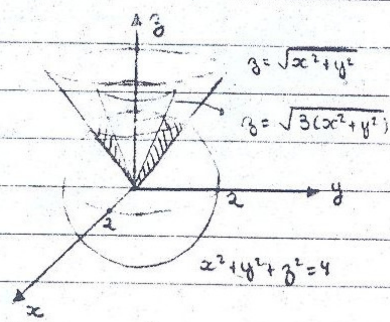
$du = -\operatorname{sen}\varphi \, d\varphi$

$$= -\int_0^{2\pi}\left[-\frac{1}{\cos\varphi}\right]_0^{\pi/4} d\theta = \int_0^{2\pi}\left(\frac{2}{\sqrt{2}}-1\right) d\theta =$$

$$= \int_0^{2\pi} \frac{(2\sqrt{2}-2)}{2}\, d\theta = \Big[(\sqrt{2}-1).\theta\Big]_0^{2\pi} = \boxed{(\sqrt{2}-1).2\pi}$$

c) $\iiint_W z\, dx\,dy\,dz \qquad z=\sqrt{x^2+y^2} \qquad z=\sqrt{3(x^2+y^2)}$

$x^2+y^2+z^2=4$



$$\begin{cases} x=\rho \operatorname{sen}\varphi \cos\theta \\ y=\rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z=\rho\cos\varphi \end{cases}$$

$x^2+y^2+z^2=4$

$\rho^2=4$

$\rho=2$

$0\le\rho\le 2 \qquad 0\le\theta\le 2\pi$

$z^2=\cancel{x^2}+y^2$	$z^2=3(\cancel{x^2}+y^2)$
$z^2=y^2$	$z^2=3y^2$
$z=\pm 1y$	$z=\pm\sqrt{3}y$
$tg\,\alpha=1$	$tg\,\alpha=\sqrt{3}$
$\alpha=\pi/4 \therefore \varphi=\pi/4$	$\alpha=\pi/3 \therefore \varphi=\pi/6$

$\pi/6\le\varphi\le\pi/4$

$$\iiint_W z\,dx\,dy\,dz = \iiint_Q \rho\cos\varphi \,.\, \rho^2\operatorname{sen}\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^2 \rho^3 \operatorname{sen}\varphi\cos\varphi \; d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\left[\frac{\rho^4}{4}\operatorname{sen}\varphi\cos\varphi\right]_0^2 d\varphi\, d\theta =$$

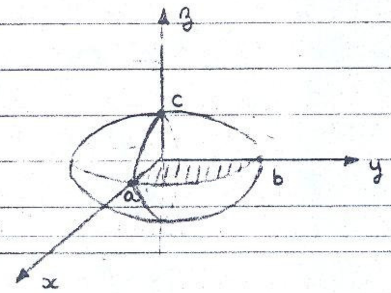
$$= 4\int_0^{2\pi}\int_{\pi/6}^{\pi/4} \operatorname{sen}\varphi\cos\varphi \; d\varphi\, d\theta = 4\int_0^{2\pi}\left[\frac{\operatorname{sen}^2\varphi}{2}\right]_{\pi/6}^{\pi/4} d\theta =$$

$$= 4\int_0^{2\pi}\left(\frac{1}{4/2}-\frac{1}{8/1}\right)d\theta = \cancel{4}\cdot\frac{1}{\cancel{8}_2}\Big[\theta\Big]_0^{2\pi} =$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\pi = \boxed{\pi}$$

d) $\displaystyle\iiint_W xyz \; dx\,dy\,dz$

$$W = \left\{ (x,y,z)\in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 ,\ \overbrace{x\geq 0,\ y\geq 0,\ z\geq 0}^{1^\circ \text{ octante}} \right\}$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$$

$\rho^2 = 1 \;\therefore\; \rho = 1$

$0\le\rho\le 1$

$0\le\theta\le\pi/2$

$0\le\varphi\le\pi/2$

$$\begin{cases} \dfrac{x}{a} = \rho\,\mathrm{sen}\,\varphi\cos\theta & \therefore\ x = a\rho\,\mathrm{sen}\,\varphi\cos\theta \\[2mm] \dfrac{y}{b} = \rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta & \therefore\ y = b\rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\[2mm] \dfrac{z}{c} = \rho\cos\varphi & \therefore\ z = c\rho\cos\varphi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} a\,\mathrm{sen}\,\varphi\cos\theta & -a\rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta & a\rho\cos\varphi\cos\theta \\ b\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta & b\rho\,\mathrm{sen}\,\varphi\cos\theta & b\rho\cos\varphi\,\mathrm{sen}\,\theta \\ c\cos\varphi & 0 & -c\rho\,\mathrm{sen}\,\varphi \end{vmatrix} =$$

$$= -abc\rho^2\,\mathrm{sen}^3\varphi\cos^2\theta - abc\rho^2\,\mathrm{sen}\,\varphi\cos^2\varphi\,\mathrm{sen}^2\theta - abc\rho^2\,\mathrm{sen}\,\varphi\cos^2\varphi\cos^2\theta$$

$$- abc\rho^2\,\mathrm{sen}^3\varphi\,\mathrm{sen}^2\theta =$$

$$= -abc\rho^2 \operatorname{sen}^3\varphi \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} - abc \operatorname{sen}\varphi \cos^2\varphi \underbrace{(\operatorname{sen}^2\theta + \cos^2\theta)}_{=1} =$$

$$= -abc\rho^2 \operatorname{sen}^3\varphi - abc \operatorname{sen}\varphi \cos^2\varphi =$$

$$= -abc\rho^2 \operatorname{sen}\varphi \underbrace{(\operatorname{sen}^2\varphi + \cos^2\varphi)}_{=1} = -abc\,\rho^2 \operatorname{sen}\varphi$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = abc\,\rho^2 \operatorname{sen}\varphi$$

$$\iiint_W xyz\; dx\,dy\,dz = \iiint_\Omega abc\rho^3 \operatorname{sen}^2\varphi \cos\varphi \operatorname{sen}\theta \cos\theta \cdot abc\,\rho^2 \operatorname{sen}\varphi \; d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 a^2b^2c^2 \rho^5 \operatorname{sen}^3\varphi \cos\varphi \operatorname{sen}\theta \cos\theta \; d\rho\, d\varphi\, d\theta =$$

$$= \frac{a^2b^2c^2}{6}\int_0^{\pi/2}\int_0^{\pi/2} \operatorname{sen}^3\varphi \underbrace{\cos\varphi}_{d(\operatorname{sen}\varphi)} \operatorname{sen}\theta \cos\theta \; d\varphi\, d\theta =$$

$$= \frac{a^2b^2c^2}{6}\int_0^{\pi/2} \left[\frac{\operatorname{sen}^4\varphi}{4}\right]_0^{\pi/2} \operatorname{sen}\theta \cos\theta \; d\theta =$$

$$= \frac{a^2b^2c^2}{24} \cdot \int_0^{\pi/2} \operatorname{sen}\theta \underbrace{\cos\theta\, d\theta}_{= d(\operatorname{sen}\theta)} =$$

$$= \frac{a^2b^2c^2}{24} \cdot \left[\frac{\operatorname{sen}^2\theta}{2}\right]_0^{\pi/2} =$$

$$= \frac{a^2b^2c^2}{24} \cdot \left(\frac{1}{2} - 0\right) = \boxed{\frac{a^2b^2c^2}{48}}\;,,$$

l) $\iiint_W (x^2+y^2+z^2)^{1/2}\; dx\,dy\,dz$

(i) $W = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2 \}$

$0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \pi$

$= 0 \leq \rho \leq a$

$x^2+y^2+z^2 = a^2$

$\rho^2 = a^2$

$\rho = a$

$f(x,y,z) = (x^2+y^2+z^2)^{1/2}$

$f(\rho,\theta,\varphi) = (\rho^2)^{1/2} = \rho$

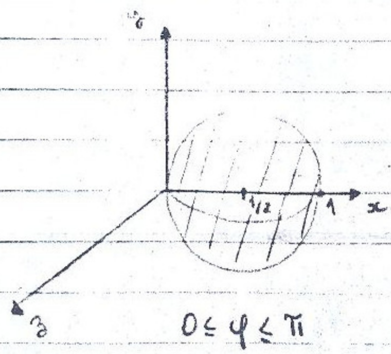
$$\iiint_W (x^2+y^2+z^2)^{1/2}\, dx\,dy\,dz = \iiint_\Omega \rho \cdot \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^4}{4} \operatorname{sen}\varphi \right]_0^a d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^4}{4} \operatorname{sen}\varphi \, d\varphi\, d\theta = \frac{a^4}{4} \int_0^{2\pi} \left[-\cos\varphi \right]_0^{\pi} d\theta =$$

$$= \frac{a^4}{4} \int_0^{2\pi} 2\, d\theta = \frac{a^4}{4} \cdot \left[2\theta \right]_0^{2\pi} = \frac{a^4}{\cancel{4}} \cdot \cancel{4}\pi = \boxed{\pi a^4}$$

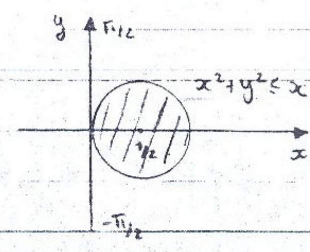
(ii) $W = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq x \}$



$x^2+y^2+z^2 = x$

$x^2 - x + y^2 + z^2 = 0$

$(x - 1/2)^2 + y^2 + z^2 = 1/4$



$0 \leq \varphi \leq \pi$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

(/ /)

$$x^2+y^2+z^2=x \qquad \begin{cases} x=\rho\,\operatorname{sen}\varphi\cos\theta \\ y=\rho\,\operatorname{sen}\varphi\,\operatorname{sen}\theta \\ z=\rho\cos\varphi \end{cases}$$

$$\rho^2=\rho\,\operatorname{sen}\varphi\cos\theta$$

$$\rho=0 \qquad \rho=\operatorname{sen}\varphi\cos\theta$$

$$\iiint_W (x^2+y^2+z^2)^{1/2}\,dx\,dy\,dz=\iiint_Q \rho.\rho^2\operatorname{sen}\varphi\,d\rho\,d\varphi\,d\theta=$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\operatorname{sen}\varphi\cos\theta}\rho^3\operatorname{sen}\varphi\,d\rho\,d\varphi\,d\theta=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\left[\rho^4\operatorname{sen}\varphi\right]_0^{\operatorname{sen}\varphi\cos\theta}d\varphi\,d\theta=$$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\operatorname{sen}^5\varphi\cos^4\theta\,d\varphi\,d\theta=\frac{1}{4}\underbrace{\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta}_{(*)}\,.\,\underbrace{\int_0^{\pi}\operatorname{sen}^5\varphi\,d\varphi}_{(**)}$$

(*) $$\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta=\int_{-\pi/2}^{\pi/2}(\cos^2\theta)^2\,d\theta=\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)^2d\theta=$$

$\cancel{\operatorname{sen}}\cos 2\theta=\cos\pi=0$

$$=\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cancel{2\cos2\theta}+\cos^2 2\theta}{4}\right)d\theta=\frac{1}{4}\left(\left[\theta\right]_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}\cos^2 2\theta\,d\theta\right)=$$

$\operatorname{sen}4\theta=\operatorname{sen}2\pi=0$

$$=\frac{1}{4}\left(\underbrace{\frac{\pi}{2}+\frac{\pi}{2}}_{\pi}+\frac{1}{2}\int_{-\pi/2}^{\pi/2}1+\cancel{\cos4\theta}\,d\theta\right)=\frac{1}{4}\left(\pi+\frac{1}{2}\left[\theta\right]_{-\pi/2}^{\pi/2}\right)=$$

$$=\frac{1}{4}\left(\frac{\pi}{1/2}+\frac{1}{2}.\pi\right)=\frac{3\pi}{8}$$

(**) $$\int_0^{\pi}\operatorname{sen}^5\varphi\,d\varphi=\int_0^{\pi}\operatorname{sen}^4\varphi.\underbrace{\operatorname{sen}\varphi\,d\varphi}_{=-d(\cos\varphi)}=-\int_0^{\pi}(1-\cos^2\varphi)^2\,d(\cos\varphi)=$$

$$=\int_0^{\pi}(1-2\cos^2\varphi+\cos^4\varphi)\,d(\cos\varphi)=-\left[\cos\varphi-\frac{2}{3}\cos^3\varphi+\frac{\cos^5\varphi}{5}\right]_0^{\pi}=$$

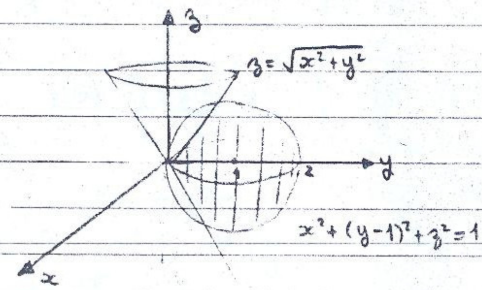
$$=-2.\left(-\frac{1}{1_{/15}}+\frac{2}{3_{/5}}-\frac{1}{5_{/3}}\right)=-2.\frac{-15+10-3}{15}=\frac{16}{15}$$

$$\circledast = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta \cdot \int_0^{\pi} \operatorname{sen}^5\varphi \, d\varphi =$$

$$= \frac{\cancel{1}^{\,1}}{\cancel{4}} \cdot \frac{3\pi}{\cancel{8}_{\,2}} \cdot \frac{\cancel{16}\,\cancel{4}^{\,1}}{\cancel{15}_{\,5}} = \boxed{\frac{\pi}{10}}$$

b) $$\iiint_W \frac{dx\,dy\,dz}{x^2+y^2+z^2}$$

$W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 2y \;,\; z \leq \sqrt{x^2+y^2} \;,\; y \geq x \;,\; x \geq 0\}$

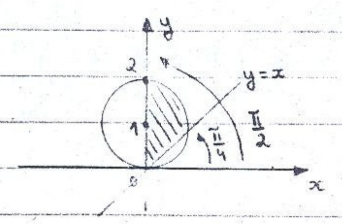


$x^2+y^2+z^2 = 2y$

$x^2+y^2-2y+z^2=0$

$x^2+(y-1)^2+z^2=1$

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$



$\frac{\pi}{4} \leq \varphi \leq \pi$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$x^2+y^2+z^2=2y$

$\rho^2 = 2\rho \operatorname{sen}\varphi \operatorname{sen}\theta$

$\rho = 0 \quad \rho = 2\operatorname{sen}\varphi \operatorname{sen}\theta$

$0 \leq \rho \leq 2\operatorname{sen}\varphi \operatorname{sen}\theta$

$$\iiint_W \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \iiint_Q \frac{1}{\cancel{\rho^2}} \cdot \cancel{\rho^2} \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\pi}\int_0^{2\operatorname{sen}\varphi\operatorname{sen}\theta} \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta = \int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\pi} 2\operatorname{sen}^2\varphi \operatorname{sen}\theta \, d\varphi\, d\theta =$$

$$= 2 \cdot \int_{\pi/4}^{\pi/2} \operatorname{sen}\theta \, d\theta \cdot \int_{\pi/4}^{\pi} \operatorname{sen}^2\varphi \, d\varphi =$$

$(*)$ $\quad$ $(**)$ $\quad$ $\bigstar$

$$(*) \int_{\pi/4}^{\pi/2} \operatorname{sen}\theta \, d\theta = \left[-\cos\theta\right]_{\pi/4}^{\pi/2} = -\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$(**) \int_{\pi/4}^{\pi} \operatorname{sen}^2\varphi \, d\varphi = \int_{\pi/4}^{\pi} \frac{1-\cos 2\varphi}{2} \, d\varphi$$

$u = 2\varphi$

$du = 2 d\varphi$

$$= \frac{1}{2} \int_{\pi/4}^{\pi} 1 - \cos 2\varphi \, d\varphi = \frac{1}{2}\left(\left[\varphi\right]_{\pi/4}^{\pi} - \int_{\pi/4}^{\pi} \cos 2\varphi \, d\varphi\right) =$$

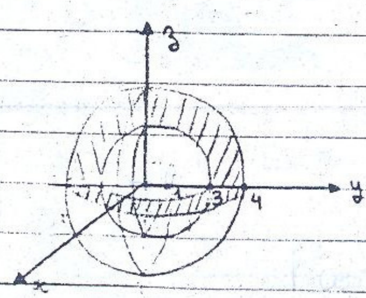
$$= \frac{1}{2}\left(\pi - \frac{\pi}{4} - \frac{1}{2}\left[\operatorname{sen} 2\varphi\right]_{\pi/4}^{\pi}\right) = \frac{1}{2}\left(\frac{3\pi}{4} - \frac{1}{2}\left(-1\right)\right) =$$

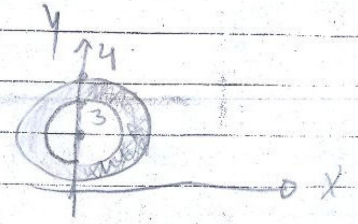
$$= \frac{1}{2} \cdot \frac{3\pi + 2}{4} = \frac{3\pi+2}{8}$$

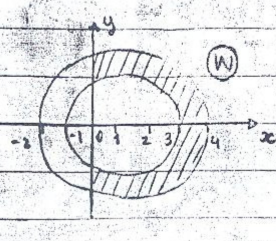
$$\bigstar = \cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{3\pi+2}{8} = \boxed{\frac{\sqrt{2}}{8} \cdot (3\pi + 2)}$$

g) $\displaystyle\iiint_W x \, dx\,dy\,dz$

$W = \{ (x,y,z) \in \mathbb{R}^3 / \ 4 \leq x^2 + (y-1)^2 + z^2 \leq 9 \ , \ x \geq 0 \ , \ z \geq 0 \}$



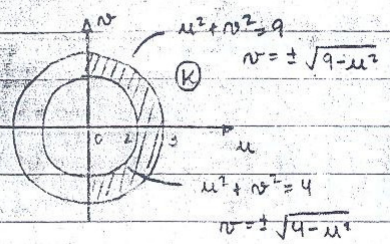




$$\begin{cases} u = x & x \geq 0 \\ v = y-1 & y \geq 0 \\ w = z \end{cases}$$

$$4 \leq x^2 + (y-1)^2 + z^2 \leq 9$$

$$4 \leq u^2 + v^2 + w^2 \leq 9$$



$$0 \leq u \leq 3$$

$$\sqrt{4-u^2} \leq v \leq \sqrt{9-u^2}$$

$$0 \leq w \leq \sqrt{9-u^2-v^2}$$

$$\begin{cases} u = \rho \,\text{sen}\, \varphi \cos\theta \\ v = \rho \,\text{sen}\, \varphi \,\text{sen}\, \theta \\ w = \rho \cos \varphi \end{cases}$$

$$4 \leq u^2 + v^2 + w^2 \leq 9$$

$$4 \leq \rho^2 \leq 9 \;\therefore\; 2 \leq \rho \leq 3$$

$$-\pi/2 \leq \theta \leq \pi/2 \qquad 0 \leq \varphi \leq \pi/2$$

$$\iiint_W x \, dx\, dy\, dz = \iiint_K u \, du\, dv\, dw = \iiint_Q \rho \,\text{sen}\,\varphi \cos\theta \cdot \rho^2 \,\text{sen}\,\varphi \; d\rho\, d\varphi\, d\theta =$$

$$= 2 \int_0^{\pi/2}\int_0^{\pi/2}\int_2^3 \rho^3 \,\text{sen}^2\varphi \cos\theta \; d\rho\, d\varphi\, d\theta =$$

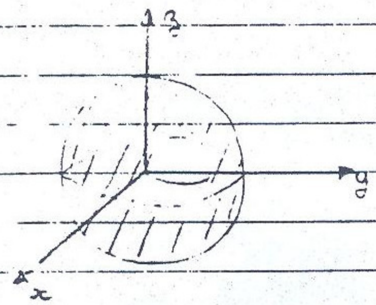
$$= \cancel{2} \cdot \frac{1}{\cancel{4}_{2}} \int_0^{\pi/2}\int_0^{\pi/2} \left[\rho^4 \,\text{sen}^2 \varphi \cos\theta \right]_2^3 d\varphi\, d\theta = \frac{1}{2} \int_0^{\pi/2}\int_0^{\pi/2} 65 \,\text{sen}^2\varphi \cos\theta \; d\varphi\, d\theta =$$

$$= \frac{65}{2} \int_0^{\pi/2}\int_0^{\pi/2} \frac{(1-\cos 2\varphi)}{2} \cdot \cos\theta \; d\varphi\, d\theta = \frac{65}{2} \cdot \frac{1}{2} \int_0^{\pi/2} \left[\varphi - \cancel{\frac{\text{sen}\, 2\varphi}{2}}^{\,0} \right]_0^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{65}{4} \cdot \frac{\pi}{2} \int_0^{\pi/2} \cos\theta \; d\theta = \frac{65\pi}{8} \cdot \left[\text{sen}\,\theta \right]_0^{\pi/2} = \frac{65\pi}{8} \cdot 1 = \boxed{\frac{65\pi}{8}}$$

5) $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} \Rightarrow$ densidade

$W = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 9 ,\ x^2+y^2+z^2 \geq 2y \}$

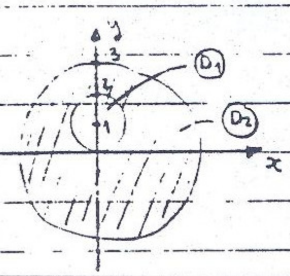


$x^2+y^2+z^2 = 2y$

$x^2+y^2-2y+z^2 = 0$

$x^2+(y-1)^2+z^2 = 1$

$$\begin{cases} x = \rho \,\text{sen}\,\varphi \cos\theta \\ y = \rho\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = \rho \cos\varphi \end{cases}$$



$x^2+y^2+z^2 = 2y$

$\rho^2 = 2\rho\,\text{sen}\,\varphi\,\text{sen}\,\theta$

$\rho = 0 \quad \rho = 2\,\text{sen}\,\varphi\,\text{sen}\,\theta$

$$\begin{cases} D_1 : \{ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq \theta \leq \pi ,\ 0 \leq \varphi \leq \pi ,\ 0 \leq \rho \leq 2\,\text{sen}\,\varphi\,\text{sen}\,\theta \} \\ D_2 : \{ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq \theta \leq 2\pi ,\ 0 \leq \varphi \leq \pi ,\ 0 \leq \rho \leq 3 \} \end{cases}$$

$\boxed{D_1}$ $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\,\text{sen}\,\varphi\,\text{sen}\,\theta} \frac{1}{\rho^2} \cdot \rho^2\,\text{sen}\,\varphi \, d\rho\, d\varphi\, d\theta = M_{D_1} =$

$\displaystyle = \int_0^{\pi}\int_0^{\pi} 2\,\text{sen}^2\varphi\,\text{sen}\,\theta \, d\varphi\, d\theta = 2 \underbrace{\int_0^{\pi} \text{sen}\,\theta\, d\theta}_{(*)} \cdot \underbrace{\int_0^{\pi} \text{sen}^2\varphi\, d\varphi}_{(**)} =$

(*) $\displaystyle\int_0^{\pi} \text{sen}\,\theta\, d\theta = \Big[-\cos\theta \Big]_0^{\pi} = 1 + 1 = 2$

$(**)$ $\displaystyle\int_0^{\pi} \operatorname{sen}^2\varphi \, d\varphi = \frac{1}{2}\int_0^{\pi} 1-\cos 2\varphi \, d\varphi =$

$= \dfrac{1}{2}\left[\varphi - \dfrac{1}{2}\operatorname{sen} 2\varphi\right]_0^{\pi} = \dfrac{1}{2}\left(\pi - 0 - 0 + 0\right) = \dfrac{\pi}{2}$

$(*)$ $= 2 \cdot \cancel{2} \cdot \dfrac{\pi}{\cancel{2}} = 2\pi$

$\boxed{D_2}$: $M_{D_2} = \displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = 3\int_0^{2\pi}\int_0^{\pi} \operatorname{sen}\varphi \, d\varphi \, d\theta =$

$= 3 \cdot \displaystyle\int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi} d\theta = 3 \cdot \int_0^{2\pi} 2 \, d\theta = 6 \cdot 2\pi = 12\pi$

$\Rightarrow M_T = M_{D_2} - M_{D_1}$

$M_T = 12\pi - 2\pi = \boxed{10\pi}$ //