§5.6 Exercícios

1. Considere a aplicação definida por

$$x = uv \qquad \text{e} \qquad y = v - u.$$

a) Determine a imagem D no plano xy do retângulo R no plano uv de vértices $(0,1), (1,1), (1,2)$ e $(0,2)$.

b) Calcule a área de D.

2. Considere a aplicação g definida pelas equações

$$x = u + v \qquad \text{e} \qquad y = v - u^2.$$

a) Utilizando g, calcule $\displaystyle\iint_D \left(x - y + \frac{1}{4}\right)^{-1/2} dxdy$, onde D é a imagem no plano xy da região Q no plano uv limitada pelas retas $u = 0$, $v = 0$ e $u + v = 2$.

b) Descreva e esboce a região D.

3. Calcule $\displaystyle\iint_D \cos\left(\frac{x-y}{x+y}\right) dxdy$, onde D é a região do plano xy limitada por $x + y = 1$, $x = 0$ e $y = 0$.

4. Calcule $\displaystyle\iint_D \frac{y + 2x}{\sqrt{y - 2x - 1}} dxdy$, onde D é a região do plano xy limitada pelas retas $y - 2x = 2$, $y + 2x = 2$, $y - 2x = 1$ e $y + 2x = 1$.

5. Calcule $\displaystyle\iint_D (2x + 1)\, dxdy$, onde D é a região no primeiro quadrante do plano xy, limitada pelas curvas $y = x^2$, $y = x^2 + 1$, $x + y = 1$ e $x + y = 2$.

6. Calcule $\displaystyle\iint_D x\, dxdy$, onde D é a região do plano xy limitada pelas parábolas $x = y^2 - 1$, $x = 1 - y^2$ e $x = 4 - \dfrac{y^2}{4}$.

Sugestão: use a mudança de variáveis $x = u^2 - v^2$, $y = uv$.

7. Calcule, usando mudança polar, as seguintes integrais:

a) $\displaystyle\iint_D \frac{dxdy}{(1 + x^2 + y^2)^2}$, onde $D = \{(x, y) \in I\!R^2 \mid x^2 + y^2 \leq 4\}$.

b) $\displaystyle\iint_D (x^2 + y^2)\, dxdy$, onde D é a região no primeiro quadrante do plano xy limitada por $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $y = x$ e $y = \dfrac{\sqrt{3}}{3}x$.

c) $\displaystyle\iint_D x^2y\,dxdy$, onde $D = \{(x,y) \in I\!R^2 \mid (x-1)^2 + y^2 \leq 1\}$.

8. Determine a área da região D do plano xy definida por $D = \{(x,y) \in I\!R^2 \mid x^2 + (y-2)^2 \leq 4 \text{ e } x^2 + y^2 \geq 4\}$.

9. Determine a área da lemniscata $\left(\dfrac{x^2}{4} + \dfrac{y^2}{9}\right)^2 = \dfrac{x^2}{4} - \dfrac{y^2}{9}$.

10. Determine o volume dos sólidos W abaixo.

a) W é limitado pelas superfícies $z = x^2 + y^2$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ e o plano $z = 10$.

b) $W = \{(x,y,z) \in I\!R^3 \mid x^2 + y^2 + z^2 \leq 25 \text{ e } x^2 + y^2 \geq 9\}$.

c) W é limitado pelas superfícies $z = 0$, $x^2 + y^2 = 2y$ e $z = \sqrt{x^2 + y^2}$.

d) W é o sólido acima do plano xy limitado pelas superfícies $z = 0$, $x + y + z = 1$ e $x^2 + y^2 = 1$.

11. Calcule $\displaystyle\iint_D \frac{e^{(x+y)/(x-y)}}{(x-y)^2}\,dxdy$, onde $D = \{(x,y) \in I\!R^2 \mid 1 \leq (x-y)^2 + (x+y)^2 \leq 4,\ y \leq 0 \text{ e } x + y \geq 0\}$.

12. a) Calcule a integral dupla

$$I(p,a) = \iint_D \frac{dxdy}{(p^2 + x^2 + y^2)^p}\ ,\ \text{onde}$$

$D = \{(x,y) \in I\!R^2 \mid x^2 + y^2 \leq a^2\}$.

b) Determine todos os valores de p para os quais $I(p,a)$ tem um limite finito quando a tende para $+\infty$.

13. Se $a > 0$, seja $I(a) = \displaystyle\int_{-a}^{a} e^{-u^2}\,du$.

a) Mostre que

$$I^2(a) = \iint_D e^{-(x^2+y^2)}\,dxdy,\ \text{onde } D = [-a,a] \times [-a,a].$$

b) Se B_1 e B_2 são as bolas fechadas inscrita e circunscrita a D, respectivamente, mostre que

$$\iint_{B_1} e^{-(x^2+y^2)}\,dxdy \leq I^2(a) \leq \iint_{B_2} e^{-(x^2+y^2)}\,dxdy.$$

c) Calcule as integrais sobre B_1 e B_2, e use b) para mostrar que $I(a)$ tende para $\sqrt{\pi}$ quando a tende para $+\infty$. Isto prova que $\displaystyle\int_0^{+\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.

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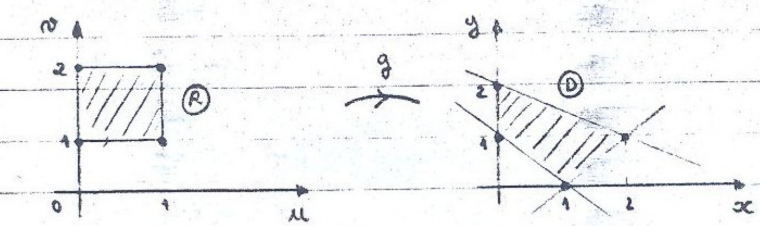
(1) $x = u . v \qquad y = v - u$

a)



$g(u,v) = (x(u,v), y(u,v)) = (u.v, v-u)$

$(0,1), (1,1), (1,2), (0,2)$

$\Downarrow g$

$(0,1), (1,0), (2,1), (0,2)$

b) $\displaystyle\iint_D f(x,y)\, dx\, dy = \iint_Q f(x(u,v), y(u,v)) . \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

$x(u,v) = u.v \qquad y(u,v) = v - u$

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ -1 & 1 \end{vmatrix} = v - (-u) = u + v$

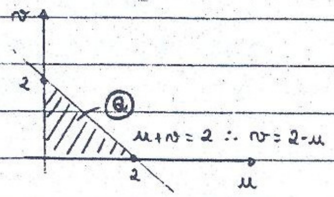
área $D = \displaystyle\iint_D 1\, dx\, dy = \iint_Q 1 . \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

$= \displaystyle\int_1^2\int_0^1 1.(u+v)\, du\, dv = \int_1^2 \left[\frac{u^2}{2} + uv \right]_0^1 dv = \int_1^2 \left(\frac{1}{2} + v \right) dv =$

$= \left[\dfrac{v}{2} + \dfrac{v^2}{2} \right]_1^2 = 1 + 2 - \left(\dfrac{1}{2} + \dfrac{1}{2} \right) = \boxed{2}$

② $x(u,v) = u + v$ $\qquad$ $y(u,v) = v - u^2$

a)



sendo Q região de tipo I:

$0 \leq u \leq 2$

$0 \leq v \leq 2-u$

$$\iint_D \left(x - y + \frac{1}{4}\right)^{-1/2} dx\,dy = ? \qquad f(x,y) = \left(x - y + \frac{1}{4}\right)^{-1/2}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_Q f(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

- $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ -2u & 1 \end{vmatrix} = 1 + 2u$

- $f(x(u,v), y(u,v)) = (u + v - (v - u^2) + 1/4)^{-1/2} = (u + \cancel{v} - \cancel{v} + u^2 + 1/4)^{-1/2}$

$= (u^2 + u + 1/4)^{-1/2} = [(u + 1/2)^2]^{-1/2} = (u + 1/2)^{-1}$

$$\Rightarrow \iint_Q \frac{1}{u + \frac{1}{2}} \cdot (1 + 2u)\,du\,dv = \iint_Q \frac{2}{\cancel{(2u+1)}} \cdot \cancel{(2u+1)}\,du\,dv =$$

$$= \iint_Q 2\,du\,dv = \int_0^2 \int_0^{2-u} 2\,dv\,du = \int_0^2 \Big[2v\Big]_0^{2-u} du =$$

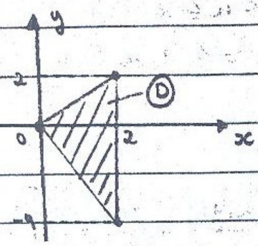
$$= \int_0^2 (4 - 2u)\,du = \Big[4u - u^2\Big]_0^2 = 8 - 4 = \boxed{4}$$

$$\iint_D f(x,y)\,dx\,dy = 4$$

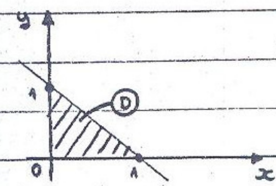
b) $x(u,v) = u + v$ $\quad$ $(0,0)\ ,\ (2,0)\ ,\ (0,2)$

$y(u,v) = v - u^2$ $\quad$ $\Downarrow g$

$(0,0)\ ,\ (2,-4)\ ,\ (2,2)$



③ $\displaystyle\iint_D \cos\left(\frac{x-y}{x+y}\right) dx\, dy$



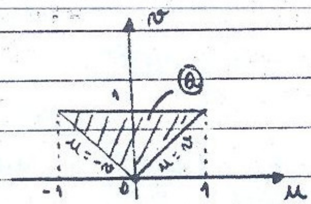
$u = x - y$

$v = x + y$

$(0,0)\ ,\ (1,0)\ ,\ (0,1)$

$\Downarrow g$

$(0,0)\ ,\ (1,1)\ ,\ (-1,1)$



sendo a região Q de tipo II:

$0 \leq v \leq 1 \qquad -v \leq u \leq v$

- $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1-(-1) = 2 \quad \therefore \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$

$$\iint_D f(x,y)\, dx\, dy = \iint_Q f(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \cdot du\, dv$$

$$\int_0^1\int_{-v}^{v} \cos\left(\frac{u}{v}\right) \cdot \frac{1}{2}\, du\, dv = \cancel{2} \cdot \int_0^1\int_0^{v} \cos\left(\frac{u}{v}\right) \cdot \frac{1}{\cancel{2}}\, du\, dv =$$

$$= \int_0^1 \int_0^v \cos\left(\frac{u}{v}\right) du\, dv = \qquad \frac{u}{v} = a \qquad da = \frac{1}{v}\, du$$

$$= \int_0^1 v \cdot \left[\operatorname{sen}\left(\frac{u}{v}\right) \right]_0^v dv = \int_0^1 v \cdot \operatorname{sen} 1 \; dv =$$

$$= \left[\frac{v^2}{2} \cdot \operatorname{sen} 1 \right]_0^1 = \boxed{\frac{\operatorname{sen} 1}{2}}\,''$$

④ $$\iint_D \frac{y+2x}{\sqrt{y-2x-1}}\, dx\, dy \qquad u = y+2x \qquad v = y-2x-1$$

$$D = \{ (x,y) \in \mathbb{R}^2 \,/\, y-2x=2,\ y+2x=2,\ y-2x=1,\ y+2x=1 \}$$

$$u = y+2x \qquad \begin{matrix} y+2x=1 \\ y+2x=2 \end{matrix} > 1 \le u \le 2$$

$$v = y-2x-1 \qquad \begin{matrix} y-2x-1=1 \\ y-2x-1=0 \end{matrix} > 0 \le v \le 1$$

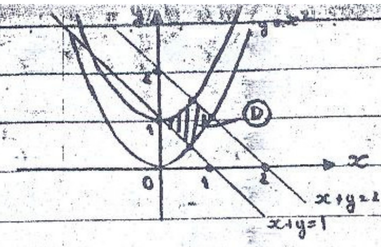
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 2-(-2) = 4 \quad \therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}$$

$$\iint_D f(x,y)\, dx\, dy = \iint_\Omega f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

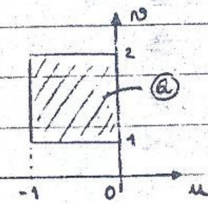
$$\int_0^1 \int_1^2 \frac{u}{\sqrt{v}} \cdot \frac{1}{4}\, du\, dv = \frac{1}{4} \int_0^1 \left[\frac{u^2}{2} \cdot v^{-1/2} \right]_1^2 dv =$$

$$= \frac{1}{4} \int_0^1 \frac{3}{2} v^{-1/2}\, dv = \frac{1}{4} \cdot \frac{3}{2} \left[\frac{v^{1/2}}{1/2} \right]_0^1 = \frac{1}{4} \cdot \frac{3}{\cancel{2}} \cdot \cancel{2} \cdot [1-0] = \boxed{\frac{3}{4}}\,''$$

⑤ $\displaystyle\iint_D (2x+1)\,dx\,dy$



$x^2-y=-1$; $x^2-y=0$ $\quad > \quad u = x^2 - y$, $\quad -1 \le u \le 0$



$x+y=1$; $x+y=2$ $\quad > \quad v = x+y$, $\quad 1 \le v \le 2$

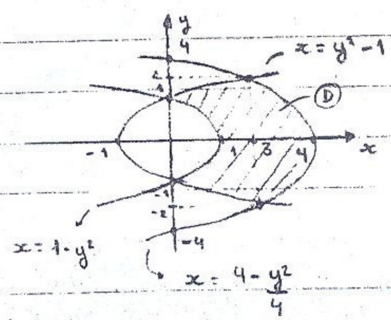
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -1 \\ 1 & 1 \end{vmatrix} = 2x-(-1) = 2x+1 \quad \therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2x+1}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_R f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\int_1^2\int_{-1}^0 (\cancel{2x+1}) \cdot \frac{1}{(\cancel{2x+1})}\,du\,dv = \int_1^2\int_{-1}^0 1\,du\,dv =$$

$$= \int_1^2 [u]_{-1}^0\,dv = \int_1^2 1\,dv = [v]_1^2 = 2-1 = \boxed{1_{\prime\prime}}$$

⑥ $\displaystyle\iint_D x\,dx\,dy.$



$$y^2 - 1 = 4 - \frac{y^2}{4}$$

$$\frac{5y^2}{4} = 5$$

$$y^2 = 4$$

$$y = \pm 2 \quad \therefore \quad x = y^2 - 1$$

$$x = 4 - 1 = 3$$

$$g \begin{cases} x = u^2 - v^2 \\ y = uv \end{cases} \qquad (0,1),\ (0,-1),\ (3,2),\ (3,-2)$$

$(0,1)$: $\quad u^2 - v^2 = 0 \quad \Rightarrow \quad u^2 - \dfrac{1}{u^2} = 0 \quad (u \neq 0)$

$\quad u \cdot v = 1 \quad \therefore \quad v = \dfrac{1}{u}$

$\quad u^4 - 1 = 0 \quad \therefore \quad u = \pm 1 \Rightarrow v = \pm 1$

$(0,-1)$ $\quad u^2 - v^2 = 0 \quad \Rightarrow \quad u^2 - \dfrac{1}{u^2} = 0 \quad \therefore \quad u = \pm 1 \Rightarrow v = \pm 1$

$\quad u \cdot v = -1 \quad \therefore \quad v = \dfrac{-1}{u}$

$(3,2)$ $\quad u^2 - v^2 = 3 \quad \Rightarrow \quad u^2 - \dfrac{4}{u^2} = 3$

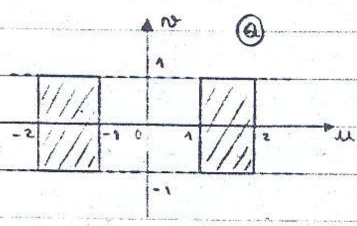
$\quad u \cdot v = 2 \quad \therefore \quad v = \dfrac{2}{u}$

$\quad u^4 - 4 - 3u^2 = 0 \qquad a = u^2$

$$a = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} \begin{cases} 4 \\ -1 \end{cases} \quad \therefore \quad u = \pm 2 \Rightarrow v = \pm 1$$

$(3,-2)$ $\quad u^2 - v^2 = 3 \quad \Rightarrow \quad u = \pm 2$

$\quad u \cdot v = -2 \quad \therefore \quad v = \dfrac{-2}{u} \qquad v = \pm 1$



$$\iint_D f(x,y)\, dx\, dy = \iint_Q f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 - (-2v^2) = 2(u^2 + v^2)$$

$$\int_{-1}^{1}\int_{1}^{2} u^2 - v^2 \cdot 2(u^2+v^2)\, du\, dv = 2\int_{0}^{1}\int_{1}^{2} 2(u^4 - v^4)\, du\, dv =$$

$$= 4\int_{0}^{1}\int_{1}^{2} u^4 - v^4\, du\, dv = 4\int_{0}^{1}\left[\frac{u^5}{5} - u \cdot v^4\right]_1^2 dv =$$

$$= 4\int_{0}^{1}\left[\frac{32}{5} - 2v^4 - \frac{1}{5} + v^4\right] dv = 4\int_{0}^{1}\left(\frac{31}{5} - v^4\right) dv =$$

$$= 4\left[\frac{31v}{5} - \frac{v^5}{5}\right]_0^1 = 4\left(\frac{31}{5} - \frac{1}{5}\right) = 4 \cdot 6 = \boxed{24}$$

⑦ a) $\displaystyle\iint_D \frac{dx\,dy}{(1+x^2+y^2)^2}$ $\quad D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$

$$\begin{cases} x = r\cos\theta & 0 \leq r \leq 2 \\ y = r\sin\theta & 0 \leq \theta \leq 2\pi \end{cases} \Big\rangle\ Q$$

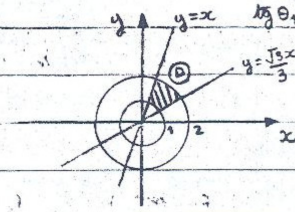
$$\iint_Q \frac{r\, dr\, d\theta}{(1+r^2\cos^2\theta + r^2\sin^2\theta)^2} = \int_0^{2\pi}\int_0^2 \frac{r}{(1+r^2)^2}\, dr\, d\theta \qquad \begin{matrix} u = 1+r^2 \\ du = 2r\, dr \end{matrix}$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\int_0^2 u^{-2}\, du\right) d\theta = \frac{1}{2}\int_0^{2\pi}\left[-\frac{1}{u}\right]_0^2 d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{-1}{1+r^2}\right]_0^2 d\theta = \frac{1}{2}\int_0^{2\pi}\left(-\frac{1}{5}+1\right) d\theta = \frac{1}{2}\int_0^{2\pi}\frac{4}{5}\, d\theta =$$

$$= \frac{1}{2}\left[\frac{4\theta}{5}\right]_0^{2\pi} = \frac{1}{\cancel{2}_1} \cdot \frac{4\,\cancel{2}\pi}{5} = \boxed{\frac{4\pi}{5}}$$

b) $\iint_D (x^2+y^2)\, dx\,dy$

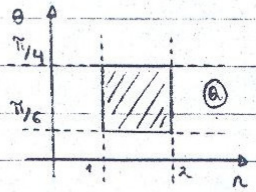


$tg\,\theta_1 = 1 \therefore \theta_1 = \pi/4$

$tg\,\theta_2 = \frac{\sqrt{3}}{3}$

$\theta_2 = \frac{\pi}{6}$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$



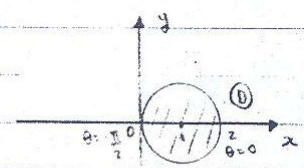
$$\iint_D f(x,y)\,dx\,dy = \iint_Q f(x(r,\theta), y(r,\theta)) \cdot r \; dr\,d\theta$$

$$\int_{\pi/6}^{\pi/4}\int_1^2 \underbrace{(r^2\cos^2\theta + r^2\,\text{sen}^2\theta)}_{= r^2(\cos^2\theta + \text{sen}^2\theta) = r^2} \cdot r \; dr\,d\theta = \int_{\pi/6}^{\pi/4}\int_1^2 r^3\; dr\,d\theta =$$

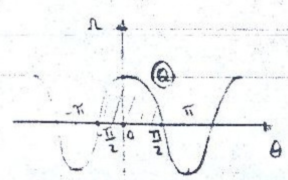
$$= \int_{\pi/6}^{\pi/4} \left[\frac{r^4}{4}\right]_1^2 d\theta = \int_{\pi/6}^{\pi/4} \left(\frac{16}{4} - \frac{1}{4}\right) d\theta = \int_{\pi/6}^{\pi/4} \frac{15}{4}\, d\theta =$$

$$= \left[\frac{15\theta}{4}\right]_{\pi/6}^{\pi/4} = \frac{15\pi}{16_{/3}} - \frac{15\pi}{24_{/2}} = \frac{45\pi - 30\pi}{48} = \frac{15\pi}{48} = \boxed{\frac{5\pi}{16}}$$

c) $\iint_D x^2 \cdot y \; dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}$



$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$



$(r\cos\theta - 1)^2 + r^2\,\text{sen}^2\theta = 1$

$r^2\cos^2\theta - 2r\cos\theta \cancel{+1} + r^2\,\text{sen}^2\theta = \cancel{1}$

$r^{\cancel{2}} - 2\cancel{r}\cos\theta = 0$

$r = 2\cos\theta$

$0 \leq r \leq 2\cos\theta$

$-\pi/2 \leq \theta \leq \pi/2$

$$\iint_D x^2 \cdot y \, dx\, dy = \iint_R r^2\cos^2\theta \cdot r\,\mathrm{sen}\,\theta \cdot r \, dr\, d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^4 \cdot \mathrm{sen}\,\theta \cdot \cos^2\theta \, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^4 \cdot \mathrm{sen}\,\theta \cdot \cos^2\theta \, dr\, d\theta =$$

$$= \int^{\pi/2} \left[\frac{r^5}{5} \cdot \mathrm{sen}\,\theta \cdot \cos^2\theta \right]_0^{2\cos\theta} dr\, d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \frac{32}{5} \cdot \mathrm{sen}\,\theta \cdot \cos^7\theta \, d\theta = \frac{32}{5} \int_{-\frac{\pi}{2}}^{\pi/2} \mathrm{sen}\,\theta \cdot \cos^7\theta \, d\theta$$

$$= -\frac{32}{5} \cdot \int_{-\frac{\pi}{2}}^{\pi/2} u^7 \, du =$$

$u = \cos\theta$

$du = -\mathrm{sen}\,\theta \, d\theta$

$$= -\frac{32}{5} \cdot \left[\frac{\cos^8\theta}{8} \right]_{-\pi/2}^{\pi/2} = -\frac{32}{5} \cdot \left(\cancelto{0}{\cos\frac{\pi}{2}} \right)^8 \cdot \frac{1}{8} + \frac{32}{5} \cdot \left(\cancelto{0}{\cos -\frac{\pi}{2}} \right)^8 \frac{1}{8} =$$

$$= \cancel{-\frac{32}{5} \cdot 0 \cdot \frac{1}{8}} + \cancel{\frac{32}{5} \cdot 0 \cdot \frac{1}{8}} = \boxed{0_{//}}$$

(8) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + (y-2)^2 \leq 4 \text{ e } x^2 + y^2 \geq 4 \}$

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

$x^2 + (y-2)^2 \leq 4$

$r^2\cos^2\theta + r^2\mathrm{sen}^2\theta - 4r\,\mathrm{sen}\,\theta + \cancel{4} \leq \cancel{4}$

$r^2 - 4r\,\mathrm{sen}\,\theta \leq 0$

$r(r - 4\,\mathrm{sen}\,\theta) = 0$

$r = 0 \quad r = 4\,\mathrm{sen}\,\theta$

$0 \leq r \leq 4\,\mathrm{sen}\,\theta$

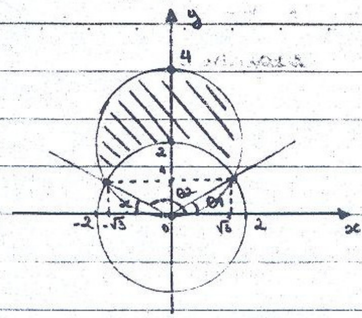
$x^2 + y^2 \geq 4$

$r^2\cos^2\theta + r^2\mathrm{sen}^2\theta \geq 4$

$r^2 - 4 \geq 0$

...ora $\quad r \leq -2$ ou $r \geq 2$

$\Downarrow$ $2 \leq r \leq 4\,\mathrm{sen}\,\theta$



$tg\ \theta_1 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\theta_1 = \pi/6$

$tg\ \alpha = \frac{\sqrt{3}}{3} \therefore \alpha = \pi/6$

$\theta_2 = \pi - \alpha = 5\pi/6$

$\pi/6 \le \theta \le 5\pi/6$

$x^2+y^2=4 \qquad x^2+(y-2)^2=4$

$\cancel{x^2}+y^2 = \cancel{x^2}+(y-2)^2$

$\cancel{y^2} = \cancel{y^2} - 4y + 4$

$4y = 4$

$y = 1 \therefore x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

⊛ $\text{área } D = \iint_D 1\, dx\, dy = \iint_Q 1 . r\, dr\, d\theta$

$$\int_{\pi/6}^{5\pi/6}\int_2^{4\,sen\,\theta} r\, dr\, d\theta = \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\right]_2^{4\,sen\,\theta} d\theta = \int_{\pi/6}^{5\pi/6} (8\,sen^2\theta - 2)\, d\theta =$$

$u = 2\theta \quad du = 2d\theta$

$$= \int_{\pi/6}^{5\pi/6} 8\,sen^2\theta\, d\theta - \int_{\pi/6}^{5\pi/6} 2\, d\theta =$$

$sen^2\theta = 1/2\,(1 - \cos 2\theta)$

$\int sen^2\theta\, d\theta = 1/2 \int d\theta - 1/2 \int \cos 2\theta\, d\theta$

$= \frac{\theta}{2} - \frac{1}{4}\,(sen\, 2\theta)$

$$= \left[8.\left(\frac{\theta}{2} - \frac{sen\,2\theta}{4}\right) - 2\theta \right]_{\pi/6}^{5\pi/6} =$$

$$= \left[4\theta - 2\,sen\,2\theta - 2\theta\right]_{\pi/6}^{5\pi/6} = \left[2\theta - 2\,sen\,2\theta\right]_{\pi/6}^{5\pi/6} =$$

$$= \cancel{2}.\frac{5\pi}{\cancel{6}_3} - 2\,sen\left(\cancel{2}.\frac{5\pi}{\cancel{6}_3}\right) - \cancel{2}.\frac{\pi}{\cancel{6}_3} + 2.\,sen\left(\cancel{2}.\frac{\pi}{\cancel{6}_3}\right) =$$

$$= \frac{5\pi}{3} - \cancel{2}.\left(-\frac{\sqrt{3}}{\cancel{2}}\right) - \frac{\pi}{3} + \cancel{2}.\left(\frac{\sqrt{3}}{\cancel{2}}\right) = \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$

(9) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2}{4}-\frac{y^2}{9}$ => região D

$\frac{x}{2}=r\cos\theta \qquad x=2r\cos\theta$

$\frac{y}{3}=r\,\text{sen}\,\theta \quad \therefore \quad y=3r\,\text{sen}\,\theta$

$(r^2\cos^2\theta+r^2\text{sen}^2\theta)^2=r^2\cos^2\theta-r^2\text{sen}^2\theta$

$(r^2)^2=r^2(\cos^2\theta-\text{sen}^2\theta)$

$(r^2)^2=r^2\cos 2\theta$

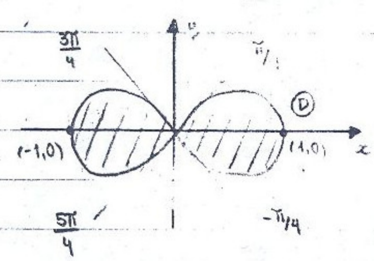
$r^2=\cos 2\theta$

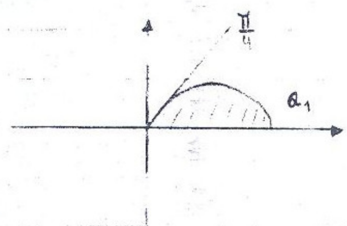
$r=\sqrt{\cos 2\theta} \qquad \therefore \qquad 0\le r\le\sqrt{\cos 2\theta}$

$\cos 2\theta\geq 0$

$-\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2} \iff -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ ou $\frac{3\pi}{4}\le\theta\le\frac{5\pi}{4}$

$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}2\cos\theta & -2r\,\text{sen}\,\theta\\ 3\,\text{sen}\,\theta & 3r\cos\theta\end{vmatrix}=6r\cos^2\theta+6r\,\text{sen}^2\theta=6r$





$D_1=\{(r,\theta)\mid 0\le\theta\le\pi/4,\ 0\le r\le\sqrt{\cos 2\theta}\}$

área de D $=\iint_D dx\,dy=\iint_{G} 6r\,dr\,d\theta=$

$=4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}6r\,dr\,d\theta=24\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}}r\,dr\,d\theta=$

$$= 24 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \overset{12}{\cancel{24}} \cdot \frac{1}{\underset{1}{\cancel{2}}} \int_0^{\pi/4} \cos 2\theta \, d\theta =$$

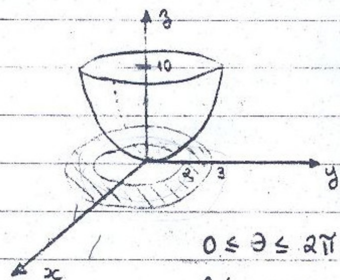
$u = 2\theta$

$du = 2\, d\theta$

$$= \overset{6}{\cancel{12}} \cdot \frac{1}{\cancel{2}} \left[\operatorname{sen} 2\theta \right]_0^{\pi/4} =$$

$$= 6 . (\underbrace{\operatorname{sen} \pi/2}_{=1} - \operatorname{sen} 0) = 6 . 1 = \boxed{6}$$

(10) a) $z = x^2 + y^2 \quad x^2 + y^2 = 4 \quad x^2 + y^2 = 9$ e $z = 10$



$$\begin{cases} x = r \cos\theta \\ y = r \operatorname{sen}\theta \end{cases}$$

$z = x^2 + y^2 = r^2 \cos^2\theta + r^2 \operatorname{sen}^2\theta$

$z = r^2 (\cos^2\theta + \operatorname{sen}^2\theta)$

$z = r^2$

$0 \leq \theta \leq 2\pi$

$2 \leq r \leq 3$

$$\iint_D f(x,y)\, dx\, dy = \iint_D 10 - (x^2 + y^2)\, dx\, dy = \iint_Q f(x(r,\theta), y(r,\theta)) . r \, dr\, d\theta$$

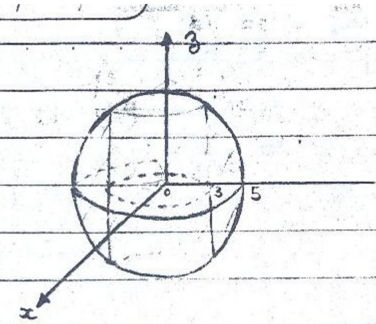
$$= \int_0^{2\pi}\int_2^3 (10 - r^2) . r \, dr\, d\theta = \int_0^{2\pi}\int_2^3 10r - r^3 \, dr\, d\theta =$$

$$= \int_0^{2\pi} \left[5r^2 - \frac{r^4}{4} \right]_2^3 d\theta = \int_0^{2\pi} \left(45 - \frac{81}{4} - 20 + 4 \right) d\theta =$$

$$= \int_0^{2\pi} \frac{35}{4}\, d\theta = \left[\frac{35\theta}{4} \right]_0^{2\pi} = \boxed{\frac{35\pi}{2}}$$

b) $W = \{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 \leq 25$ e $x^2 + y^2 \geq 9 \}$

$$\begin{cases} x = r \cos\theta \\ y = r \operatorname{sen}\theta \end{cases}$$



$3 \leq r \leq 5$

$0 \leq \theta \leq 2\pi$

$z^2 = 25 - x^2 - y^2$

$z = \pm\sqrt{25 - (x^2 + y^2)}$

$z = \pm\sqrt{25 - r^2}$

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(x(r,\theta), y(r,\theta)) \cdot r\,dr\,d\theta =$$

$$= 2\int_0^{2\pi}\int_3^5 \sqrt{25 - r^2} \cdot r\,dr\,d\theta =$$

$u = 25 - r^2$

$du = -2r$

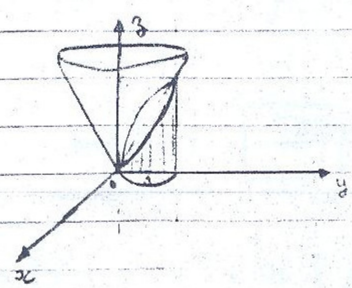
$$= \cancel{2}\left(-\frac{1}{\cancel{2}}\right)\int_0^{2\pi}\int_3^5 u^{1/2}\,du\,d\theta = -\int_0^{2\pi}\left[\frac{2(25-r^2)^{3/2}}{3}\right]_3^5 d\theta =$$

$$= -\int_0^{2\pi} -\frac{128}{3}\,d\theta = \frac{128}{3}\Big[\theta\Big]_0^{2\pi} = \frac{128 \cdot 2\pi}{3} = \boxed{\frac{256\pi}{3}}$$

c) $z = 0$ $\quad x^2 + y^2 = 2y$ $\quad z = \sqrt{x^2 + y^2}$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

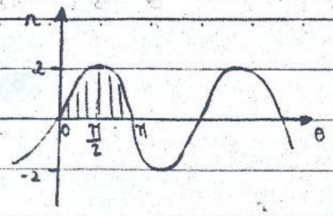
$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$

$r^2 - 2r\sin\theta = 0$

$r = 2\sin\theta$

$0 \leq r \leq 2\sin\theta$



$0 \le \theta \le \pi$

$z = \sqrt{x^2+y^2}$

$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$z = \sqrt{r^2} = r$

$$\iint_D f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^{2\sin\theta} r\cdot r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta =$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{2\sin\theta} d\theta = \int_0^{\pi}\frac{8\sin^3\theta}{3}\,d\theta = \frac{8}{3}\int_0^{\pi}\sin^3\theta\,d\theta = \circledast$$

$$\int \sin^3\theta\,d\theta = \int \sin^2\theta\cdot\sin\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta =$$

$$= \int \sin\theta\,d\theta - \int \sin\theta\cdot\cos^2\theta\,d\theta = \int \sin\theta\,d\theta + \int u^2\,du$$

$$= -\cos\theta + \frac{\cos^3\theta}{3}$$

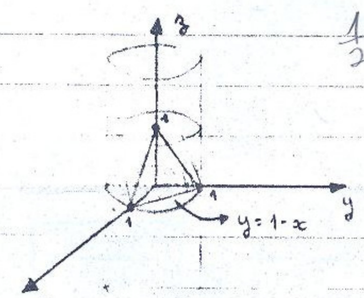
$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$\circledast = \frac{8}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{8}{3}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) =$$

$$= \frac{8}{3}\cdot\left(\frac{2}{1} - \frac{2}{3}\right) = \frac{8}{3}\cdot\frac{4}{3} = \boxed{\frac{32}{9}}\ //$$

d) $z = 0$. $x + y + z = 1$ $\quad x^2 + y^2 = 1$



$\frac{1}{2}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$0 \le r \le 1 \qquad 0 \le \theta \le 2\pi$

$z = 1 - x - y \ \therefore\ f(x,y) = 1 - x - y$

$\iint$ $dr\,d\theta$

$$\iint_D f(x,y)\,dx\,dy = \int_{\frac{\pi}{2}}^{2\pi}\int_0^1 f(x(r,\theta), y(r,\theta)).\,r\,dr\,d\theta + \int_0^1\int_0^{1-x} f(x,y)\,dy\,dx$$

A B

$$A \quad \int_{\frac{\pi}{2}}^{2\pi}\int_0^1 (1 - r\cos\theta - r\sin\theta).\,r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{2\pi}\int_0^1 r - r^2\cos\theta - r^2\sin\theta\,dr\,d\theta =$$

$$= \int_{\pi/2}^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right]_0^1 d\theta =$$

$$= \int_{\pi/2}^{2\pi} \frac{1}{2} - \frac{\cos\theta}{3} - \frac{\sin\theta}{3}\,d\theta = \left[\frac{\theta}{2} - \frac{\sin\theta}{3} + \frac{\cos\theta}{3}\right]_{\pi/2}^{2\pi} =$$

$$= \pi \cancel{-0} + 1/3 - \pi/4 + 1/3 \cancel{-0} = 3\pi/4 + 2/3$$

$$B \quad \int_0^1\int_0^{1-x} 1 - x - y\,dy\,dx = \int_0^1\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx =$$

$$= \int_0^1 1 - x - x(1-x) - \frac{(1-x)^2}{2}\,dx = \int_0^1 1 - x - x + x^2 - \frac{(1-x)^2}{2}\,dx =$$

$$= \frac{1}{2}\int_0^1 x^2 - 2x + 1\,dx = \frac{1}{2}\left[\frac{x^3}{3} - x^2 + x\right]_0^1 =$$

$$= \frac{1}{2}\left(\frac{1}{3} \cancel{-1} \cancel{+1}\right) = \frac{1}{6}$$

$$V_T = V_A + V_B = \frac{3\pi}{4_3} + \frac{2}{3_4} + \frac{1}{6_2} = \frac{9\pi + 8 + 2}{12} = \boxed{\frac{9\pi + 10}{12}}$$

(11) $\displaystyle\iint_D \frac{e^{(x+y)/(x-y)}}{(x-y)^2}\,dx\,dy$

$D=\{(x,y)\in\mathbb{R}^2 \mid 1\le (x-y)^2+(x+y)^2\le 4,\ y\le 0 \text{ e } x+y\ge 0\}$

$1\le x^2-\cancel{2xy}+y^2+x^2+\cancel{2xy}+y^2\le 4$

$1\le 2x^2+2y^2\le 4$

$\frac{1}{2}\le x^2+y^2\le 2$

$\begin{cases} u=x+y & \therefore\ y=u-x \\ v=x-y & \therefore\ y=x-v \end{cases}$

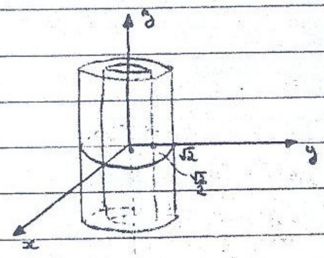
$1\le v^2+u^2\le 4 \qquad u\ge 0$

$u-x\le 0 \qquad x-v\le 0$

$x\ge u \qquad v\ge x$

$u\le x\le v$

$0\le u\le v \qquad v\ge 0$



$1\le u^2+v^2\le 4 \ \therefore\ 1\le v\le 2$

$\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-1-1=-2 \ \therefore\ \dfrac{\partial(x,y)}{\partial(u,v)}=-\dfrac{1}{2}$

$\displaystyle\int_1^2\int_0^v \frac{e^{u/v}}{v^2}\cdot\left|-\frac{1}{2}\right|du\,dv=$

$\int e^{x/a}dx=a\int e^u du=ae^u=a\ldots$

$u=x/a \ \therefore\ 1/a\,dx=du$

$\displaystyle=\frac{1}{2}\int_1^2\left[\frac{\cancel{v}}{v^{\cancel{2}}}\cdot e^{u/v}\right]_0^v dv=\frac{1}{2}\int_1^2 \frac{1}{v}\cdot e-\frac{1}{v}\,dv=$

$\displaystyle=\frac{1}{2}\Big[e\cdot\ln|v|-\ln|v|\Big]_1^2=\frac{1}{2}\left(e\cdot\ln 2-\ln 2-e\cancel{\ln 1}^{\,0}+\cancel{\ln 1}^{\,0}\right)=$

$=\dfrac{1}{2}(e\ln 2-\ln 2)=\boxed{\dfrac{\ln 2}{2}\cdot(e-1)}$

(12) a) $I(p,a) = \iint_D \frac{dx\,dy}{(p^2+x^2+y^2)^p}$

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a^2\}$

$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$ $\quad 0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi$

- $I(p,a) = \int_0^{2\pi}\int_0^a \frac{r\,dr\,d\theta}{(p^2+r^2)^p}$ $\qquad u = p^2+r^2$, $du = 2r\,dr$

$= \frac{1}{2}\int_0^{2\pi}\int_0^a \frac{du}{u^p}\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^a u^{-p}\,du\,d\theta =$

$= \frac{1}{2}\int_0^{2\pi}\left[\frac{u^{-p+1}}{-p+1}\right]_0^a d\theta = \frac{1}{2}\int_0^{2\pi}\left[\frac{(p^2+r^2)^{-p+1}}{-p+1}\right]_0^a d\theta =$

$= \frac{1}{2}\int_0^{2\pi} \frac{(p^2+a^2)^{-p+1}}{-p+1} - \frac{(p^2)^{-p+1}}{-p+1}\,d\theta =$

$= \frac{1}{2}\left[\theta \cdot \left(\frac{(p^2+a^2)^{-p+1} - (p^2)^{-p+1}}{-p+1}\right)\right]_0^{2\pi} =$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \left(\frac{(p^2+a^2)^{-p+1} - (p^2)^{-p+1}}{-p+1}\right) =$

$= \boxed{\pi \cdot \left(\frac{(p^2+a^2)^{-p+1} - p^{-2p+2}}{-p+1}\right)}$ $\qquad p \neq 1$

- se $p=1$ $\quad I(p,a) = I(1,a) = \int_0^{2\pi}\int_0^a \frac{r\,dr\,d\theta}{(1+r^2)^1}$

$u = 1 + r^2 \qquad du = 2r\,dr$

$$= \frac{1}{2} \int_0^{2\pi}\int_0^{a} \frac{du}{u}\, d\theta = \frac{1}{2} \int_0^{2\pi} \left[\ln(1+r^2) \right]_0^a d\theta =$$

$$= \frac{1}{2} \int_0^{2\pi} \ln(1+a^2)\, d\theta = \frac{1}{2} \cdot \left[\theta \cdot \ln(1+a^2) \right]_0^{2\pi} =$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \cdot \ln(1+a^2) = \boxed{\pi \cdot \ln(1+a^2)_{\,//}} \qquad p = 1$$

b) $I(p,a) = \pi \cdot \left(\dfrac{(p^2+a^2)^{-p+1} - p^{-2p+2}}{-p+1} \right)$

- $a \to +\infty \;\therefore\; I(p,a) \to \pi \cdot (p^2+a^2)^{\frac{-p+1}{x}} \to +\infty$

limite infinito

- se $x < 0 \;\therefore\; I(p,a) \to \pi \cdot \dfrac{1}{(p^2+a^2)^x} \to 0$

limite finito

Logo, $-p+1 < 0 \;\therefore\; \boxed{p > 1_{\,//}}$

(13) $I(a) = \displaystyle\int_{-a}^{a} e^{-u^2}\, du \qquad a > 0$

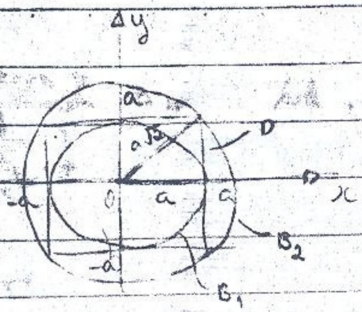
a) $I^2(a) = \displaystyle\iint_D e^{-(x^2+y^2)}\, dx\,dy \qquad$ onde $D = [-a,a] \times [-a,a]$

$$\int_{-a}^{a}\int_{-a}^{a} e^{-x^2} \cdot e^{-y^2}\, dx\,dy = \int_{-a}^{a} e^{-y^2} \left(\int_{-a}^{a} e^{-x^2}\, dx \right) dy =$$

$$= \int_{-a}^{a} e^{-y^2} \cdot I(a)\, dy = I(a) \int_{-a}^{a} e^{-y^2}\, dy = I(a) \times I(a) = I^2(a)_{\,//}$$

c.q.d.

b)



$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

B_1: $\displaystyle\int_0^{2\pi}\int_0^{a} e^{-r^2}\, dr\, d\theta$

B_2: $\displaystyle\int_0^{2\pi}\int_0^{a\sqrt{2}} e^{-r^2}\, dr\, d\theta$