4.1 *Compute the determinant using the Laplace expansion (using the first row) and the Sarrus rule for*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

a. using the Laplace expansion along first row.

$$det(A) = (-1)^{1+1}\begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$
$$= 4 - 8 + 4 = 0$$

b. using Sarrus rule

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow 16 + 20 + 0 - 0 - 12 - 24 = 0$$

4.2 *Compute the following determinant efficiently:*

$$\begin{vmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 6$$

4.3 *Compute the eigenspaces of*

a.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} x = 0$$

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

$$thus,\ \lambda = 1$$

if we solve homogenous equation,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x = 0$$

$$solution\ space\ E_1 = span[\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$$

b.

$$B = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

$$Bx = \lambda x$$

$$p(\lambda) = \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$$

$$\lambda = 2,\ -3$$

$$for\ \lambda = 2,\ if\ we\ solve$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x = 0$$

$$E_2 = span[\begin{bmatrix} 1 \\ 2 \end{bmatrix}]$$

$$for\ \lambda = -3,\ if\ we\ solve$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = 0$$

$$E_{-3} = span[\begin{bmatrix} 2 \\ -1 \end{bmatrix}]$$

4.4 *Compute all eigenspaces of*

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 1-\lambda & -2 & 3 \\ 2 & -1 & -\lambda & 0 \\ 1 & -1 & 1 & -\lambda \end{vmatrix} = -1/\lambda^2 \begin{vmatrix} -\lambda & -1 & 1 & 1 \\ 0 & -\lambda & -1 & 3-\lambda \\ 0 & 0 & (\lambda+1)^2 & \lambda\left(-\lambda^2-2+\lambda\right)+\lambda-3 \\ 0 & 0 & 0 & (\lambda-1)(\lambda-2) \end{vmatrix}$$

$$= -(\lambda+1)^2(\lambda-1)(\lambda-2)$$

$$thus,\ \lambda = -1, 1, 2$$

$$E_1 \rightarrow \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 0 & -2 & 3 \\ 2 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0,\ E_1 = span[\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}]$$

$$E_2 \rightarrow \begin{bmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & 3 \\ 2 & -1 & -2 & 0 \\ 1 & -1 & 1 & -2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0,\ E_2 = span[\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}]$$

$$E_{-1} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 2 & -2 & 3 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0,\ E_{-1} = span[\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}]$$

4.5

Diagonalizability of a matrix is unrelated to its invertibility. Determine for the following four matrices whether they are diagonalizable and / or invertible.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

indeed diagonalizable and invertible.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

using eigen decomposition, we can find eigen vectors of B.

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0, \ thus\ \lambda = 0,\ 1$$

$$E_0 = span[\begin{bmatrix} 1 \\ 0 \end{bmatrix}],\ E_1 = span[\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$$

since $E_0,\ E_1$ *is linearly independent, it can be diganolizable but not invertible*

$$c. \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = 0, \ thus\ \lambda = 0,\ 2$$

$$E_0 = span[\begin{bmatrix} 0 \\ 0 \end{bmatrix}],\ E_2 = span[\begin{bmatrix} 0 \\ 0 \end{bmatrix}]$$

since $E_0,\ E_2$ *is linearly dependent, it is not diagonlizable but invertible.*

$$d. \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

since it has only one eigen value, it is not diagonalizable and not invertible.

4.6

Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

$$a.\ A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1/(2-\lambda)\begin{vmatrix} 2-\lambda & 3 & 0 \\ 0 & (\lambda-1)(\lambda-5) & 3(2-\lambda) \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-5)(1-\lambda) = 0$$

$$thus,\ \lambda = 1,\ 5$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} x = 0,\ E_1 = span[\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}]$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} x = 0,\ E_5 = span[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}]$$

since there are two linear independent basis, it is not diagonalizable.

$$b.\ A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3(1-\lambda)$$

$$thus,\ \lambda = 0,\ 1$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = 0,\ E_0 = span[\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = 0,\ E_0 = span[\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ then\ it\ forms\ four\ basis,\ so\ it\ is\ diagonalizable.$$

$$4.7$$

Are the following matrices diagonalizable?
If yes, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal.
If no, give reasons why they are not diagonalizable.

$$a.\ \ A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8 = (\lambda-2)^2 + 4$$

so there is no real root λ, *determines it is not dagonalizable.*

$$b.\ \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$1/(1-\lambda)^2(\lambda-2)\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & \lambda(\lambda-2) & -\lambda \\ 0 & 0 & \lambda(\lambda-1)(\lambda-3) \end{vmatrix} = \lambda^2(\lambda-3)$$

$$\lambda = 0, 3$$

$$E_0 = span[\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}]$$

$$E_3 = span[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]$$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1}$$

$$c.\ \ A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 4 & 2 & 1 \\ 0 & 1-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & 0 \\ 1 & 1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda-4)^2/(5-\lambda)\begin{vmatrix} 5-\lambda & 4 & 2 & 1 \\ 0 & 1-\lambda & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (2-\lambda)(\lambda-4)^2(1-\lambda),\ \lambda = 1, 2, 4$$

$$E_1 = \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = span[\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}]$$

4.8 *Find the SVD of the matrix.*

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = (\lambda - 25)(\lambda - 9)$$

$$E_{25} = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} = span[\begin{bmatrix} 1 \\ 1 \end{bmatrix}]$$

$$E_9 = span[\begin{bmatrix} 1 \\ -1 \end{bmatrix}]$$

$$AA^T = USV^TVSU^T = US^2U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$1/(13-\lambda) \begin{vmatrix} 13-\lambda & 12 & 2 \\ 0 & 25-\lambda & 6\lambda - 50 \\ 0 & 0 & \lambda^2 - 9\lambda \end{vmatrix} = \lambda(25-\lambda)(\lambda - 9)$$

$$E_{25} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = span[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}]$$

$$E_9 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = span[\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}]$$

$$E_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = span[\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}]$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$A^TA = VSU^TUSV^T = VS^2V^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{} & \frac{1}{} \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ 2 & 2 & 1 \end{bmatrix}$$

4.9 *Find the singular value decomposition of*

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda = 8,\ 2$$

$$E_8 = span[\begin{bmatrix} 1 \\ 0 \end{bmatrix}],\ E_2 = span[\begin{bmatrix} 0 \\ 1 \end{bmatrix}],\ U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (\lambda-2)(\lambda-8)$$

$$E_8 = span[\begin{bmatrix} 1 \\ 1 \end{bmatrix}],\ E_2 = span[\begin{bmatrix} 1 \\ -1 \end{bmatrix}],\ V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$4.10\ \textit{Find the rank} - 1\ \textit{approximation of}$$
$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$
$$\textit{from previous result},\ S_1 = 5$$
$$A_1 = 5\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2.5 & 2.5 & 0 \\ 2.5 & 2.5 & 0 \end{bmatrix}$$
$$A_2 = 5\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$4.11\ \textit{Show that for any}\ A \in \mathbb{R},$$
$$m \times n\ \textit{the matrices}\ A^T A\ \textit{and}\ AA^T\ \textit{possess the same nonzero eigenvalues}.$$

$$A^T A\, x = \lambda x$$
$$AA^T A\, x = A\lambda x$$
$$AA^T(Ax) = \lambda(Ax)$$
$$\textit{so they share same eigen values}\ \lambda.$$