2.1

We consider $(\mathbb{R}\setminus\{-1\},\ \star)$, *where*

$$a \star b \ := \ ab + a + b, \qquad a, b \ \in \ \mathbb{R}\setminus\{-1\}$$

a. *Show that* $(\mathbb{R}\setminus\{-1\},\ \star)$ *is an Abelian group.*

– Associativity

$$a \star (b \star c) \ = \ a(bc + b + c) + a + (bc + b + c) \ = abc + ab + ac + bc + a + b + c$$
$$(a \star b) \star c \ = \ (ab + a + b) * c + ab + a + b + c = abc + ac + bc + ab + a + b + c$$

– Closure

$$ab + a + b = -1$$
$$\Rightarrow a(b+1) = \ -(b+1)$$

the only solution is $a = -1$ *or* $b = -1$ *which contradicts condition.*

– Neutral Element

$$x \star 0 \ = \ x * 0 \ + \ x \ + \ 0 \ = \ x$$
$$0 \star x \ = \ 0 * x \ + \ 0 \ + \ x \ = \ x$$

– Inverse Element

$$x \star x^{-1} = x * x^{-1} \ + \ x \ + \ x^{-1} \ = \ 0$$
$$x^{-1} \star x \ = x^{-1} * x \ + x^{-1} + x = 0$$
$$\Rightarrow x^{-1} \ = \ \frac{-x}{x+1}$$

$if\ x^{-1} \ = \ -1, \ \ -x + x - 1 = 0$ *which is contradiction.*

$x \neq -1$ $by\ definition\ therefore$ *the inverse always exists and it's in the set.*

– Commutative

$$a \star b = \ ab + a + b$$
$$b \star a = ba + b + a$$

which is true.

b. *Solve,* $3 \star x \star x = 15$
in the Abelian group $(\mathbb{R}\setminus\{-1\}, \star)$

$$x^2 + 2x - 3 = 0$$
$$\Rightarrow (x-1)(x+3) = 0$$
$$\Rightarrow x = 1, \ -3$$

2.2

$$Let\ n\ be\ in\ \mathbb{N} \setminus \{0\}. Let\ k,\ x\ be\ in\ \mathbb{Z}.$$
$$We\ define\ the\ congruence\ class\ \overline{k}\ of\ the\ integer\ k\ as\ the\ set$$

$$\overline{k} = \{x \in \mathbb{Z} \mid x - k = 0\ (\mathrm{mod} n)\}$$
$$= \{x \in \mathbb{Z} \mid \exists a \in \mathbb{Z} : (x - k = n \cdot a)\}$$

$$We\ now\ define\ \mathbb{Z}/n\mathbb{Z}\ (sometimes\ written\ \mathbb{Z}n)\ as\ the\ set\ of\ all\ congruence\ classes\ modulo\ n.$$
$$Euclidean\ division\ implies\ that\ this\ set\ is\ a\ finite\ set\ containing\ n\ elements:$$

$$\mathbb{Z}n = \{\overline{0}, \overline{1}, ..., \overline{n - 1}\}$$

$$For\ all\ \overline{a}, \overline{b} \in \mathbb{Z}n,\ we\ define$$

$$\overline{a} \oplus \overline{b} := \overline{a + b}$$

$$a.\ Show\ that\ (\mathbb{Z}n, \oplus)\ is\ a\ group. Is\ it Abelian?$$

$$-\ Associativity$$
$$(\overline{a} \oplus \overline{b}) \oplus \overline{c} = \overline{a} \oplus (\overline{b} \oplus \overline{c}) = \overline{a + b + c}$$

$$-\ Closure$$
$$(a + b)\ mod\ n\ is\ also\ in\ \{\overline{0}, \overline{1}, ..., \overline{n - 1}\}$$

$$-\ Neutral\ Element$$
$$\overline{a} \oplus \overline{0} = \overline{0} \oplus \overline{a} = \overline{a + 0} = \overline{a}$$

$$-\ Inverse\ Element$$
$$\overline{a} \oplus \overline{i} = \overline{0}$$
$$\Rightarrow \overline{a + i} = \overline{0}$$
$$\Rightarrow \overline{i} = \overline{-a}$$

$$-\ Commutative$$
$$\overline{a} \oplus \overline{b} = \overline{b} \oplus \overline{a} = \overline{a + b} = \overline{b + a}$$

$$So\ it's\ an\ Abelian.$$

$b.$ *We now define another operation* $\otimes$ *for all* $\overline{a}$ *and* $\overline{b}$ *in* $\mathbb{Z}n$ *as*

$$\overline{a} \otimes \overline{b} = \overline{a \times b}, \quad (2.135)$$

where $a \times b$ *represents the usual multiplication in* $\mathbb{Z}$.

Let $n = 5$. *Draw the times table of the elements of* $\mathbb{Z}_5 \backslash \{\overline{0}\}$ *under* $\otimes$,

i.e., calculate the products $\overline{a} \otimes \overline{b}$ *for all a and b in* $\mathbb{Z}_5 \backslash \{\overline{0}\}$.

⊗	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Hence, show that $\mathbb{Z}_5 \backslash \{\overline{0}\}$ *is closed under* $\otimes$ *and possesses a neutral element for* $\otimes$.

$\Rightarrow$ *as shown in the above times table, it's closed under* $\otimes$, *and a neutral element is* $\overline{1}$.

Display the inverse of all elements in $\mathbb{Z}_5 \backslash \{\overline{0}\}$ *under* $\otimes$.

$$\overline{1}^{-1} = \overline{1}$$
$$\overline{2}^{-1} = \overline{3}$$
$$\overline{3}^{-1} = \overline{2}$$
$$\overline{4}^{-1} = \overline{4}$$

Conclude that $(\mathbb{Z}_5 \backslash \{\overline{0}\}, \otimes)$ *is an Abelian group.*

$\Rightarrow$ *usual multiplication holds commutative law, so it is an Abelian group.*

$c.$ *Show that* $(Z_8 \backslash \{0\}, \otimes)$ *is not a group.*

$\Rightarrow$ *we can't get inverse of even element.* $\{\overline{2}, \overline{4}, \overline{6}\}$

$d.$ *We recall that the B·ezout theorem states that two integers a and b are relatively prime* $(i.e.,\ gcd(a,\ b)\ =\ 1)$ *if and only if there exist two integers u and v such that* $au\ +\ bv\ =\ 1$. *Show that* $(\mathbb{Z}n \setminus \{\bar{0}\},\ \otimes)$ *is a group if and only if* $n \in \mathbb{N} \setminus \{0\}$ *is prime.*

if $\mathbb{Z}n$ *has to have inverses, there must exist two integer* $a, b \in \mathbb{Z}n$ *that holds*

$$a * b = u * n + 1, \qquad u \in \mathbb{Z}$$

$$\Rightarrow ab + (-u)n = 1$$

where b is an inverse of a, and vice versa.

as $(-u)$ *is also an integer, n is relatively prime for both a, b by the B·ezout theorem.*

that said, $gcd(a, n) = gcd(b, n) = 1$ *for any* $a, b \in \mathbb{Z}n$.

therefore n should be prime number.

2.3

Consider the set $\mathbb{G}$ *of* 3×3 *matrices defined as follows:*

$$\mathbb{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ *as the standard matrix multiplication.*

Is $(\mathbb{G},\ \cdot)$ *a group? If yes, is it Abelian? Justify your answer.*

– Associative by the matrix multiplication property.

– Closure but not commutative

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & z + z' + xy' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix}$$

– Has neutral element, Identity matrix

– Inverse

$$\left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy - z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So it's a group but not abelian.

2.4 *Compute the following matrix products, if possible:*

a. not solvable

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

b. column operation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

c. row operation

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

d.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

e.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ -21 & 2 \end{bmatrix}$$

2.5 *Find the set S of all solutions in x of the following inhomogeneous linear systems $Ax = b$, where A and b are defined as follows:*

a.

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -2 & 2 & -1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

so there is no solution set.

b.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

let free variable $x_3,\ x_5 = 0$, then the special solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

and to solve homogeneous solution, make rref,

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and do minus -1 trick,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

then entire solution set of $Ax = b$ is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

$$2.6 \textit{ Using Gaussian elimination, find all solutions of the inhomogeneous equation system } Ax = b \textit{ with,}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

let free variable $x_1,\ x_3,\ x_6 = 0$, *the special solution is*

$$\mathbb{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

if we do minus 1 *trick to get homogeneous solution,*

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\textit{then entire set is, } \mathbb{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

2.7 *Find all solutions in* $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ *of the equation system* $Ax = 12x$, *where*

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}, \quad \Sigma_{i=1}^{3} x_i = 1 \, .$$

we can get two equations,

$$\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$$

first equation is

$$\begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\textit{then, homogeneous solution is } \mathbb{x} = \lambda \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

$$\textit{given } x_1 + x_2 + x_3 = 1, \textit{ the solution is } \lambda = \frac{1}{8}$$

2.8 *Determine the inverses of the following matrices if possible :*

a.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so there's no inverse

b.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$

2.9 *Which of the following sets are subspaces of* $\mathbb{R}^3$?

$$a.\ A = \left\{\left(\lambda, \lambda + \mu^3, \lambda - \mu^3\right) \mid \lambda, \mu \in \mathbb{R}\right\}$$

let arbitary another set be $\left(\lambda', \ \lambda' + \mu'^3, \lambda' - \mu'^3\right)$

$\left(\lambda + \lambda', \ \lambda + \lambda' + \mu^3 + \mu'^3, \lambda + \lambda' - \mu^3 - \mu'^3\right)$ *should be in the set.*

that said, $\mu^3 + \mu'^3$ *should be* μ''^3 $(\mu'' \in \mathbb{R})$

since $\sqrt[3]{\mu^3 + \mu'^3}$ *is also a real number, it is a subspace*

$$b.\ B = \left\{\left(\lambda^2, -\lambda^2, 0\right) \mid \lambda \in \mathbb{R}\right\}$$

if we multiply -1, $\left(-\lambda^2, \ \lambda^2, \ 0\right)$, *there is no real number that satisfy* $\lambda'^2 = -\lambda^2$

so it is not a subspace.

c. Let γ *be in* $\mathbb{R}$. $C = \left\{(\xi_1, \ \xi_2, \ \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\right\}$

given that $(0, 0, 0)$ *is origin,* γ *should be* 0.

let $(\xi_1', \ \xi_2', \ \xi_3')$ *be another set.*

since $\xi_1 + \xi_1' - 2(\xi_2 + \xi_2') + 3(\xi_3 + \xi_3') = 2\gamma = 0$, *so it is a subspace.*

$$d.\ D = \left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\right\}$$

if we scale set by real number, ξ_2 *might not be an integer which can't be a subspace.*

2.10 *Are the following sets of vectors linearly independent?*

a.

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},\ x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},\ \textit{which is linearly dependent.}$$

b.

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which all column is pivot, so it is linearly independent.

2.11

$$\textit{Write, } y = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

as linear combination of

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\textit{so, } y = -6x_1 + 3x_2 + 2x_3$$

2.12

Consider two subspaces of $\mathbb{R}^4$:

$$U_1 = span[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}],\ U_2 = span[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}]$$

Determine a basis of $U_1 \cap U_2$.

$$U_1 = span[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}],\ since \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$similary, U_2 = span[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}]$$

$$then\ we\ can\ get\ linear\ equation \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -3 & 0 \\ 1 & -1 \end{bmatrix} v = \begin{bmatrix} -1 & 2 \\ -2 & -2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} w$$

$$using\ Gausian\ elimination, \begin{bmatrix} 1 & 2 & 1 & -2 \\ 1 & -1 & 2 & 2 \\ -3 & 0 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\frac{4}{9} \\ 0 & 1 & 0 & -\frac{10}{9} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} = \mathbb{0}$$

$$thus,\ w_1 = -\frac{2}{3} w_2$$

$$let\ w_2 = 3,\ w_1 = -2$$

$$therefore,\ U_1 \cap U_2 = span[-2\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}] = span[\begin{bmatrix} 8 \\ -2 \\ -4 \\ -2 \end{bmatrix}] = span[\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}]$$

2.13

Consider two subspaces U_1 and U_2, where

U_1 is the solution space of the homogeneous equation system $A_1x = 0$ and U_2 is the solution space of the homogeneous equation system $A_2x = 0$ with

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

a. Determine the dimension of $U1$, $U2$.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where } n - rk(A_1) = 1$$

$$A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where } n - rk(A_2) = 1$$

b. Determine bases of U_1 and U_2.

$$U_1 = span[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}], \quad U_2 = span[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}]$$

c. Determine a basis of $U_1 \cap U_2$.

$$U_1 \cap U_2 = span[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}]$$

2.14

Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of A_1 and U_2 is spanned by the columns of A_2 with

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

a. Determine the dimension of $U1$, $U2$.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad where\ rk(A_1) = dim(A_1) = 2$$

$$A_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad where\ rk(A_2) = dim(A_2) = 2$$

b. Determine bases of U_1 and U_2.

$$U_1 = span[\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}],\ U_2 = span[\begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix}]$$

c. Determine a basis of $U_1 \cap U_2$.

$$\begin{bmatrix} 1 & 0 \\ 1 & -2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} v = \begin{bmatrix} 3 & -3 \\ 1 & 2 \\ 7 & -5 \\ 3 & -1 \end{bmatrix} w$$

$$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 1 & -2 & -1 & -2 \\ 2 & 1 & -7 & 5 \\ 1 & 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad that\ said\ w_4\ should\ be\ zero.$$

$$therefore,\ U_1 \cap U_2 = span[\begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}]$$

2.15

$$Let\ F = \left\{(x, y, z)\ \in\ \mathbb{R}^3 \mid x + y - z = 0\right\}\ and\ G = \{(a - b, a + b, a - 3b) \mid a, b \in \mathbb{R}\}.$$

$$a.\ Show\ that\ F\ and\ G\ are\ subspaces\ of\ \mathbb{R}^3.$$

$$both\ set\ includes\ origin\ vector.$$

$$for\ F,\ let\ arbitary\ another\ set\ (x', y', z'),\ then$$

$$(x + x', y + y', z + z')\ \Rightarrow\ x + x' + y + y' - z - z' = 0$$

$$so\ it\ is\ a\ subspace.$$

$$similary\ for\ G,\ let\ arbitary\ set\ (a' - b', a' + b', a' - 3b'),\ then$$

$$(a + a' - (b + b'), a + a' + b + b', a + a' - 3(b + b'))\ which\ is\ also\ closed, which\ is\ a\ subspace.$$

$$b.\ Calculate\ F \cap G\ without\ resorting\ to\ any\ basis\ vector.$$

$$a - b + a + b - a + 3b = 0$$

$$a + 3b = 0,\ a = -3b$$

$$\Rightarrow (-4b,\ -2b,\ -6b),\ F \cap G = span[\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}]$$

$$c.\ Find\ one\ basis\ for\ F\ and\ one\ for\ G,$$

$$calculate\ F \cap G\ using\ the\ basis\ vectors\ previously\ found$$

$$and\ check\ your\ result\ with\ the\ previous\ question.$$

$$F\ is\ a\ homogenous\ solution\ space\ of$$

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} f = \mathbb{0}$$

$$F = span[\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}]$$

$$G\ by\ definition\ span[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}]$$

$$then,\ we\ solve \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} v = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} w$$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$if\ we\ perfome\ back\ substitution,\ w_1 = -3w_2,\ v_2 = 3v_1$$

$$F \cap G = span[-3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}] = span[\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}]\ which\ is\ same\ to\ previous\ result.$$

2.16 *Are the following mappings linear?*

a.

Let $a, b \in \mathbb{R}$.

$$\Phi : L^1([a, b]) \rightarrow \mathbb{R}$$

$$f \rightarrow \Phi(f) = \int_a^b f(x)dx,$$

where L^1 $([a,\ b])$ *denotes the set of integrable functions on* $[a,\ b]$.

$$since \int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$and \int_a^b cf(x)dx = c\int_a^b f(x)dx$$

it is a linear mapping.

b.

$$\Phi : C^1 \rightarrow C^0$$

$$f \rightarrow \Phi(f) = f',$$

where for $k \geqslant 1$, C^k *denotes the set of k times continuously differentiable functions, and* C^0 *denotes the set of continuous functions.*

since, $d(au + bv) = adu + bdv$ *where* a, b *is constant.*

it is a linear mapping.

c.

$$\Phi : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \rightarrow \Phi(x) = cos(x)$$

since $cos(\frac{\pi}{2} + 0) \neq cos(\frac{\pi}{2})$, *it is not a linear mapping. same apply to scalar.*

d.

$$\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$x \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$$

since, matrix multiplication is distributive and scalable, it is a linear mapping.

e.

Let θ *be in* $[0, 2\pi]$ *and*

$$\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$x \rightarrow \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} x$$

apparently, it is a linear mapping.

2.17 *Consider the linear mapping*

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

$a.$ *Find the transformation matrix* A_Φ.

$$A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$b.$ *Determine* $rk(A_\Phi)$.

$$rk(A_\Phi) = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 3$$

$c.$ *Compute the kernel and image of* Φ.
What are $dim(ker(\Phi))$ *and* $dim(Im(\Phi))$?

$$ker(\Phi) = \mathbb{0},\ Im(\Phi) = span[\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}]$$

$by\ rank - nullity\ theorem,\ dim(Im(\Phi)) = rk(A) = 3\ and\ dim(ker(\Phi)) = 3 - rk(A) = 0$

2.18

Let E be a vector space. Let f and g be two automorphisms on E such that
$f \circ g \;=\; id_E$ *(i.e., $f \circ g$ is the identity mapping id_E).*
Show that $ker(f) = ker(g \circ f)$, $Im(g) = Im(g \circ f)$ and that $ker(f) \cap Im(g) = \{\mathbb{0}_E\}$.

let $dim(E) = n$. since f, g is automorphism, transformation matrix are $F^{n\times n}$, $G^{n\times n}$.
also, given that $F \times G = I^{n\times n}$, F and G are regular which means they have full rank.
then, $dim(E) = rk(id_E) = rk(F) = rk(G) = n$
since $f \circ g = g \circ f = id_E$, $ker(f) = ker(g \circ f) = ker(id_E) = \mathbb{0}_E$
we can find out that, $Im(g \circ f) = Im(id_E) = E$
as G has full rank, column space of g has n basis vector which spans whole E.
thus, $Im(g) = Im(g \circ f) = E$ where $ker(f) \cap Im(g) = \{\mathbb{0}_E\}$ is true.

2.19

$$\textit{Consider an endomorphism } \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \textit{ whose transformation matrix}$$
$$\left(\textit{with respect to the standard basis in } \mathbb{R}^3\right) \textit{ is}$$

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$a.\ \textit{Determine } ker(\Phi) \textit{ and } Im(\Phi).$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ker(\Phi) = \mathbb{0},\ Im(\Phi) = \mathbb{R}^3$$

$$b.\ \textit{Determine the transformation matrix } \tilde{A_\Phi} \textit{ with respect to the basis}$$

$$B = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}),$$

$$i.e.,\ \textit{perform a basis change toward the new basis } B.$$

$$\textit{let standard basis transformation matrix } S \textit{ be } B \to I$$

$$\textit{since } \Phi \textit{ is endomorphism, } \tilde{A_\Phi} = S^{-1} A_\Phi S$$

$$S^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\tilde{A_\Phi} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

2.20

Let us consider $b_1,\ b_2,\ b'_1,\ b'_2$, *4 vectors of* $\mathbb{R}^2$ *expressed in the standard basis of* $\mathbb{R}^2$ *as*

$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ b'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},\ b'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (b_1,\ b_2)$ *and* $B' = (b'_1,\ b'_2)$ *of* $\mathbb{R}^2$.

a. Show that B *and* B' *are two bases of* $\mathbb{R}^2$ *and draw those basis vectors.*

clearly, B *and* B' *bases are linearly independent so that they are bases of* $\mathbb{R}^2$.

b. Compute the matrix P_1 *that performs a basis change from* B' *to* B.

$$b'_1 = a_1 b_1 + a_2 b_2\ ,\ b'_2 = a_3 b_1 + a_4 b_2 \quad \text{where } P_1 = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

solve two linear system, $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},\ \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & -1 & -2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right] = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

c. We consider $c_1,\ c_2,\ c_3$, *three vectors of* $\mathbb{R}^3$ *defined in the standard basisof* $\mathbb{R}^3$ *as*

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ c_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},\ c_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (c_1,\ c_2,\ c_3)$.

(*i*) *Show that* C *is a basis of* $\mathbb{R}^3$, *e.g., by using determinants* (*see Section* 4.1).

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix},\ \det(C) = 4 \text{ which means it is invertible.}$$

thus, it is full rank and spans whole $\mathbb{R}^3$

(*ii*) *Let us call* $C' = (c'_1,\ c'_2,\ c'_3)$ *the standard basis of* $\mathbb{R}^3$.

Determine the matrix P_2 *that performs the basis change from* C *to* C'.

$$P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

$d.$ *We consider a homomorphism* $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, *such that*

$$\Phi(b_1 + b_2) = c_2 + c_3$$
$$\Phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$$

where $B = (b_1, b_2)$ *and* $C = (c_1, c_2, c_3)$ *are ordered bases of* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, *respectively. Determine the transformation matrix* A_Φ *of* Φ *with respect to the ordered bases* B *and* C.

$$A_\Phi b_1 = c_1 + 2c_3$$
$$A_\Phi b_2 = -c_1 + c_2 - c_3$$
$$A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

$e.$ *Determine* A', *the transformation matrix of* Φ *with respect to the bases* B' *and* C'.

$$A' = P_2 A_\Phi P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$

$f.$ *Let us consider the vector* $x \in \mathbb{R}^2$ *whose coordinates in* B' *are* $[2, 3]^\top$. *In other words,* $x = 2b'_1 + 3b'_2$.

(i) *Calculate the coordinates of* x *in* B.

$$\begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

(ii) *Based on that, compute the coordinates of* $\Phi(x)$ *expressed in* C.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$$

(iii) *Then, write* $\Phi(x)$ *in terms of* c'_1, c'_2, c'_3.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

(iv)

Use the representation of x *in* B' *and the matrix* A' *to find thisresult directly.*

$$\begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$