3.1

*Show that* $\langle \cdot, \cdot \rangle$ *defined for all* $x = [x_1, x_2]^\top \in \mathbb{R}^2$ *and* $y = [y_1, y_2]^\top \in \mathbb{R}^2$

*by* $\langle x,\ y \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$

*is an inner product.*

*to be an inner product it has to be a positive definite and symmetric bilinear mapping.*

$$\Omega(\lambda x + \psi y,\ z) =$$

$$\lambda x_1 z_1 + \psi y_1 z_1 - (\lambda x_1 z_2 + \psi y_1 z_2 + \lambda x_2 z_1 + \psi y_2 z_1) + 2(\lambda x_2 z_2 + \psi y_2 z_2)$$

$$= \lambda \Omega(x, z) + \psi \Omega(y, z)$$

*simliarlly,*

$$\Omega(x,\ \lambda y + \psi z) = \lambda \Omega(x,\ y) + \psi \Omega(x,\ z)\ .$$

*so it's bilinear mapping.*

$\Omega(x, y) = \Omega(y, x)$ *is true so it's symmetric.*

$$\langle x,\ x \rangle := x_1 x_1 - (x_1 x_2 + x_2 x_1) + 2(x_2 x_2)$$

$$= (x_1 - x_2)^2 + {x_2}^2$$

*which is also positive definite.*

3.2

*Consider* $\mathbb{R}^2$ *with* $\langle \cdot, \ \cdot \rangle$ *defined for all* $x$ *and* $y$ *in* $\mathbb{R}^2$ *as*

$$\langle x,\ y \rangle := x^\top \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} y$$

*Is* $\langle \cdot, \ \cdot \rangle$ *an inner product?*

*no it's not as* $A$ *is not a symmetric matrix.*

3.3

*Compute the distance between*

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

*using*

$$a. \; \langle x, y \rangle := x^\top y$$

$$d(x, y) = \sqrt{(x-y)^\top (x-y)} = \sqrt{4+9+9} = \sqrt{22}$$

$$b. \; \langle x, y \rangle := x^\top A\, y, \quad A := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$d(x, y) = \sqrt{(x-y)^\top \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} (x-y)}$$

$$= \sqrt{9 \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = 3\sqrt{7}$$

3.4

*Compute the angle between*

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

*using*

$$a. \; \langle x, y \rangle := x^\top y$$

$$cos\omega = \frac{\langle x, y \rangle}{||x|| \; ||y||} = \frac{-3}{\sqrt{5}\sqrt{2}} = -\sqrt{0.9}$$

$$b. \; \langle x, y \rangle := x^\top A \, y, \quad A := \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$cos\omega = \frac{\langle x, y \rangle}{||x||_A \; ||y||_A} = \frac{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}}{\sqrt{\begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -3 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}}} = \frac{-11}{\sqrt{18 * 7}} = -\sqrt{\frac{121}{126}}$$

3.5

*Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $x \in \mathbb{R}^5$ are given by*

$$U = span[\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}], \; x = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

*a. Determine the orthogonal projection $\pi_u(x)$ of $x$ onto $U$*

$$B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\Pi = B\left(B^\top B\right)^{-1} B^\top, \;\; p = \Pi\, x$$

$$thus, \; p = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

*b. Determine the distance $d(x, U)$*

$$d(x, U) = \|x - p\| = \sqrt{\begin{bmatrix} -2 & -4 & 0 & -6 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 0 \\ -6 \\ 2 \end{bmatrix}} = \sqrt{60}$$

$$3.6$$

*Consider* $\mathbb{R}^3$ *with the inner product*

$$\langle x, y\rangle := x^\top \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

*Furthermore, we define* $e_1,\ e_2,\ e_3$ *as the standard / canonical basis in* $\mathbb{R}^3$.

a. *Determine the orthogonal projection* $\pi_U(e_2)$ *of* $e_2$ *onto*

$$U = span[e_1,\ e_3]\ .$$

*Hint: Orthogonality is defined through the inner product.*

*we need to find a coordinate of the projection result.*

*which is determined by*

$$\pi_U(x) = \lambda_1 b_1 + \lambda_2 b_2 = B\lambda$$

*by orthogonality,*

$$\langle x - B\lambda, b_1\rangle = \langle x - B\lambda, b_2\rangle = 0$$

$$b_1{}^\top A(x - B\lambda) = 0$$

$$b_2{}^\top A(x - B\lambda) = 0$$

$$\Rightarrow B^\top A(x - B\lambda) = 0$$

$$B^\top Ax = B^\top AB\lambda$$

*since A is positive definite matrix which means it has 0 null space dimension,*
*A has full rank and is invertible by rank – nullity theorem.*
*since B is lineary indepedent,* $B^\top AB$ *becomes positive definite matrix as well.*

$$thus,\ \lambda = \left(B^\top AB\right)^{-1} B^\top Ax$$

$$\Pi = B\left(B^\top AB\right)^{-1} B^\top A$$

$$given\ that\ A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & -0.5 & 1 \end{bmatrix},\ p = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix}$$

b. *Compute the distance* $d(e_2, U)$.

$$d(e2, U) = \|e_2 - p\| = \begin{bmatrix} -0.5 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \\ 0.5 \end{bmatrix} = 1$$

c. *Draw the scenario: standard basis vectors and* $\pi_U(e_2)$

3.7

$Let\ V\ be\ a\ vector\ space\ and\ \pi\ an\ endomorphism\ of\ V.$

$a.\ Prove\ that\ \pi\ is\ a\ projection\ if\ and\ only\ if\ id_v - \pi\ is\ a\ projection,$
$where\ id_v\ is\ the\ identity\ endomorphism\ on\ V.$

$$y(x) = id_v(x) - \pi(x) = x - \pi(x)$$
$$if\ y(x)\ is\ a\ projection,\ y(y(x)) = y(x)$$
$$\Rightarrow x - \pi(x) - \pi(x - \pi(x)) = x - \pi(x)$$
$$\pi(x - \pi(x)) = 0$$
$$by\ linearlity,\ \pi(x) = \pi(\pi(x))\ so\ \pi\ is\ a\ projection.$$
$$reversely, if\ \pi(x)\ is\ a\ projection$$
$$y(y(x)) = x - \pi(x) - \pi(x - \pi(x)) = x - \pi(x) - \pi(x) + \pi(\pi(x))$$
$$= x - \pi(x) - \pi(x) + \pi(x)$$
$$= x - \pi(x) = y(x)$$
$$so\ y(x)\ is\ a\ projection.$$

$b.\ Assume\ now\ that\ \pi\ is\ a\ projection.$
$Calculate\ Im(id_v - \pi)\ and\ ker(id_v - \pi)\ as\ a\ function\ of\ Im(\pi)\ and\ ker(\pi).$

$$again,\ let\ y(x) = id_v(x) - \pi(x) = x - \pi(x)$$
$$then\ kernel\ is\ a\ homogenous\ solution\ space\ of\ equation,\ x - \pi(x) = 0$$
$$thus,\ ker(id_v - \pi) = Im(\pi)$$
$$since,\ \pi(x - \pi(x)) = \pi(x) - \pi(\pi(x)) = 0$$
$$Im(id_v - \pi) = ker(\pi)\ effectively.$$

3.8

*Using the Gram – Schmidt method, turn the basis* $B = (b_1\ ,\ b_2)$ *of a two – dimensional subspace* $U \subseteq R^3$ *into an ONB* $C = (c_1, c_2)$ *of* $U$*, where*

$$b_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ b_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

*let* $u_1 = b_1$*,*

$$u_2 = b_2 - \pi_{u1}(b_2) = b_2 - \frac{u_1 u_1^\top b_2}{\|u_1\|^2} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}{3}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -4/3 \\ 5/3 \\ -1/3 \end{bmatrix} = \frac{3}{\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$$

$$\textit{thus, } c_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ c_2 = \frac{3}{\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$$