Deep Learning and Inverse Problems
Summer 2021
Lecturer: Reinhard Heckel

Problem Set 2

Issued: Wednesday April 21, 2021, Due: Friday April 30 2021.

We covered much of what is necessary for this exercise already on April 21, but will cover a bit more on incoherence on April 28, therefore this is due on Friday, April 30.

Problem 1 (Compressive sensing with spikes and random vectors)**.** In this problem we consider recovery of a signal that is sparse in the concatenation of the identity basis and a random basis, i.e., in the dictionary

$$\mathbf{D} = [\mathbf{B}, \mathbf{I}] \in \mathbb{R}^{n \times 2n}.$$

Here, $\mathbf{B} \in \mathbb{R}^{n \times n}$ is random Gaussian matrix, with entries iid $\mathcal{N}(0, 1/n)$, and $\mathbf{I} \in \mathbb{R}^{n \times n}$ contains the standard basis vectors, i.e., the k-th column of $\mathbf{I}$ is the unit vector $\mathbf{e}_k$ with k-th entry equal to one.

1. First, we are given the measurement

$$\mathbf{y} = [\mathbf{b}_0, \mathbf{e}_0, \ldots, \mathbf{e}_{n-1}]\mathbf{x},$$

   where $\mathbf{x}$ is s-sparse. Furthermore, suppose the support set of $\mathbf{x}$, i.e., the position of the non-zeros is known. How large can s be such that recovery of every s-sparse vector from the corresponding measurement $\mathbf{y}$ is possible, with probability one, and why?

Next, we study recovery of $\mathbf{x}$ from the measurement

$$\mathbf{y} = \mathbf{D}\mathbf{x}.$$

Towards this goal, we first study the incoherence of the matrix $\mathbf{D}$. Recall from class, that a matrix $\mathbf{A}$ with unit norm columns is μ-incoherent if μ is the largest number such that for all pairs of columns $\mathbf{a}_i, \mathbf{a}_j$, $i \neq j$ of $\mathbf{A}$,

$$|\langle \mathbf{a}_i, \mathbf{a}_j \rangle| \leq \mu.$$

Also recall that for a Gaussian random variable $x \sim \mathcal{N}(0, 1)$, we have that, for $\beta \geq \frac{1}{2\pi}$ that

$$\mathrm{P}\left[x \geq \beta\right] \leq e^{-\beta^2/2}.$$

2. What is the distribution of $\langle \mathbf{e}_i, \mathbf{b}_j \rangle$, and given this result, what is bound on the probability that the absolute values of this random variable exceeds a constant β'?

3. Building on the result from the previous part, what is the coherence parameter of the matrix $\mathbf{D}$, with high probability (say with probability at least $1 - \delta$, where $\delta > 0$ is a parameter)? Since $\|\mathbf{b}_i\|_2 \approx 1$, with high probability, you can assume $\|\mathbf{b}_i\|_2 = 1$ for simplicity.

4. Recall from class that provided a matrix $\mathbf{A}$ is μ-incoherent, and $\mathbf{x}$ is s-sparse with $s < \frac{1}{2\mu}$, then ℓ_1-minimization provably recovers the vector $\mathbf{x}$ from the measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$. Based on this result, what is the maximum sparsity under which recovery with ℓ_1-minimization provably succeeds? Is this result pessimistic, i.e., is there a matrix that allows the sparsity s to be significantly larger, if yes, specify that matrix and specify what the allowed level of sparsity is.