Deep Learning and Inverse Problems
Summer 2021
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Problem Set 8
Issued: Wednesday June 16, 2021, Due: Wednesday June 23, 2021.

Problem 1 (Robustness of regularized least squares)**.** Suppose our goal is to recover a vector $\mathbf{x}$ from a noisy measurement $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}, m < n$, is a generic measurement matrix and $\mathbf{e}$ is additive noise. We first study the worst-case robustness of the ℓ_2-regularized least-squares estimator:

$$\hat{\mathbf{x}}_{2,\lambda}(\mathbf{y}) = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda\frac{1}{2}\|\mathbf{x}\|_2^2.$$

(a) Give a closed form expression of $\hat{\mathbf{x}}_{2,\lambda}$ in terms of the singular values and vectors of the matrix $\mathbf{A}$.

(b) Find a worst-case ℓ_2 perturbation

$$\hat{\mathbf{e}} = \arg\max_{\mathbf{e}\colon \|\mathbf{e}\|_2 \leq \epsilon} \|\hat{\mathbf{x}}_{2,\lambda}(\mathbf{A}\mathbf{x} + \mathbf{e}) - \mathbf{x}\|_2^2,$$

and state the maximal mean-squared reconstruction error that the worst-case perturbation induces. How does this error behave as a function of the regularization parameter λ?

(c) Finally, we study how the ℓ_1-regularized least-squares estimator

$$\hat{\mathbf{x}}_{1,\lambda}(\mathbf{y}) = \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

behaves under random (average-case) perturbations. Towards this end, choose $\mathbf{A} \in \mathbb{R}^{500 \times 2000}$ as a Gaussian random matrix with $\mathcal{N}(0, 1/500)$ iid entries, and choose $\mathbf{x}$ as an 50-sparse vector with non-zeros drawn from the distribution $\mathcal{N}(0, 1/50)$. Plot the reconstruction mean-squared error $\|\hat{\mathbf{x}}_{1,\lambda}(\mathbf{A} + \mathbf{e}) - \mathbf{x}\|_2^2$ for a random noise vector $\mathbf{e}$ with iid $\mathcal{N}(0, 0.5/500)$ entries as a function of the regularization parameter λ.