

DEEP LEARNING AND INVERSE PROBLEMS
SUMMER 2021

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Problem Set 4

Issued: Wednesday May 5, 2021, Due: Wednesday May 12, 2021.

Problem 1 (sparse signal recovery). In this problem, we solve the following least-squares regression problem with ℓ_1 -norm regularization numerically:

$$\text{minimize}_{\mathbf{x}} g(\mathbf{x}), \quad g(\mathbf{x}) = \frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{x}\|_1.$$

We take $\mathbf{A} \in \mathbb{R}^{2000 \times 1000}$ as a Gaussian random matrix with entries of zero mean and unit variance, and let $\mathbf{y} = \mathbf{Ax} + \mathbf{e}$, where \mathbf{e} is zero mean Gaussian noise with variance 0.1, and \mathbf{x} is 100-sparse with Gaussian entries, again zero mean and unit variance. Choose λ so that the solution is sparse.

- (a) An algorithm to solve this optimization problem numerically is the Iterated Soft Thresholding Algorithm (ISTA), which is an proximal gradient descent algorithm. Define $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2$, and $\nabla f(\mathbf{x})$ denotes the gradient of f at \mathbf{x} . The ISTA algorithm starts at some random initial point \mathbf{x}^0 and then performs the iterates:

$$\mathbf{x}^k = \tau_{\lambda\eta}(\mathbf{x}^k - \eta \nabla f(\mathbf{x}^k)).$$

Here, τ is the soft thresholding operator, which is applied entrywise and is defined as

$$\tau_{\lambda}(x) = \begin{cases} x + \lambda, & \text{if } x < -\lambda, \\ 0, & \text{if } x \in [-\lambda, \lambda], \\ x - \lambda, & \text{if } x > \lambda. \end{cases}$$

Moreover, α is a stepsize parameter. Implement the ISTA algorithm, and provide a plot of convergence of the ISTA algorithm. In this plot, you plot the value of $g(\mathbf{x}^k) - g(\mathbf{x}^*)$ over the number of iterations, k . Here, \mathbf{x}^* is obtained by running the algorithm until convergence.

- (b) Next, take $\mathbf{A} \in \mathbb{R}^{500 \times 2000}$ as a Gaussian random matrix with zero-mean entries. Generate random k -sparse vectors $\mathbf{x}^* \in \mathbb{R}^{2000}$ (as before, the non-zeros are zero-mean Gaussian) for k ranging from 1 to 500 in some suitably chosen steps, and explore numerically for which values of k , recovery of \mathbf{x}^* is possible from the underdetermined measurement $\mathbf{y} = \mathbf{Ax}^*$ by solving the least-squares regression problem numerically. Note that you have to tune the parameter λ appropriately. One approach to do so is to perform a grid search of the best parameter λ on a logarithmic scale.