CPSC 449: PROGRAMMING PARADIGMS

Assignment Three

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1(E). PRINCIPLE OF STRUCTURAL INDUCTION FOR DATA TYPE POLY

To prove that $P(p)$ holds for all finite polynomials Poly p, we must prove the following:

- Base Case: $P(\text{Constant } c) \wedge P(\text{Variable } v)$
- Induction Step: $P(p_1) \wedge P(p_2) \rightarrow P(Add\ p_1\ p_2) \wedge P(Mult\ p_1\ p_2) \wedge P(Exp\ p_1\ p_2)$

2. PROOF OF $size\ t_r < 2^{depth\ t_r}$ BY STRUCTURAL INDUCTION

Given:

$depth :: Tree\ a \rightarrow \text{Integer}$
$depth\ NilT = 0$ (depth.1)
$depth\ (Node\ n\ t_1\ t_2) = 1 + \max\ (depth\ t_1)\ (depth\ t_2)$ (depth.2)

$size\ NilT = 0$ (size.1)
$size\ (Node\ x\ t_1\ t_2) = 1 + size\ t_1 + size\ t_2$ (size.2)

Prove for all finite $nTree$s, t_r, $size\ t_r < 2^{depth\ t_r}$

Proof. $size\ t_r < 2^{depth\ t_r}$

Base Case $P(NilT)$:

$sizeNilT < 2depthNilT$

$0 < 2^{depth\ NilT}$ (by size.1)

$0 < 2^0$ (by depth.1)

$0 < 1$

Inductive Hypothesis $P(t1) \wedge P(t2)$:

$size\ t_1 < 2^{depth\ t_1}$ (hyp.1)

$size\ t_2 < 2^{depth\ t_2}$ (hyp.2)

Assume Inductive Hypothesis and Prove Inductive Step $P(Node\ n\ t_1\ t_2)$:

$size\ (Node\ n\ t_1\ t_2) < 2^{depth(Node\ n\ t_1\ t_2)}$

LHS - $size\ (Node\ n\ t_1\ t_2)$:

$size\ (Node\ x\ t_1\ t_2) = 1 +\ size\ t_1 + size\ t_2$ (by size.2)

$1 + size\ t_1 + size\ t_2 < 1 + 2^{depth\ t1} + size\ t_2$ (by hyp.1)

$1 + size\ t_1 + size\ t_2 < 1 + 2^{depth\ t_1} + 2^{depth\ t_2}$ (by hyp.2)

RHS - $2^{depth\ (Node\ n\ t_1\ t_2)}$:

$depth\ (Node\ n\ t_1\ t_2) = 2^{1+max\ (depth\ t_1)\ (depth\ t_2)}$ (by depth.2)

$2^{1+max\ (depth\ t_1)\ (depth\ t_2)} = 2 \times 2^{max\ (depth\ t_1)\ (depth\ t_2)}$

If $max\ (depth\ t_1)\ (depth\ t_2) = (depth\ t_1)$:

$1 + size\ t_1 + size\ t_2 < 1 + 2^{depth\ t_1} + 2^{depth\ t_2} \leq 2 * 2^{depth\ t_1}$

Since $depth\ t_1 > depth\ t_2$, $depth\ t_1 - 1 \geq depth\ t_2$.

So $1 +\ size\ t_1 + size\ t_2 < 1 + 2^{depth\ t_1} + 2^{depth\ t_2} \leq 1 + 2^{depth\ t_1} + 2^{depth\ t_1 - 1} \leq 2 * 2^{depth\ t_1}$

To prove this inequality, we need to prove that $1 + 2^n + 2^{n-1} \leq 2 * 2^n$ is indeed true.

Proof. $1 + 2^n + 2^{n-1} \leq 2 * 2^n$

Base $n = 1$:

$1 + 2^1 + 2^0 <= 2 * 2^1\ 4 <= 4$

Assume this is true for $1 \leq n \leq k$ where $n \in \mathbb{N}$.

Prove $n = k + 1$

$1 + 2^{k+1} + 2^k \leq 2 * 2^{k+1}$

$1 + 3 * 2^k \leq 4 * 2^k$

□

So since $1 + size\ t_1 + size\ t_2 < 1 + 2^{depth\ t_1} + 2^{depth\ t_1 - 1} \leq 2 * 2^{depth\ t_1}$, $1 + size\ t_1 + size\ t_2 < 2^{1+depth\ t1}$

The proof proceeds much the same way for if $max\ (depth\ t_1)\ (depth\ t_2) = (depth\ t_2)$.

WLOG, the proof is complete. $\square$