Contents

Week 9: Concrete Formulation II

This week continues to focus on the English description and the concrete formulation, which are the crux of optimization modeling. By the end of this week, you will be able to utilize auxiliary decision variables and creative problem solving to tackle more complex problems.

Session 17: Effectively Utilizing Auxiliary Decision Variables

In-Class Exercise: Food Production

A food factory requires 2000 tons of canola oil every month as a raw ingredient. The price of canola oil fluctuates from month to month due to market conditions. The predicted prices for the next six months are as follows

Month	1	2	3	4	5	6
Price ($) per ton	150	160	180	170	180	160

The factory's supplier for canola oil delivers it on the first day of every month, and charges the prices above. The factory can decide how much oil to buy each month from the supplier. At the end of each month, the factory can also store unused oil for future use, but the inventory of oil (the total amount stored) cannot exceed 1000 tons at any given time. (The current inventory of oil before the shipment in Month 1 is zero.) Formulate a linear optimization problem to decide how much canola oil to buy for each of the six months in order to minimize the total purchase cost over these six months.

English Description

Decision:

Objective:

Constraints:

Concrete Formulation

Decision Variables:

Objective:

Constraints:

In-Class Exercise: Adding Fixed Costs

Suppose that in each month in which any amount of oil is bought, there is a fixed cost of \$50,000. Furthermore, if the amount of oil bought in a month is not zero, then it must be at least 200 tons. Modify the above formulation to incorporate these considerations.

Modified Concrete Formulation

Decision Variables:

Objective:

Constraints:

Exercise 9.1: Investment Planning

Finco Investment Corporation wishes to determine an investment strategy for the firm for the next 3 years. At present (Year 0), $100,000$ is available for investment. The goal is to maximize the cash on hand at the end of Year 3.

There are five investment options, each of which allows you to put in an arbitrary amount of principal at a given time, and will payoff a certain percentage of the principal at a later time. (The payoff includes all of the money you will get back; you won't get back the principal at a later time.) The five options are summarized below:

Investment option	Time of investment	Payoff schedule
A	Year 0	50% in Year 1 and 100% in Year 2
B	Year 1	50% in Year 2 and 100% in Year 3
C	Year 0	120% in Year 1
D	Year 0	190% in Year 3
E	Year 2	150% in Year 3

To ensure that the company's portfolio is diversified, Finco required that at most $75,000 be placed in any single investment option. Payoffs happen at the beginning of the year, so can be reinvested in the same year. For example, the positive cash flow received from Option C in Year 1 can be reinvested immediately in Option B. However, Finco cannot borrow funds, so net cash on hand must be non-negative in all years. Formulate this as a linear optimization problem.

English Description

Decision:

Objective:

Constraints:

Concrete Formulation

Decision Variables:

Objective:

Constraints:

Exercise 9.2: Nurse Scheduling Revisited

This question modifies the nurse-scheduling probelm from last week by incorporating multiple shift lengths.

Hospital administrators must schedule nurses so that the hospital's patients are provided with adequate care. At the same time, in the face of tighter competition in the health care industry, they must pay careful attention to keeping costs down.

From historical records, administrators estimated the minimum number of nurses to have on hand for the various times of the day, as shown in the following table.

Shift	Time	Minimum number of nurses needed
1	Midnight-4am	5
2	4am-8am	12
3	8am-noon	14
4	noon-4pm	8
5	4pm-8pm	14
6	8pm-Midnight	10

In a given day, a nurse can either work for one shift, or for two consecutive shifts. The hourly pay for a four hour shift is 60 dollars/hour, while the hourly pay for an eight hour shift (two consecutive shifts) is 50 dollars/hour. As a result, in each shift, there are two types of nurses: those working for two shifts that started in the previous shift (and are now working their second shift), and those that just started in this shift (some of whom may be working in the next shift as well). Note that a nurse working two shifts who starts at the 8pm-Midnight shift would finish work after the next day's Midnight-4am shift.

Formulate a linear optimization problem to **minimize the total cost per day for hiring nurses** subject to being able to fulfill all business constraints. (For this problem, you only need to submit the concrete formulation, but don't need to submit the English description.)

Decision variable:

Objective

Constraints:

Session 18: Creative Problem Solving

Tips for Solving Difficult Problems

1. Be in a relaxed state of mind. If needed, take a break and go to the bathroom, or work on something else and come back later.

2. Make sure you understand the problem. It might be helpful to sketch a drawing to visualize what is going on at a high level, or talk yourself through a numerical example to make sure you internalize the key tradeoffs.

3. Brainstorm a few ideas before committing. The first idea you think of is often not the best one. Spend a few minutes to jot down 3-5 different ideas of how to approach the problem before exploring any idea in depth. For tricky constraints, think of several ways of phrasing the constraint in English, before trying to formulate anything using variables.

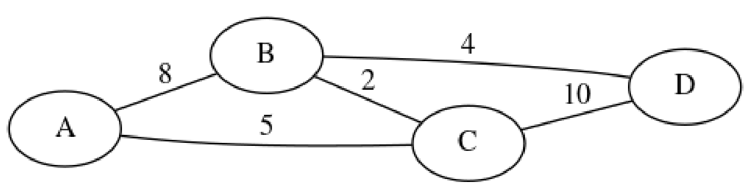
4. Clarify the formulation in English first. Before writing any mathematical expressions, ensure that the English description of your formulation is precise, complete and succinct, and try to use helpful keywords as much as possible.

5. Use auxiliary decision variables correctly. Auxiliary decision variables can be helpful for clarifying complex logic or for expressing non-linear logic in a linear way. When you use these variables, ensure that you connect them to the main decision variables using appropriate linear constraints.

6. Plug in concrete numbers as a sanity check. After writing down a mathematical formulation, make up some values for the decision variables and plug them in, to see whether the objective and constraints are what they should be. In particular, check whether the constraints always rule out what should be ruled out and always allow what should be allowed. Plug in several sets of numbers until you are convinced that the logic of your formulation is correct.

Sample Problem: Maximizing Supply Chain Throughput

A company manufactures a type of heavy machinery in city A and would like to determine the fastest rate at which it can deliver machines to customers in city D. (Rate, or throughput, is measured in the average number of machines delivered per day.) The bottleneck is that the company must use a special type of truck to ship the machine, and a limited number of these trucks travel between two adjacent cities each day. Each truck can carry only one machine at a time, and each truck only makes trips between two specified cities and will not go anywhere else. The following figure shows which cities are adjacent and how many trucks travel between each pair of adjacent cities in either directions each day.



For example, 8 trucks travel from A to B per day, and all 8 return from B to A on the same day. Since a truck can bear load when travelling in either directions, the rate at which machines travel between A and B is at most 8 per day in either directions. Machines that arrive at city B must be immediately unloaded from the truck it came from (as the truck is going back to city A); later on, that machine can be loaded unto other trucks that travel for example to city C or D. Because all the demand are in city D, the rate at which machines arrive into city B must equal the rate at which machines leave city B, and similarly for city C.

i) Formulate a linear optimization model to determine the fastest rate at which the company can satisfy demand in city D.

Decision Variables:

Objective and Constraints:

ii) Suppose that there is an additional constraint: if the company uses any trucks that travel directly between A and C, then it cannot use trucks that travel directly between B and D. **Define additional decision variables and linear constraints to implement this.**

Additional Decision Variables:

Additional Linear Constraints:

Exercise 9.3: Optimal Box Selection

A company sells items of various sizes and ships them to customers using special boxes. While the sizes of the items are fixed, the company can decide what sized boxes to use for shipping. The following table lists the types of items, along with the minimum box size needed for each item, as well as the number of each item that needs to be shipped.

Item type	1	2	3
Minimum box size (in cubit feet)	1.5	1.7	2.0
Demand	400	500	200

For simplicity, the company limits the set of possible box sizes to be exactly the sizes listed in the table above. In order to satisfy demand, the company can always use a larger box to ship a smaller item. For example, a type-1 item can be shipped with a box of size 1.5, but can also be shipped using boxes of sizes 1.7 or 2.0.

While larger boxes are more flexible, they are also more expensive to make: **the variable cost** (in dollars) of making each box is exactly equal to the box size. However, the higher variable cost might be worth it since to produce a box of a certain size, the company needs to pay a **fixed cost** of 1000 to create the mold. So using larger boxes might allow the company to make do with fewer box types, which would lower the total fixed cost. For example, using boxes of all three types would incur a fixed cost of 3000, while using only boxes of size 2.0 would incur a fixed cost of 1000.

Write a linear optimization formulation to help the company determine which box types to produce, as well as how many boxes of each size to produce, in order to minimize the total cost while satisfying all demand. (The total cost is the sum of the total variable cost and the total fixed cost.)

English Description

Decision:

Objective:

Constraints:

Concrete Formulation

Decision Variables:

Objective and Constraints:

Exercise 9.4: Course Selection

Aithne is currently enrolled in a Masters program at USC and is planning her courses for the next 2 semesters. There are five elective courses she would like to take, which we refer to as Courses A, B, C, D, and E. Based on her conversations with past students and prospective employers, she has estimated an "importance score" for each course, as well as the "workload" in terms of hours of work needed per week. Moreover, the schedules for the next two semesters have already been published, and this gives her information about scheduling conflicts as well as how much time she can afford to spend on these electives after accounting for her mandatory courses and other responsibilities. **Each course is a single semester long and can be taken only once, but the same course may be offered in both semesters, so she can choose when to take each course as well as whether to take it.** These information are summarized in the three tables below.

Course	**A**	**B**	**C**	**D**	**E**
Importance Score	5	3	2	4	5
Workload (hours/week)	15	10	10	5	10

Schedule	Semester 1	Semester 2
A	Tue/Thu 11-12:20	Mon/Wed 12:30-13:50
B	Tue 9-12:00	Tue 9:00-12:00
C	Mon/Wed 12:30-13:50	Not offered
D	Mon 12:00-15:00	Tue/Thu 11:00-12:20
E	Mon/Wed 14:00-15:50	Tue/Thu 11:00-12:20
Total time she can afford to spend	20 hours/week	15 hours/week

Furthermore, **Course A is a pre-requisite of Course D**, which means that if Aithne wishes to take Course D, she must take it in Semester 2 after taking Course A in Semester 1. Moreover, **Course A is a co-requisite of Course E**, which means that if she takes Course E in a semester, she must either be concurrently taking Course A or has already completed the course in a previous semester. (However, Course A can be taken by itself, without concurrently taking Course E.)

Aithne would like to plan a schedule that maximizes the total importance score of courses she takes. For example, if she takes only courses A and B in the two semesters, the total importance score would be $5+3=8$. **Write a concrete formulation of a linear optimization model to help Aithne plan her course selection for the two semesters. The objective and all constraints must be linear.**

Decision Variables:

Objective:

Constraints:

Instructions for Quiz 3

Quiz 3 will take place on Thursday next week, in the first fifteen minutes of class. The quiz will be open-notes but closed-computer. **You will be asked to identify whether or not a given expression is a correct linear constraint encoding a desired logic.** Learning to recognize what is correct or incorrect will deeper your grasp of linear optimization modeling, and increase your confidence on whether or not your own formulation is correct.

As always, the quiz is worth 4 percent of your final grade. If you miss the quiz, the weight will automatically be transferred to the final exam. After you are done, do not share the quiz questions or your answers with other students, including those in other sections. If you finish the quiz early, you can quietly stay in your seats or leave the room. **Do not use a cell phone, tablet or computer in the classroom before all the quizzes are handed in.**

Sample Quiz 3

Name: ____________________________________ Section: 12:30pm or 2pm

Each of the following questions asks you to identify **linear constraints** that correctly encode the desired logic, which is either expressed in English or in terms of Python code. **For each question, select the single best answer** by circling it or by writing the corresponding letter next to the question.

Q1. Suppose that X, Y and Z are all binary variables.

```
X=(Y and Z)
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A) $Y \leq X \leq Z$.

B) $2X \leq Y + Z \leq X + 1$.

C) $X \leq Y + Z \leq 2X$.

D) $X = YZ$.

E) None of the above is a correct linear constraint.

F) More than one of the above options are correct.

Q2. Suppose that W, X, Y, and Z are all binary variables.

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If W and X are both True, then Y must be no greater than Z; otherwise, there's no requirement.
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A) $WX(Y - Z) \leq 0$.

B)

$W + Y \leq Z + 1$,
$X + Y \leq Z + 1$.

C) $2 - (W + X) \leq Y - Z$.

D) $2(W + X) + Y \leq Z + 4$.

E) None of the above is a correct linear constraint.

F) More than one of the above options are correct.

Q3. Suppose that X is a binary variable; Y and Z are continuous variables bounded between 0 and 100.

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If X is true, then Y<=Z; otherwise, there is no requirement.
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A) $X(Z - Y) \geq 0$.

B) $Y \leq Z + 100X$.

C) $X \leq Z - Y$.

D) $100X + Y \leq Z + 100$.

E) None of the above is a correct linear constraint.

F) More than one of the above options are correct.