Math311, Quiz 5, 07/25/14

First Name: ......Keys...... Last Name: ....

Grade: ....

SHOW ALL WORK!

Problem 1. For the vectors:

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix},$$

(a) (40%) determine whether they are linearly dependent or independent.

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3)\bar{c} = \bar{0} \Rightarrow \left(\begin{array}{ccc|c} 3 & -3 & -6 & 0 \\ -2 & 2 & 4 & 0 \\ 4 & -4 & -8 & 0 \end{array}\right) \Rightarrow \begin{array}{l} R_1 \to R_1 \\ R_2 + \frac{2}{3}R_1 \to R_2 \\ R_4 - \frac{4}{3}R_1 \to R_4 \end{array} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow 3c_1 - 3c_2 - 6c_3 = 0, \text{ set } c_2 = \alpha, c_3 = \beta, c_1 = \alpha + 2\beta$$

$$\bar{c} = \begin{pmatrix} \alpha + 2\beta \\ \alpha \\ \beta \end{pmatrix} \forall \alpha, \beta \quad \bar{c} \text{ can be non zero}$$

$\Rightarrow \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ is linearly dependent

(b) (10%) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

$$\text{span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3\} = \text{span}\{\bar{x}_1\} \Rightarrow \dim(\text{span}(\bar{x}_1)) = 1.$$

Problem 2. (50%) Determine whether the vectors $1, e^x + e^{-x}, e^x - e^{-x}$ linearly dependent or independent.

$$W = \begin{vmatrix} 1 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} e^x - e^{-x} & e^x + e^{-x} \\ e^x + e^{-x} & e^x - e^{-x} \end{vmatrix} = (e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})$$

$$= e^{2x} - 2e^x e^{-x} + e^{-2x} - (e^{2x} + 2e^x e^{-x} + e^{-2x})$$

$$= e^{2x} - 2 + e^{-2x} - e^{2x} - 2 - e^{-2x} = -4 \neq 0$$

Wronskian

$\Rightarrow$ linearly independent

Math311, Quiz 6, 07/29/14

First Name: Keys. .... Last Name: ....

Grade: .... Keys

SHOW ALL WORK!

Problem 1. Find a basis for the row space, a basis for the column space, and a basis for the null space of the matrix A is given by

$$A = \begin{array}{c} \begin{bmatrix} 3 & 1 & 5 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

Sol

$$\begin{array}{l} R_1 \to R_1 \\ 3R_2 - R_1 \to R_2 \\ 3R_3 - R_1 \to R_3 \\ R_4 \to R_4 \end{array} \Rightarrow \begin{pmatrix} 3 & 1 & 5 & 2 & 0 \\ 0 & -1 & -2 & -2 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 \end{pmatrix} \begin{array}{l} R_1 \to R_1 \\ -1 \cdot R_2 \to R_2 \\ -1 \cdot R_3 + R_2 \to R_3 \\ R_4 + R_2 \to R_4 \end{array} \Rightarrow \begin{pmatrix} 3 & 1 & 5 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} R_1 - R_2 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ R_4 \to R_4 \end{array} \Rightarrow \begin{pmatrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \frac{1}{3} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ R_4 \to R_4 \end{array} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A basis of $\text{row}(A) = \text{span}\{(1,0,1,0,0)^T, (0,1,2,2,0)^T, (0,0,0,0,1)^T\}$

A basis of $\text{col}(A) = \text{span}\{(\bar{a}_1, \bar{a}_2, \bar{a}_5)\} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Nullspace: $\begin{cases} x_1 + x_3 = 0 \\ x_2 + 2x_3 + 2x_4 = 0 \\ \boxed{x_5 = 0} \end{cases} \Rightarrow$ $\boxed{x_1 = -x_3}$ $\boxed{x_2 = -2x_3 - 2x_4}$

$\boxed{\begin{array}{l} x_3 = \alpha \\ x_4 = \beta \end{array}}$ - free variables

$$\Rightarrow \bar{x} = \begin{pmatrix} -\alpha \\ -2\alpha - 2\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \forall \alpha, \beta$$

$$\Rightarrow \text{Null}(A) = \text{Span}\left\{ \underbrace{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{are bases for the nullspace.}} \right\}$$

7

Math311, Quiz [redacted] 07/31/14

First Name: .... Last Name: Keys.

Grade: ....

SHOW ALL WORK!

Problem 1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ and defined as

$$L(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix}, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

Find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$, $\forall \mathbf{x} \in \mathbb{R}^3$.

Insert the basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \in \mathbb{R}^3$ into the operator:

$$\left.\begin{aligned} L(\bar{e}_1) &= L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \\ L(\bar{e}_2) &= L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} \\ L(\bar{e}_3) &= L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \end{aligned}\right\} \Rightarrow A = \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix} \Rightarrow L(\bar{x}) = A\bar{x}, \quad \forall \bar{x} \in \mathbb{R}^3$$

Problem 2. Let $L : \mathbb{P}_2 \to \mathbb{R}^2$ and defined as

$$L(p(x)) = \begin{pmatrix} \int_0^1 p(x)dx \\ p(0) \end{pmatrix}, \quad \forall p \in \mathbb{P}_2.$$

Find a matrix A such that $L(\alpha + \beta x) = A \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Test your answer.

(Hint: use the set $\{1, x\}$ as an ordered basis of P_2.

Compute $L(1)$ and $L(x)$.

$$\left.\begin{aligned} L(1) &= \begin{pmatrix} \int_0^1 1\,dx \\ 0 \end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \\ L(x) &= \begin{pmatrix} \int_0^1 x\,dx \\ 0 \end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix} \end{aligned}\right\} \Rightarrow A = \begin{pmatrix}1 & \frac{1}{2} \\ 0 & 0\end{pmatrix};$$

<u>Test</u> $L(\alpha+\beta x) = \begin{pmatrix} \int_0^1 (\alpha+\beta x)dx \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha + \frac{1}{2}\beta \\ 0 \end{pmatrix}$

$$L(\alpha+\beta x) = A\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1 & \frac{1}{2}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha + \frac{1}{2}\beta \\ 0\end{pmatrix}$$

Math311, Quiz 8, 08/04/14

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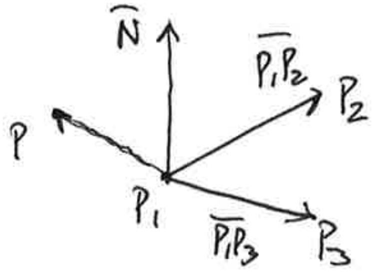
Grade: ....

SHOW ALL WORK!

Problem 1. Find the equation of the plane that passes through the points

$$P_1 = (2,3,1), \quad P_2 = (5,4,3), \quad P_3 = (3,4,4).$$

$$\overline{P_1P_2} = \begin{pmatrix} 5-2 \\ 4-3 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad ; \quad \overline{P_1P_3} = \begin{pmatrix} 3-2 \\ 4-3 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$



$$\overline{N} = \overline{P_1P_2} \times \overline{P_1P_3} = \begin{vmatrix} \overline{e}_1 & \overline{e}_2 & \overline{e}_3 \\ 3 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= (-1)^{1+1}\overline{e}_1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + (-1)^{1+2}\overline{e}_2 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} + (-1)^{1+3} e_3 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (3-2)\overline{e}_1 - (9-2)\overline{e}_2 + (3-1)\overline{e}_3$$

$$= 1 \cdot \overline{e}_1 - 7\overline{e}_2 + 2\overline{e}_3$$

$$\Rightarrow \overline{N} = \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} \text{ — the normal vector.}$$

Take $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \overline{P_1P} = \begin{pmatrix} x-2 \\ y-3 \\ z-1 \end{pmatrix} \Rightarrow \overline{N}^T\, \overline{P_1P} = 0$, must be 0.

$$\Rightarrow \underbrace{1(x-2) - 7(y-3) + 2(z-1) = 0}_{\text{The equation of the plane.}} \quad \text{or} \quad x - 7y + 2z = 17$$

Math311, Quiz 9, 08/06/14

First Name: Keys. .... Last Name: ....

Grade: ....

SHOW ALL WORK!

Problem 1. (a) 80% Find the best least squares fit by a linear function $y = a + bx$, to data $(x, y) = (-1, 0), (0, 1), (1, 3), (2, 9)$.

x	y
-1	0
0	1
1	3
2	9

$y = a + bx$ find a, b.

$$\begin{cases} 0 = a + b(-1) \\ 1 = a + b\cdot 0 \\ 3 = a + b\cdot 1 \\ 9 = a + b\cdot 2 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\bar{c}} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}}_{b}$$

$A\bar{c} = \bar{b}$ - overdetermined

$\underbrace{A^T A \bar{c} = A^T \bar{b}}_{\text{least squares.}}$

$$\Rightarrow A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}; \quad A^T \bar{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}$$

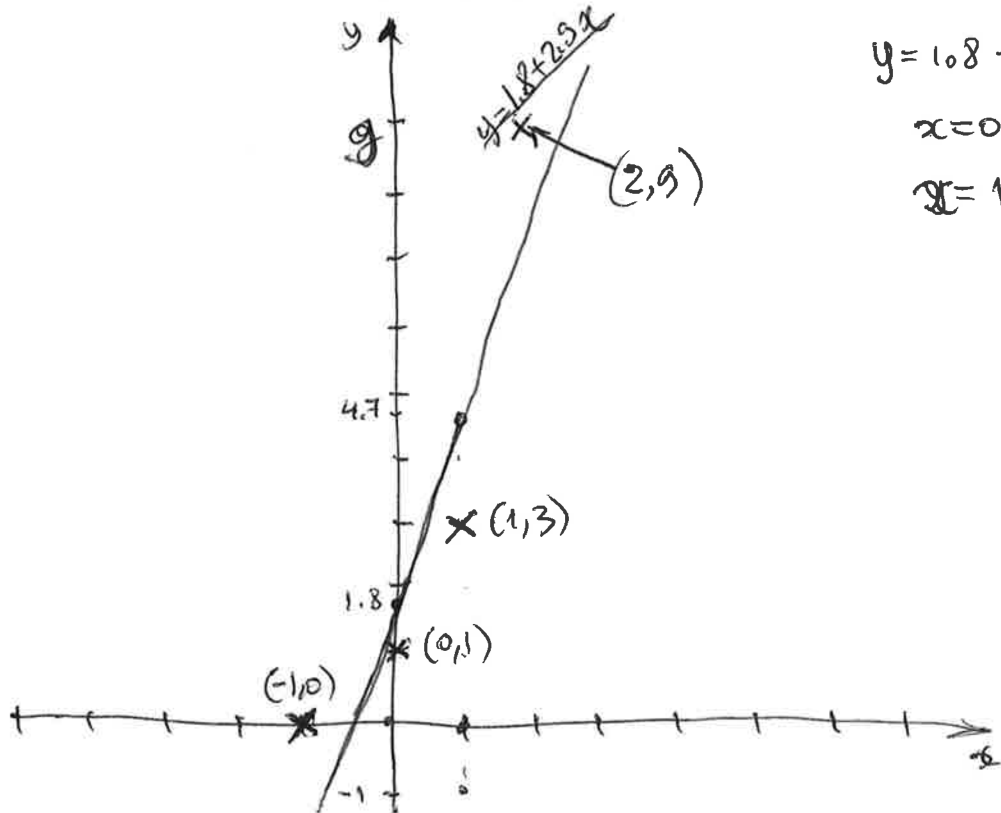
$$A^T A \bar{c} = A^T \bar{b} \Rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 4 & 2 & 13 \\ 2 & 6 & 21 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 4 & 2 & 13 \\ 0 & -10 & -29 \end{array}\right) \Rightarrow -10b = -29$$

$$\boxed{b = 2.9}$$

$4a + 5.8 = 13$

$4a = 7.2 \Rightarrow a = 1.8 \Rightarrow \boxed{y = 1.8 + 2.9x}$

(b) 20% Plot the line you found in part (a) together with the data on a coordinate system.



$y = 1.8 + 2.9x$

$x = 0 \Rightarrow y = 1.8$

$x = 1 \Rightarrow$

$y = 1.8 + 2.9 = 4.7$