# Math311, Midterm exam, 07/23/14

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"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work"
Signature: ________________________ UIN: __________________

**Rules: no books, no notes, no tablets, no calculators. Show all work for full credit! Solve the easiest problems first. Good luck!**

SHOW ALL WORK!

**Problem 1.** **20%** Solve the following system of equations using Gaussian elimination. If there are free variables find all solutions.

$$\begin{aligned} x_1 + \ x_2 + 2x_3 + 3x_4 + 2x_5 &= \ \ 1, \\ x_1 + \ x_2 \qquad\quad + 2x_4 + 5x_5 &= -1, \\ 2x_1 + 2x_2 + 6x_3 + 7x_4 + 2x_5 &= \ \ 0. \end{aligned} \qquad (1)$$

Solution. We write an augmented matrix as.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 0 & 2 & 5 & -1 \\ 2 & 2 & 6 & 7 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_1 \\ R_2 - R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & -2 & -1 & 3 & -2 \\ 0 & 0 & 2 & 1 & -2 & -2 \end{array}\right)$$

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 + R_2 \to R_3 \end{array} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & -2 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right) \Rightarrow \left\{\begin{aligned} c_1 + c_2 + 2c_3 + 3c_4 + 2c_5 &= 1, \\ -2c_3 - c_4 + 3c_5 &= -2, \\ \underline{c_5 = -4}, \end{aligned}\right.$$

$$\Rightarrow \left\{\begin{aligned} c_1 + c_2 + 2c_3 + 3c_4 &= 9, \\ -2c_3 - c_4 &= 10, \\ \boxed{c_5 = -4} \end{aligned}\right. \Rightarrow \text{set } c_4 \text{ as a free variable} \Rightarrow \boxed{c_3 = \tfrac{1}{2}c_4 - 5}$$

set $c_2$ as a free variable

$$\left.\begin{aligned} \Rightarrow c_1 &= -c_2 - 2c_3 - 3c_4 + 9 \\ &= -c_2 - c_4 + 10 - 3c_4 + 9 \end{aligned}\right\} \Rightarrow \boxed{c_1 = -c_2 - 4c_4 + 19}$$

$$\Rightarrow \bar{x} = \begin{pmatrix} -c_2 - 4c_4 + 19 \\ c_2 \\ \frac{1}{2}c_4 - 5 \\ c_4 \\ -4 \end{pmatrix}.$$

$\forall\, c_2$ and $c_4$

Test | $c_2 = 1$, $c_4 = 0 \Rightarrow \bar{x} = \begin{pmatrix} 18 \\ 1 \\ -5 \\ 0 \\ -4 \end{pmatrix}$ insert into (1)

$$\begin{aligned} \Rightarrow\ & 18 + 1 - 10 + 0 - 8 = 1, \checkmark \\ & 18 + 1 + 0 + 0 - 20 = -1, \checkmark \\ & 36 + 2 - 30 + 0 - 8 = 0, \checkmark \end{aligned}$$

Problem 2. 15% Find $A^{-1}$ (inverse of the matrix A) if

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

<u>Solution</u> Write an augmented matrix $(A \mid I)$ and transform it into $(I \mid B)$; $B = A^{-1}$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_3 - 3R_2 \Rightarrow R_3 \quad \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_3 \cdot \left(-\frac{1}{2}\right) \to R_3 \\ R_4 + R_3 \to R_4 \end{array}$$

$$\Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 1 & 1 \end{array}\right) \quad R_3 + \frac{1}{2}R_4 \to R_3 \quad \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -3 & 1 & 1 \end{array}\right) \quad R_2 - R_3 \to R_2$$

$$\Rightarrow \Bigg( \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I} \;\Bigg|\; \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -3 & 1 & 1 \end{matrix}}_{A^{-1}} \Bigg)$$

<u>Test:</u>

$$A^{-1} \cdot A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \checkmark$$

**Problem 3.** The matrix $A$ is given by

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$$A = \begin{bmatrix} 2 & 0 & x & 8 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 0 & 2 & x \\ 2 & 2 & 7 & 9 \end{bmatrix}$$

where $x$ is unknown.

(a) 10% Compute the determinant of $A$ as a function of $x$;

Expand along the ~~first~~ second row:

$$\det(A) = 1 \cdot (-1)^{2+1} \begin{vmatrix} 0 & x & 8 \\ 0 & 2 & x \\ 2 & 7 & 9 \end{vmatrix} = - \begin{vmatrix} 0 & x & 8 \\ 0 & 2 & x \\ 2 & 7 & 9 \end{vmatrix},$$

now expand along the first column:

$$= -2 \cdot (-1)^{3+1} \begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = -2(x^2 - 16).$$

(b) 5% Find all values of $x$ for which the determinant equals 0.

$$\det(A) = 0 \Rightarrow -2(x^2-16) = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$

**Problem 4.** For each of the following pairs of matrices, find an elementary matrix $E$ such that $EA = B$.

(a) 5%

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{bmatrix}$$

It is easy to see that the first and third rows are interchanged.

therefore

$$E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Test $E \cdot A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix}$ ✓

(b) 5%

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 7 & 1 \end{bmatrix}$$

It is easy to see that $\begin{cases} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 + 3R_2 \to R_3 \end{cases} \Rightarrow \begin{cases} 1 \cdot R_1 + 0 \cdot R_2 + 0 \cdot R_3 \to R_1, \\ 0 \cdot R_1 + 1 \cdot R_2 + 0 \cdot R_3 \to R_2, \\ 0 \cdot R_1 + 3 \cdot R_2 + 1 \cdot R_3 \to R_3 \end{cases}$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}}_{E} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \to \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}.$$

Test: $E \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 7 & 1 \end{pmatrix}$ ✓

**Problem 5.1.** Determine whether the following are subspaces of $\mathbb{R}^3$.

(a) 5% $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\} = S$

(i) $S$ is not empty since it has at least $\bar{x} = (1,0,0)^T$.

(ii) Take $\alpha$ - a scalar, $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ 1-x_1 \end{pmatrix} \Rightarrow \alpha \cdot \bar{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha(1-x_1) \end{pmatrix} \Rightarrow \alpha x_1 + \alpha(1-x_1) = \alpha x_1 - \alpha x_1 + \alpha = \alpha \neq 1$ if $\alpha \neq 1$

$\Rightarrow$ it is not a subspace.

(b) 5% $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\} = S$

(i) $(0,0,0) \in S \Rightarrow S \neq \emptyset$.

(ii) $\alpha \cdot \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha x \\ \alpha x \end{pmatrix} \in S$ ; (iii) $\bar{x} + \bar{y} = \begin{pmatrix} x \\ x \\ x \end{pmatrix} + \begin{pmatrix} y \\ y \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix} \in S$, it is a subspace.

**Problem 5.1**: Determine whether the following are subspaces of $C[-1, 1]$.

(a) 5% The set of functions $f$ in $C[-1, 1]$ such that $f(-1) = f(1)$ : $S$

(i) $f(x) = 0$ belongs to the subspace $\Rightarrow$ it is not empty.

(ii) $\alpha$ - scalar ; $f \in S$ , $(\alpha f)(x) = \alpha f(x)$ ; so if $f(-1) = f(1)$ , $\alpha f(-1) = \alpha f(1)$ ✓
$\Rightarrow (\alpha f)(x) \in S$.

(iii) $f, g \in S \Rightarrow (f+g)(x) = f(x) + g(x) \Rightarrow \begin{matrix} f(-1) = f(1) \\ g(-1) = g(1) \end{matrix} \Rightarrow \begin{matrix} f(-1) + g(-1) = f(1) + g(1) \\ (f+g)(-1) = (f+g)(1) \end{matrix}$ ✓

$S$ is a subspace.

(b) 5% The set of continuous non-decreasing functions in $[-1, 1]$, i.e. $\forall a, b \in [-1, 1]$ such that $a \le b$ we have $f(a) \le f(b)$ : $S$

(i) $f(x) = 0 \in S \Rightarrow S \neq \emptyset$

(ii) $(\alpha f)(x) = \alpha f(x)$ ; if $f(a) \le f(b) \Rightarrow \alpha f(a) \le \alpha f(b)$ if $\alpha > 0$

$\alpha f(a) \geq \alpha f(b)$ if $\alpha < 0$

That means that $(\alpha f)(x)$ can be non-increasing

$\Rightarrow (\alpha f) \notin S$ , if $\alpha < 0$



$\Rightarrow S$ is not a subspace.

**Problem 6.**

(a) **10%** Show that the vectors

$$\mathbf{x}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}3\\1\\7\end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix}0\\2\\2\end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

linearly dependent.

A linear combination of the vectors reads:

$$c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3 + c_4\bar{x}_4 + c_5\bar{x}_5 = 0$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 2 & 1 & 7 & 2 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \begin{cases} c_1 + 3c_3 = 0 \\ c_2 + c_3 + 2c_4 + c_5 = 0 \\ \boxed{c_5 = 0} \end{cases} \Rightarrow \begin{cases} c_1 + 3c_3 = 0 \\ c_2 + c_3 + 2c_4 = 0 \end{cases} \Rightarrow$$

take $c_4$ and $c_2$ as a free var.

$$\Rightarrow \boxed{c_3 = -c_2 - 2c_4}, \ \boxed{c_1 = 3c_2 + 6c_4} \quad \bar{c} = \begin{pmatrix} 3c_2 + 6c_4 \\ c_2 \\ -c_2 - 2c_4 \\ c_4 \\ 0 \end{pmatrix}$$

$\forall c_2, c_4$, that can be non zero. ✓

(b) **10%** Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis of $\mathbb{R}^3$.

Take now $c_2 = 1, c_4 = 0 \Rightarrow \bar{c} = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 3\bar{x}_1 + \bar{x}_2 - \bar{x}_3 = 0 \Rightarrow \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ lin. depen.

Take $c_2 = 0, c_4 = 1 \Rightarrow \bar{c} = \begin{pmatrix} 6 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow 6\bar{x}_1 - 2\bar{x}_3 + \bar{x}_4 = 0 \Rightarrow \{\bar{x}_1, \bar{x}_3, \bar{x}_4\}$ lin dep.

$\Rightarrow \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5\}$; The possible basis are:

$\{\bar{x}_1, \bar{x}_2, \bar{x}_4\}$

$\{\bar{x}_1, \bar{x}_3, \bar{x}_4\}$

$\{\bar{x}_1, \bar{x}_4, \bar{x}_5\}$

$\{\bar{x}_2, \bar{x}_3, \bar{x}_4\}$

$\{\bar{x}_2, \bar{x}_4, \bar{x}_5\}$

$\{\bar{x}_3, \bar{x}_4, \bar{x}_5\}$