Piecewise n-th order spline approximation

- let $x_0, \dots, x_N$ be a sequence of $N+1$ data points with (numerical) derivatives $x_0^{(k)}, \dots, x_N^{(k)}$ up to the k-th order; assumptions is that these data points are results of an underlying function $f(t)$ at instants $t_0, \dots, t_N$ equidistantly distributed with $h = \Delta t$
- sought are N piecewise spline representations $p_0(t), \dots, p_{N-1}(t)$ of $f(t)$
- let $p_s(t)$ be the sought n-th order spline representation of the segment $[t_s, t_{s+1}]$

$$p_s(t) = \sum_{i=0}^{n} a_{s,i} T_s^i(t), \quad T_s(t) = \frac{t - t_s}{h} \quad \Rightarrow \quad p_s^{(k)}(t) = \frac{1}{h^k} \sum_{i=k}^{n} \frac{i!}{(i-k)!} a_{s,i} T_s^{i-k}(t) \tag{1}$$

- in particular, the derivatives at joint knots read

$$p_s^{(k)}(t_s) = \frac{k!}{h^k} a_{s,k}, \quad p_s^{(k)}(t_{s+1}) = \frac{1}{h^k} \sum_{i=k}^{n} \frac{i!}{(i-k)!} a_{s,i} \tag{2}$$

Data approximation constraints

- interpolation of given data points up to the m-th order derivative yields $2(m+1)$ constraints

$$p_s^{(0)}(t_s) = x_s^{(0)}, \quad p_s^{(0)}(t_{s+1}) = x_{s+1}^{(0)}, \quad \dots, \quad p_s^{(m)}(t_s) = x_s^{(m)}, \quad p_s^{(m)}(t_{s+1}) = x_{s+1}^{(m)} \tag{3}$$

- expressed in matrix notation

$$\begin{pmatrix} x_s^{(0)} \\ \vdots \\ x_s^{(m)} \\ x_{s+1}^{(0)} \\ \vdots \\ x_{s+1}^{(m)} \end{pmatrix} = \begin{pmatrix} \frac{0!}{h^0} & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & \frac{m!}{h^m} & 0 & \dots & 0 \\ \frac{0!}{0!h^0} & \dots & \frac{m!}{m!h^0} & \frac{(m+1)!}{(m+1)!h^0} & \dots & \frac{n!}{n!h^0} \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \frac{m!}{0!h^m} & \frac{(m+1)!}{1!h^m} & \dots & \frac{n!}{(n-m)!h^m} \end{pmatrix} \begin{pmatrix} a_{s,0} \\ \vdots \\ a_{s,n} \end{pmatrix} \tag{4}$$

Spline continuity constraints

- continuity up to the k-th order derivative gives k constraints ($k = 0$ is trivially fulfilled)

$$p_s^{(1)}(t_s) = p_{s-1}^{(1)}(t_s), \quad \dots, \quad p_s^{(k)}(t_s) = p_{s-1}^{(k)}(t_s) \tag{5}$$

- expressed in matrix notation

$$\begin{pmatrix} \frac{1}{h^1} \sum_{i=1}^{n} \frac{i!}{(i-1)!} a_{s-1,i} \\ \vdots \\ \frac{1}{h^k} \sum_{i=k}^{n} \frac{i!}{(i-k)!} a_{s-1,i} \end{pmatrix} = \begin{pmatrix} \frac{1!}{h^1} & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & \frac{k!}{h^k} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_{s,0} \\ \vdots \\ a_{s,n} \end{pmatrix} \tag{6}$$