Contents

1. ABSTRACT

In the first part of this paper we shall study the underlying theory of function analysis in several complex variables. We will state generalisations of results in single variable complex analysis, but also illustrate fundamental differences between the two fields and explain why these differences arise. We will also introduce and motivate pseudoconvex domains which play a key role in the understanding of the subject.

In the second part of the paper we will consider the Catlin multitype [3] and the associated boundary system. We rewrite the steps outlined in Catlins origional paper in a (hopefully) more digestible way. We shall also give examples of the computation of the boundary system which are seemingly fairly scarce in published literature. We will finish by looking at some theorems considering the multitype in recent litrature

2. Preliminaries

We may identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ using the map

$$(z_1, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n) \mapsto (x_1, y_1, ..., x_n, y_n)$$

We note that $x_j = \frac{z_j + \bar{z}_j}{2}$ and $y_j = \frac{z_j - \bar{z}_j}{2i}$ hence given any $f(x_1, y_1, ..., x_n, y_n)$ we may write it as $f(z_1, z_2..., z_n)$.

We Now define:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial y_j}\right) \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right)$$

These definitions are forced by the relations:

$$\frac{\partial z_i}{\partial z_j} = \delta_{i,j} \qquad \frac{\partial \bar{z}_i}{\partial \bar{z}_j} = \delta_{i,j}$$

$$\frac{\partial \bar{z}_i}{\partial z_j} = 0 \qquad \frac{\partial z_i}{\partial \bar{z}_j} = 0$$

We also define:

$$dz_j = dx_j + idy_j \qquad d\bar{z}_j = dx_j - idy_j$$

Which gives the relations:

$$< dz_i, \frac{\partial}{\partial z_j} >= \delta_{i,j} \qquad < d\bar{z}_i, \frac{\partial}{\partial \bar{z}_j} >= \delta_{i,j}$$

$$< d\bar{z}_i, \frac{\partial}{\partial z_j} >= 0 \qquad < dz_i, \frac{\partial}{\partial \bar{z}_j} >= 0$$

$$dz_i \wedge d\bar{z}_i = 2idx_i \wedge dy_i$$

The volume form on $\mathbb{C}^n$ is derived from the standard volume form on $\mathbb{R}^{2n}$:

$$dV_n(z) = dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n = \frac{1}{(2i)^n} dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n$$

The open ball is defined as

$$B(z_0, r) = \{z \in \mathbb{C}^n \ : |z - z_0| < r\}$$

The open polydisk is defined as

$$D^n(z_0, r) = \{z \in \mathbb{C}^n : |z_j - z_{0\,j}| < r\}$$

We set $D(z_0, r) := D^1(z_0, r)$ and $D := D(0, 1) = B^1(0, 1)$.

A Domain will be any open connected set.

2.1. **Multi-indexes.** A *multi-index* α is an element of $\mathbb{Z}^n_{\geq 0}$, i.e $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_1, ..., \alpha_n$ are non-negative integers. If $z = (z_1, z_2, ..., z_n)$ then:

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} ... \, z_n^{\alpha_n}$$

$$\alpha! := \alpha_1!\alpha_2!...\alpha_n!$$

$$\left(\frac{\partial}{\partial z}\right)^\alpha := \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial z_2}\right)^{\alpha_2} ... \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$$

$$\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha := \left(\frac{\partial}{\partial \bar{z}_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial \bar{z}_2}\right)^{\alpha_2} ... \left(\frac{\partial}{\partial \bar{z}_n}\right)^{\alpha_n}$$

$$dz^\alpha := dz_{\alpha_1} \wedge dz_{\alpha_2} \wedge ... \wedge dz_{\alpha_n}$$

$$d\bar{z}^\alpha := d\bar{z}_{\alpha_1} \wedge d\bar{z}_{\alpha_2} \wedge ... \wedge d\bar{z}_{\alpha_n}$$

2.2. **Defining functions.** Let Ω be a domain in $\mathbb{R}^n$ and U be a neighbourhood of $\partial\Omega$. We say $\rho : U \to \mathbb{R}$ is a C^k defining function if $\rho \in C^k(U)$ and :

$$\forall x \in \Omega \cap U, \;\; \rho(x) < 0$$

$$\forall x \in \bar{\Omega}^c \cap U, \;\; \rho(x) > 0$$

$$\forall x \in \partial\Omega \cap U, \;\; \nabla\rho(x) \neq 0$$

If Ω has a C^k defining function then we say it has a C^k boundary.

3. Holomorphic functions

We give four definitions of holomorphic functions in several variables. All of them are inspired by their single variable counter parts and all of them are equivalent. We will assume all functions considered are locally integrable.

Let $\Omega \subset \mathbb{C}$ be a domain.

Definition 3.1. *A function $f : \Omega \to \mathbb{C}$ is holomorphic if it is holomorpic in each variable in our classical one variable sense. i.e* $\forall 0 \leq k \leq n \quad \forall z_1, ..., z_{k-1}, z_{k+1}, ..., z_n$

$$g_k(\zeta) = f(z_1, ..., z_{k-1}, \zeta, z_{k+1}, ..., z_n)$$

is holomorphic on

$$\{\zeta \in \mathbb{C} : ((z_1, ...z_{k-1}, \zeta, z_{k+1}, ...z_n) \in \Omega\}$$

Definition 3.2. *A function $f : \Omega \to \mathbb{C}$ that is continuously differentiable in each complex variable separately is holomorphic if it satisfies the cauchy riemann equations in each variable separately.*

Definition 3.3. *A function $f : \Omega \to \mathbb{C}$ is holomorphic if* $\forall z_0 \in \Omega\ \exists r > 0 : \bar{D}_n(z_0, r) \subset \Omega$ *and*

$$f(z) = \sum_{(0,..,0)\leq\alpha} a_\alpha (z - z_0)^\alpha \qquad \forall z \in D^n(z_0, r)$$

Where the series converges absolutely and uniformly.

Definition 3.4. *Suppose $f : \Omega \to \mathbb{C}$ is continuous in each variable separately and is locally bounded. Then f is holomorphic if* $\forall w \in \Omega \exists r > 0 : \bar{D}_n(z_0, r) \subset \Omega$ *and*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n|} .... \int_{|\zeta_1 - w_1|} \frac{f(\zeta_1, ..., \zeta_n)}{(\zeta_1 - w_1)...(\zeta_n - w_n)} d\zeta_1....d\zeta_n \qquad \forall z \in D^n(z_0, r)$$

It is easy to see all four are equivalent if we assume f is continuously differentiable, however it is quite hard to show that if f is only holomorphic in each variable then it is continuous. This fact is also quite surprising as its real variable equivalent is false.

4. Integral formulas

We can write the differential of any form $\omega = \sum_{\alpha,\beta} \omega_{\alpha,\beta}\, dz^\alpha \wedge d\bar{z}^\beta$ as:

$$d\omega = \sum_{i=1}^{n} \sum_{\alpha,\beta} \frac{\partial \omega_{\alpha,\beta}}{\partial z_i} dz_i \wedge dz^\alpha \wedge d\bar{z}^\beta + \sum_{i=1}^{n} \sum_{\alpha,\beta} \frac{\partial \omega_{\alpha,\beta}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz^\alpha \wedge d\bar{z}^\beta$$

This naturally leads to the definitions:

$$\partial \omega = \sum_{i=1}^{n} \sum_{\alpha,\beta} \frac{\partial \omega_{\alpha,\beta}}{\partial z_i} dz_i \wedge dz^\alpha \wedge d\bar{z}^\beta$$

$$\overline{\partial}\omega = \sum_{i=1}^{n}\sum_{\alpha,\beta}\frac{\partial\omega_{\alpha,\beta}}{\partial\overline{z}_i}d\bar{z}_i \wedge dz^{\alpha} \wedge d\overline{z}^{\beta}$$

giving us $d = \partial + \overline{\partial}$. Note that if f is A $(0,0)-$form, i.e. a function, we have:

$$\overline{\partial}\omega = 0 \iff \sum_{i=1}^{n}\frac{\partial f}{\partial\overline{z}_i}d\overline{z}_i = 0 \iff \forall i \quad \frac{\partial f}{\partial\overline{z}_i} = 0 \iff f \text{ is holomorphic}$$

It is natural, given the definitions of holomorphic functions, to ask whether or not there exists an equivalent formulation of the Cauchy-Green formula in several variables. There is. It is called the Bochner-Martinelli formula.[5]

Theorem 4.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^1 boundary and let $f \in \mathbb{C}^1(\overline{\Omega})$. Then $\forall z \in \Omega$ we have:*

$$f(z) = \frac{1}{nW(n)}\int_{\partial\Omega}\frac{f(\zeta)\eta(\bar{\zeta}-\bar{z})\wedge\omega(\zeta)}{|\zeta-z|^{2n}} - \frac{1}{nW(n)}\int_{\Omega}\frac{\bar{\partial}f(\zeta)}{|\zeta-z|^{2n}}\wedge\eta(\bar{\zeta}-\bar{z})\wedge\omega(\zeta)$$

where

$$\omega(z) := dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$$

$$\eta(z) := \sum_{j=1}^{n}(-1)^{j+1}z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n$$

and

$$W(n) := \int_{B(0,1)}\omega(\bar{z})\wedge\omega(z) = (-1)^{q(n)}\cdot(2i)^n\cdot\Big(\text{ volume of the unit ball in } \mathbb{C}^n \approx \mathbb{R}^{2n}\Big)$$

Note that when $n = 1$ we get the Cauchy-Green formula and also if f is holomorphic then $\overline{\partial}f = 0$ and we get

$$f(z) = \frac{1}{nW(n)}\int_{\partial\Omega}\frac{f(\zeta)\eta(\bar{\zeta}-\bar{z})\wedge\omega(\zeta)}{|\zeta-z|^{2n}}$$

which for $n = 1$ gives the regular Cauchy integral formula. As the kernel of the Cauchy integral is holomorphic when $n = 1$, we could use it to construct holomorphic functions with certain properties on $\Omega \subset \mathbb{C}$ by defining f on $\partial\Omega$. However in more than one variable the kernal is no longer necessarily holomorphic and so we lose this useful method for constructing holomorphic functions on $\Omega \subset \mathbb{C}^n$.

It would be useful to have some integral formulas that could be used to construct holomorphic functions. One such integral formula is given by the so called Bergman kernel. We now give a brief overview of its construction.

Definition 4.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We define the Bergman space to be*

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \left(\int_\Omega |f(z)|^2 dV(z) \right)^{1/2} \equiv \|f\|_{A^2(\Omega)} < \infty \right\}$$

It can be shown that $A^2(\Omega)$ is a Hilbert subspace of $L^2(\Omega)$ w.r.t the inner product

$$\langle f, g \rangle \equiv \int_\Omega f(z)\overline{g(z)} dV(z)$$

Lemma 4.1. [1] *For each fixed $z \in \Omega$, the functional*

$$\Phi_z : f \mapsto f(z), \quad f \in A^2(\Omega)$$

is a linear continuous functional on $A^2(\Omega)$.

We may now use the Rieze representation theorem to see that there exists $k_z \in A^2(\Omega)$ that represents Φ_z such that

$$f(z) = \langle f, k_z \rangle$$

We set $K(z, \zeta) = \overline{k_z(\zeta)}$ so that

$$f(z) = \int K(z, \zeta) f(\zeta) dV(\zeta), \quad \forall f \in A^2(\Omega)$$

We define $K(z, \zeta)$ to be the Bergman kernel.

Theorem 4.2. [1] *If Ω is a bounded domain in $\mathbb{C}^n$, then the mapping*

$$P : f \mapsto \int_\Omega K(\cdot, \zeta) f(\zeta) dV(\zeta)$$

Is the Hilbert space orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$.

This means we can feed this integral any $f \in L^2(\Omega)$ and it will return a holomorphic function $h \in A^2(\Omega)$. This would be quite useful however the this argument is non-constructive and it is quite hard to find the Bergman kernel for an arbitrary domain $\Omega \subset \mathbb{C}^n$. However Krantz[1] calculates a couple of simple examples:

Example 4.1. *The Bergman kernel on the unit ball $B \subset \mathbb{C}^n$ is*

$$K_B(z, \zeta) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \bar{\zeta})^{n+1}}$$

where $z \cdot \bar{\zeta} = z_1\bar{\zeta}_1 + z_2\bar{\zeta}_2 + \cdots + z_n\bar{\zeta}_n$

Example 4.2. *The Bergman kernel for the polydisc $D^n(0, 1) \subseteq \mathbb{C}^n$ is*

$$K(z, \zeta) = \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{\left(1 - z_j\bar{\zeta}_j\right)^2}$$

5. Domains of Holomorphy

Definition 5.1. *We call an open set $U \subset \mathbb{C}^n$ a domain of holomorphy if there **does not** exist open sets $U_1, U_2 \subset \mathbb{C}^n$ s.t:*

- *U_2 is connected.*
- *$U_2 \not\subset U$*
- *$U \cap U_2 \neq \emptyset$*
- *$U_1 \subset U \cap U_2$*
- *$\forall h \in \mathcal{O}(U) \quad \exists h_2 \in \mathcal{O}(U_2): \quad h(z) = h_2(z) \quad \forall z \in U_1$*

A simpler stricter definition would be to say: "$U \subset \mathbb{C}^n$ is a domain of holomorphy if there does not exist an open set U' s.t $U \subset U'$ and every holomorphic function on U can be analytically extended to a function on U'". Note however this definition would not allow for sets whose boundary intersects itself. For example if $U = \mathbb{C} - (-\infty, 0]$, it would not be sufficient to say: "U is not a domain of holomorphy as $ln: U \to \mathbb{C}$ cannot be analytically extended to a larger set".

Example 5.1. *Let $0 < \alpha < \beta$ and consider the domain $\Omega = D^2(0,\beta) - \bar{D}^2(0,\alpha) \subset \mathbb{C}^2$. Suppose $h \in \mathcal{O}(\Omega)$. Then for fixed z_1 such that $|z_1| < \beta$ we get the Laurent series expansion:*

$$h_{z_1}(z_2) := h(z_1, z_2) = \sum_{n=-\infty}^{+\infty} a_n(z_1) z_2^n$$

where the coefficients age given by:

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=\gamma} \frac{h(z_1,\zeta)}{\zeta^{n+1}} \, d\zeta$$

for any $\alpha < \gamma < \beta$.
We know $h(z_1,\zeta)$ is holomorphic $\forall \zeta: \ |\zeta| < \beta$ when $\alpha < z_1 < \beta$. Hence by Cauchy's theorem we have $a_n(z_1) = 0$ when $n < 0$ and $\alpha < |z_1| < \beta$. Hence by analytic extension we have $a_n(z_1) = 0$ when $n < 0$ and $|z_1| < \beta$.
We may conclude that:

$$h(z_1, z_2) = \sum_{n=0}^{+\infty} a_n(z_1) z_2^n$$

Which defines a holomorphic extension of h on all of $D^2(0,\beta)$. Hence we may conclude Ω is not a domain of holomorphy.

Example 5.2. *Let $U \subset \mathbb{C}$ a simply connected domain and suppose $U \neq \mathbb{C}$. Then by the Riemann mapping theorem there exists a biholomorphism from U the unit disk D. Hence if we prove the unit disk is a* *domain of holomorphy we will also prove U is a domain of holomorphy. Consider the function $f : D \to \mathbb{C}$:*

$$f(z) = \sum_{k=0}^{\infty} z^{k!}$$

We have that $(e^{\frac{2a\pi i}{b}})^{k!} = 1$ when $b \geq k$ where $a, b \in \mathbb{Z}$. We see that the singularities of f at are located densely on ∂D and therefore we may not analytically extend f beyond D. We conclude that D is a domain of holomorphy which implies U is also a domain of holomorphy.

6. Pluriharmonic functions

It is natural to ask what properties do the real and imaginary parts of holomorphic functions in several variables have. It is obvious from the first definition of holomorphic functions that they will be harmonic in each complex variable separately. We can however say more than this.

Definition 6.1. *Let $a, b \in \mathbb{C}^n$. We define the set*

$$\{a + b\zeta : \zeta \in \mathbb{C}\}$$

to be a complex line in $\mathbb{C}^n$.

A complex line is essentially an embedding of $\mathbb{C}$ into $\mathbb{C}^n$. It can be viewed as a plane in $\mathbb{R}^{2n}$ however it should be noted that not every plane in $\mathbb{R}^{2n}$ is a complex line.

We can see from the first definition of holomorphic functions that if f is holomorphic then any restriction of f to any complex line which is perpendicular to a complex axis will also be holomorphic. It turns out that this fact can be generalised.[1]

Theorem 6.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic. Then for every complex line $\ell = \{a + b\zeta : \zeta \in \mathbb{C}\} \subset \mathbb{C}^n$ we have*

$$g(\zeta) = f(a + b\zeta)$$

is also holomorphic in the single variables sense on the set
$\Omega_\ell = \{\zeta \in \mathbb{C} : a + b\zeta \in \Omega\}$

Note that if f is holomorphic on every complex line, then f is harmonic on every complex line. This leads us to the following definition.

Definition 6.2. *Let $\Omega \subset \mathbb{C}^n$. A C^2 function $f : \Omega \to \mathbb{C}$ is said to be pluriharmonic if for every complex line $\ell = \{a + b\zeta\}$ we have that*

$$g(\zeta) = f(a + b\zeta)$$

is harmonic on the set $\Omega_\ell = \{\zeta \in \mathbb{C} : a + b\zeta \in \Omega\}$.

Pluriharmonic functions in several variables play the role of harmonic functions in one variable, as illustrated by the following theorem.[1]

Theorem 6.2. *Let $D^n(P,r) \subset \mathbb{C}^n$ be a polydisc and assume that $f : D^n(P,r) \to \mathbb{R}$ is C^2. Then f is pluriharmonic on $D^n(P,r)$ if and only if f is the real part of a holomorphic function on $D^n(P,r)$.*

It would obviously be quite tricky to check that a function is harmonic on every complex line. The following theorem gives us an easier method of checking whether or not a function is pluriharmonic.[1]

Theorem 6.3. *Let $\Omega \subset \mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ be a C^2 function. Then*

$$f \text{ is pluriharmonic } \iff \partial\overline{\partial} f \equiv 0$$

Definition 6.3. *Let $\Omega \subset \mathbb{R}^n$. A function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be upper-semi continuous at $p \in \Omega$ if:*

$$\limsup_{x \to p} f(x) \leq f(p)$$

7. Convexity

Convexity and related notions play a key role in function theory of several complex variables. Here we give some results and proofs from Krantz[1] that have been rewritten to be more intuitive.

Definition 7.1. *A subset $S \subset \mathbb{R}^n$ is said to be geometrically convex if $\forall P, Q \in S$ we have $tP + (1-t)Q \in S \;\; \forall t \in [0,1]$*

Geometric convexity can be difficult to prove from the definition. It is of much benefit to develop an analytical definition of convexity. To do this we shall use the tangent space of the domain of interest.

Definition 7.2. *Let $\Omega \subset \mathbb{R}^n$, ρ be a C^1 defining function for Ω and $P \in \partial\Omega$. The tangent space of $\partial\Omega$ at P is defined as:*

$$T_P(\partial\Omega) := \{w \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j} w_j = 0\}$$

This definition is independent of the choice of ρ as:

$$w \in T_p(\partial\Omega) \iff w \perp \nabla\rho \iff w \perp \nu_P$$

Where ν_P is the normal vector to $\partial\Omega$ at P which is independent of choice of ρ.

Definition 7.3. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with a C^2 boundary and let ρ be a defining function for Ω. Then $\partial\Omega$ is said to be weakly convex at $P \in \partial\Omega$ if*

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0 \qquad \forall w \in T_P(\partial\Omega)$$

Definition 7.4. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with a C^2 boundary and let ρ be a defining function for Ω. Then $\partial\Omega$ is said to be strongly convex at $P \in \partial\Omega$ if*

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k > 0 \qquad \forall w \in T_P(\partial\Omega) \setminus \{0\}$$

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^n$ be strongly convex. Then there exists a defining function $\tilde{\rho}$ and a constant $C > 0$ s.t*

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) w_j w_k \geq C|w|^2 \qquad \forall w \in \mathbb{R}^n$$

Proof. The statement is obvious when $w = 0$ so we assume WLOG $w \neq 0$ We first prove the result for fixed P. Rearranging the statement we get:

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) \frac{w_j}{|w|} \frac{w_k}{|w|} \geq C \qquad \forall w \in \mathbb{R}^n \setminus \{0\}$$

which is equivalent to

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k \geq C \qquad \forall v \in \{w \in \mathbb{R}^n : |w| = 1\}$$

for some $C > 0$. This set is the unit sphere (which we will here denote as S) which is known to be compact in $\mathbb{R}^n$. Hence it suffices to show that there exists $\tilde{\rho}$ such that

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k > 0 \qquad \forall v \in S$$

. Let ρ be some defining function on Ω. We wish to construct a $\tilde{\rho}$ from ρ. One way of doing this is to let $\tilde{\rho} := f \circ \rho$ for some differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

- $f(0) = 0$
- $f'(0) \neq 0$
- $f(x) > 0 \qquad \forall x > 0$
- $f(x) < 0 \qquad \forall x < 0$

These properties ensure $\tilde{\rho}$ will also be a defining function of Ω. By the chain rule we get:

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k = \sum_{j,k} f''(0) \frac{\partial \rho}{\partial x_j}(P) \frac{\partial \rho}{\partial x_k}(P) v_j v_k + \sum_{j,k} f'(0) \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) v_j v_k$$

Suppose that $f'(0) = 1$ and set $f''(0) := \lambda > 0$. Then we have:

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k = \lambda \left(\sum_j \frac{\partial \rho}{\partial x_j}(P) v_j \right)^2 + \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) v_j v_k$$

Let

$$X := \{ v \in S : \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) v_j v_k \leq 0 \}$$

If $v \notin X$ then our expression is clearly positive. It remains to show the expression is positive when $v \in X$.
As Ω is strongly convex we have $\forall v \in X$ we have that

$$\sum_j \frac{\partial \rho}{\partial x_j}(P) v_j \neq 0 \implies \left(\sum_j \frac{\partial \rho}{\partial x_j}(P) v_j \right)^2 > 0$$

Hence as X is compact $\exists C_1 > 0$ s.t:

$$\left(\sum_j \frac{\partial \rho}{\partial x_j}(P) v_j \right)^2 \geq C_1 \qquad \forall v \in X$$

Again as X is compact we also have

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k \geq C_2$$

for some $C_2 < 0$. Hence we have

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) v_j v_k \geq \lambda C_1 + C_2$$

If we choose $\lambda > -\frac{C_2}{C_1}$ then we get $C := \lambda C_1 + C_2 > 0$. This almost proves the statement for fixed P however we still need to find a function f that satisfies all of our desired properties. There are many that we could choose but a simple one is :

$$f(x) = \frac{e^{\lambda x} - 1}{\lambda}$$

Now we note that all of our estimates hold in a neighbourhood of P hence as $\partial\Omega$ is compact we can choose a C that works $\forall P \in \partial\Omega$. $\square$

Proposition 7.1. *Ω strongly convex $\implies \Omega$ geometrically convex.*

Proof. Let $S = \{(P_1, P_2) : \forall t \in (0,1)\ tP_1 + (1-t)P_2 \in \Omega\}$.
Ω is assumed to be a domain $\implies$ Ω is connected $\implies$ $\Omega \times \Omega$ is connected $\implies$ If $S \subset \Omega \times \Omega$ is non-empty and both open and closed then $S = \Omega \times \Omega$ which will prove the result.
Let $\tilde{\rho}$ be defined as in the lemma. Then note that:

$$S = \{(P_1, P_2) : \forall t \in (0,1)\ \tilde{\rho}(tP_1 + (1-t)P_2) < 0\}$$

S is clearly open and non-empty.
Suppose $(P_1, P_2) \in \bar{S} \setminus S$. Then by the continuity of $\tilde{\rho}$ we have:

$$\tilde{\rho}(tP_1 + (1-t)P_2) \leq 0 \qquad \forall t \in (0,1)$$

and as $(P_1, P_2) \notin S$ we also have:

$$\exists t_0 \in (0,1) \text{ s.t } \tilde{\rho}(t_0 P_1 + (1-t_0)P_2) \geq 0$$

Hence we may conclude that $f(t) = \tilde{\rho}(tP_1 + (1-t)P_2)$ has a maximum of 0 at $t_0 \implies \frac{d^2 f}{dt^2}(t_0) \leq 0$

$$0 \geq \frac{d^2 f}{dt^2}(t_0) = \sum_{j,k} \frac{\partial^2 \tilde{\rho}(P_0)}{\partial x_j \partial x_k}(P_1 - P_2)_j (P_1 - P_2)_k > 0$$

Where the last inequality is trueby the lemma as $P_0 := t_0 P_1 + (1 - t_0)P_2 \in \partial\Omega$ as $\tilde{\rho}(t_0 P_1 + (1-t_0)P_2) = 0$. Hence we have a contradiction so we may conclude $\bar{S} = S$ i.e S is closed. ☐

Theorem 7.1. [1] *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with C^2 boundary. Then:*

$$\Omega \text{ weakly convex} \iff \Omega \text{ geometrically convex}$$

Note that this means strong convexity implies both weak and geometric convexity. The converse is not true as if we consider the square with rounded corners, with a suitable choice of coordinates, the bottom edge can be give by $y = 0$. One possible defining function in a neighbourhood of the bottom edge is $\rho(x, y) = -y$. The real Hessian is clearly zero on the bottom edge and hence our domain is not strongly convex even though it is clearly geometrically convex.

8. Pseudoconvexity

We wish to construct a notion of convexity that is invariant under biholomorphisms. If we write our real tangent space in terms of complex coordinates we get:

$$T_P(\partial\Omega) := \{v \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial \rho}{\partial x_j} v_j = 0\} = \{w \in \mathbb{C}^n : Re\left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j\right) = 0\}$$

This is clearly not invariant under multiplication by i. We would like our definition of the complex tangent space to be a subspace of $T_P(\partial\Omega)$ that is invariant under multiplication by i. The largest subspace that satisfies this property is

$$\mathcal{T}_P(\partial\Omega) := \{w \in \mathbb{C}^n : \left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j} w_j\right) = 0\}$$

Example 8.1. *Consider the tangent spaces of the unit disc $\Omega = D \subset \mathbb{C}$ at $P = 1$. Then $\rho(z) = |z|^2$ and $\frac{\partial\rho}{\partial z} = \bar{z} \implies \frac{\partial\rho}{\partial z}(z) = \bar{z} \implies \frac{\partial\rho}{\partial z}(1) = 1$ hence:*

$$T_P(\partial\Omega) = \{w \in \mathbb{C} : Re(w) = 0\} = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$
$$\mathcal{T}_P(\partial\Omega) = \{0\}$$

It is worth noting that for any domain $\Omega \subset \mathbb{C}$ with C^1 boundary we have $\mathcal{T}_P(\partial\Omega) = 0$. This result is key in explaining why complex analysis in one variable is fundamentally so different to complex analysis in multiple variables.

Example 8.2. *Consider the tangent spaces of the unit ball $\Omega = B(0, 1) \subset \mathbb{C}^2$ at $P = (1, 0)$. Then $\rho(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1$. and $\frac{\partial\rho}{\partial z_i} = \bar{z}_i \implies \frac{\partial\rho}{\partial z_1}(P) = 1$ and $\frac{\partial\rho}{\partial z_2}(P) = 0$ hence:*

$$T_P(\partial\Omega) = \{(w_1, w_2) \in \mathbb{C}^2 : Re(w_1) = 0\} = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : x_1 = 0\}$$

$$\mathcal{T}_P(\partial\Omega) = \{(w_1, w_2) \in \mathbb{C}^2 : w_1 = 0\} = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : x_1 = 0, y_1 = 0\}$$

Let $\Omega \subset \mathbb{C}^n$ be a domain with C^1 boundary. Suppose $\Phi : \Omega \to \Phi(\Omega)$ $\Omega \subset \mathbb{C}^n$ is a biholomorphism. Then:

$$(w_1, ..., w_n) \in \mathcal{T}_P(\partial\Omega) \iff \left(\sum_{j=1}^{n} \frac{\partial \Phi_1}{\partial z_j} w_j, ..., \sum_{j=1}^{n} \frac{\partial \Phi_n}{\partial z_j} w_j\right) := (w'_1, ..., w'_n) \in \mathcal{T}_{\Phi(P)}(\partial(\Phi(\Omega)))$$

We now look at our convexity condition in complex coordinates. It is easy to show that:

$$0 \leq \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P)\zeta_j\zeta_k + 2\sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial y_k}(P)\zeta_j\eta_k + \sum_{j,k} \frac{\partial^2 \rho}{\partial y_j \partial y_k}(P)\eta_j\eta_k$$

becomes:

$$0 \leq Re\left(\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) w_j w_k\right) + \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k$$

Where $w_j = \zeta_j + i\eta_j$.
The first of these terms does not transform nicely under a biholomorphism, however the second is invariant under biholomorphism leading us to the following definition:

Definition 8.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^2 boundary and let $P \in \partial\Omega$. We say $\partial\Omega$ is Levi pseudoconvex at P if:*

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0 \qquad \forall w \in \mathcal{T}_P(\partial\Omega)$$

We say it is strongly Levi pseudoconvex if the inequality is strict when $w \neq 0$. We say Ω is Levi pseudoconvex if $\partial\Omega$ is Levi pseudoconvex at all points.

Example 8.3. *Consider $B^n(0,1)$.*
The defining function is given by:

$$\rho(z_1, ..., z_n) = \sum_{i=1}^{n} |z_i|^2 - 1$$

hence

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k = \sum_{i=1}^{n} |w_i|^2 > 0$$

when $w \neq 0$. Hence we see the unit ball is strongly pseudoconvex.

The main reason why Pseudoconvexity is of such interest in the function theory of several complex variables, is illustrated by the following theorem.[1]

Theorem 8.1. *Let Ω be a domain with C^2 boundary. Then:*

$$\Omega \textit{ is a domain of holomorphy} \iff \Omega \textit{ is levi pseudoconvex}$$

This gives us a relatively easy way of deducing weather or not a domain is a domain of holomorphy. For example as we have shown the unit ball in $\mathbb{C}^n$ is strongly pseudoconvex and hence we know it is also a domain of holomorphy. Note also that given any domain $\Omega \subset \mathbb{C}$ we have $\forall\ P \in \partial\Omega,\ \mathcal{T}_P(\partial\Omega) = \{0\}$, hence Ω is Levi pseudoconvex so we may conclude that every domain $\Omega \subset \mathbb{C}$ is a domain of holomorphy.

The forward direction of this theorem was relatively easy to prove. The reverse direction was an open problem for some time until it was proven by Kiyoshi Oka in 1958[2]. This theorem immediately proved a huge amount about domains of holomorphy as much more is known about pseudoconvex domains.

Corollary 8.1. [1] *Let* $\{\Omega_\alpha\}_{\alpha\in A}$ *be domains of holomorphy in* $\mathbb{C}^n$*. If* $\Omega \equiv \cap_{\alpha\in A}\Omega_\alpha$ *is open, then* Ω *is a domain of holomorphy.*

Corollary 8.2. [1] *If* Ω *is geometrically convex, then* Ω *is a domain of holomorphy.*

It should be noted that the previous two corollaries only requires the forward direction of the theorem and hence did not require the solving of the Levi problem. The next theorem (known as the Behnke-Stein theorem[1]) requires the use of both directions of the equivalence.

Theorem 8.2. [6] *Let* $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ *be domains of holomorphy. Then* $\Omega \equiv \cup_j\Omega_j$ *is a domain of holomorphy.*

The next two theorems [1] will give a somewhat more geometric classification of pseudoconvex domains and hence a classification of domains of holomorphy.

Theorem 8.3. *Let* $\Omega \subset \mathbb{C}^n$ *be a pseudoconvex domain. Then*

$$\Omega = \cup_j \, \Omega_j$$

where each $\Omega_j \subset \mathbb{C}^n$ *are bounded strongly pseudoconvex domains such that* $\Omega_j \subset \Omega_{j+1}$*.*

Example 8.4. *Suppose*

$$\Omega = \{(z_1, ..., z_n) \in \mathbb{C}^n : \; Re\{z_1\} > 0\}$$

Then a defining function of Ω *is*

$$\rho(z_1, z_2...z_n) = -2Re\{z_1\} = -z_1 + -\overline{z}_1$$

hence given $P \in \partial\Omega$ *we have*

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k = 0 \quad \forall w \in \mathcal{T}_P(\partial\Omega)$$

so we can conclude Ω *is pseudoconvex. Now let*

$$\Omega_k = B((k, 0, 0..., 0), k)$$

which we know to be strongly pseudoconvex and by the triangle inequality we have

$$\Omega_k \subset \Omega_{k+1}$$

It is clear that $\cup_k \Omega_k \subset \Omega$ *so it remains to show* $\Omega \subset \cup_k \Omega_k$*. i.e. given* $(z_1, ...z_n) = (x_1 + iy_1, ..., x_n + y_n)$ *with* $x_1 > 0$ *we need to find* k *such that*

$$(z_1, ..., z_n) \in \Omega_k \iff |(z_1 - k, ..., z_n)| < k$$

$$\iff \sqrt{(x_1 - k)^2 + y_1^2 + x_2^2 + y_2^2 + ... + x_n^2 + y_n^2} < k$$

$$\iff x_1^2 - 2kx_1 + k^2 + y_1^2 + x_2^2 + y_2^2 + ... + x_n^2 + y_n^2 < k^2$$

$$\iff \frac{x_1^2 + y_1^2 + ...x_2^2 + y_2^2}{2x_1} < k$$

where we have used the assumption $x_1 > 0$. We hence can choose k to be large enough so that $(z_1, ...z_n) \in \Omega_k$ and hence we have

$$\Omega = \cup_k B((k, 0, 0..., 0), k)$$

Let $\mathcal{H}$ denote the set of all domains of holomorphy. Using the previous two theorems we can deduce that for any domain $\Omega \in \mathcal{H}$ we have that

$$\Omega = \cup_j \Omega_j$$

where $\Omega_j \subset \Omega_{j+1}$ and the boundary of each Ω_j can be paved by a finite number of open sets $U_{j,k}$ such that $U_{j,k} \cap \Omega_j$ is biholomorphic to a convex domain. Here "paving the boundary" means that $\partial\Omega_j \subset \cup_k U_{j,k}$.

9. The $\overline{\partial}$-Neumann problem

Hence we have for any (p, q)-form ω, we have

$$0 = d^2\omega = \partial^2\omega + (\partial\bar{\partial} + \bar{\partial}\partial)\omega + \bar{\partial}^2\omega$$

Now we know $\partial^2\omega$ is a $(p + 2, q)-$form, $\bar{\partial}^2\omega$ is a $(p, q + 2)-$form and $(\partial\bar{\partial} + \bar{\partial}\partial)\omega$ is a $(p + 1, q + 1)-$form. Hence we may conclude:

- $\partial^2\omega = 0$
- $(\partial\bar{\partial} + \bar{\partial}\partial)\omega = 0$
- $\bar{\partial}^2\omega = 0$

The $\overline{\partial}$-Neumann problem is (roughly speaking): "Given a $(p, q + 1)$-form ω defined on a domain Ω we want to find a (p, q)-form f such that $\overline{\partial} f = \omega$." There is the extra caveat that we require our solution to be orthogonal to the space of L^2-Holomorphic functions. The reason for this requirement is rather technical, however our goal here is to mainly motivate the definition of the multitype and understanding this technicallity would lead us too far afield.

It is of interest to know about the regularity of solutions to the $\overline{\partial}$-Neumann problem. (Regularity basically means the solution is somewhat well behaved) David Catlin showed that for pseudoconvex domains regularity of the solution can be guaranteed everywhere on the domain if we have the existence of certain invariants.[7] One of these invariants is the so called multitype.

10. The Multitype

The multitype is an n-tuple of rational numbers that characterises the vanishing of the defining function In order to define the multitype we will need several preliminary definitions.

Definition 10.1. *A multi-weight $\mu = (\mu_1, ...\mu_n)$ is an n-tuple of real numbers such that $\mu_1 = 1 \geq \mu_2... \geq \mu_n \geq 0$.*

Definition 10.2. *A monomial $a_\alpha z^\alpha \overline{z}^\beta$ is said to have weight w w.r.t the multi-weight μ if $w = (\alpha + \beta|\mu)$*

Definition 10.3. *Let μ be a multi-weight. A polynomial $f(z_1, ..., z_n) = \sum_{\alpha,\beta} z^\alpha \overline{z}^\beta$ is said to be weighted homogenous with weight w if*

$$f(\mu_1 z_1, ..., \mu_n z_n) = w f(z_1, ..., z_n) \qquad \forall z \in \mathbb{C}^n$$

$\therefore$

Definition 10.4. *An inverse weight is defined as any n-tuple $\lambda = (\lambda_1, ...\lambda_n)$ s.t $\lambda = \mu^{-1} := (\mu_1^{-1}, ...\mu_n^{-1})$ for some multi-weight μ.*

Definition 10.5. *An inverse weight is called admissible if $\forall i : 0 \leq i \leq n$ either*
$\lambda_i = \infty$, or $\exists\, a_1, a_2..., a_i \in \mathbb{Z}_{\geq 0}$ with $a_i > 0$ such that $\sum_{j=1}^{i} a_i \lambda_i^{-1} = 1$.

$$\Gamma_n := \{\lambda \in \mathbb{R}^n : \lambda \textit{ is an admissable inverse weight}\}$$

Definition 10.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth boundary with defining function r. Then $\Lambda = (\lambda_1, ...\lambda_n) \in \Gamma_n$ is said to be distinguished at $z_0 \in \partial\Omega$ if there exists a holomorphic change of coordinates such that z_0 is mapped to the origin and*

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} < 1 \implies D^\alpha \bar{D}^{\bar{\beta}} r = 0$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial z^\alpha}$ and $D^{\bar{\beta}} = \dfrac{\partial^{|\beta|}}{\partial \overline{z}^\beta}$

$$\Gamma(z_0) := \{\Lambda = (\lambda_1, ...\lambda_n) \in \Gamma_n : \Lambda \textit{ is distinguished at } z_0\}$$

Definition 10.7. *The multitype of $\partial\Omega$ at z_0 is defined as:*

$$\mathcal{M}(z_0) = \max \Gamma(z_0)$$

where $\Gamma(z_0)$ is ordered lexicographically.

Example 10.1. *Consider the sum of squares domain $\Omega \subset \mathbb{C}^{n+1}$ given by the defining function*

$$\rho(z_1, ..., z_{n+1}) = -2Rez_1 + \sum_{k=1}^{n} |z_{k+1}|^{2k}$$

We wish to find $\mathcal{M}(0) := (\lambda_1, ..., \lambda_{n+1})$. Note that any mixed partial derivatives will automatically be zero. Hence it is sufficient to find the largest such $(\lambda_1, ..., \lambda_{n+1})$ such that:

$$\frac{a+b}{\lambda_j} < 1 \implies \frac{\partial^{a+b}\rho}{\partial \overline{z}_j^a \partial z_j^b}(0) = 0$$

It is easy to see that if either $a < j$ or $b < j$ then

$$\frac{\partial^{a+b}\rho}{\partial \overline{z}_j^a \partial z_j^b}(0) = 0$$

and if $a = b = i$ we get

$$\frac{\partial^{a+b}\rho}{\partial \overline{z}_j^a \partial z_j^b}(0) = k!^2 \neq 0$$

hence

$$\frac{a+b}{2j} < 1 \implies \frac{\partial^{a+b}\rho}{\partial \overline{z}_j^a \partial z_j^b}(0) = 0$$

however this statement is not true if we replace $2j$ with $2j+1$. We conclude that $\mathcal{M}(0) \geq (1, 2, 4, ..., 2n)$.

In the previous example it is tempting to say that $M(0) = (1, 2, 4, ..., 2n)$, however we must remember that although this is the smallest admissible weight in the original coordinate system, there may be a holomorphic change of coordinates with a smaller admissible weight.
An alternative way to calculate the multitype for pseudoconvex domains is to find Catlin's commutator multitype.
Catlin's multitype is a tuple of integers that arises from studying a geometric object called the boundary system. The boundary system is a collection of vector fields and funtions obtained by differentiating the defining function by lists of these vector fields.

11. Vector Fields

Any vector field L on $\Omega \subset \mathbb{C}^n$ can be written as:

$$L = \sum_{i=1}^{n} a_i(z_1, ..., z_n)\frac{\partial}{\partial z_i} + \sum_{i=1}^{n} b_i(z_1, ..., z_n)\frac{\partial}{\partial \bar{z}_i}$$

where a_i and b_i are smooth $\forall\, i$. *(Not necessarily holomorphic)*

We say that L is a vector field of type $(1,0)$ if $b_i = 0\ \forall\, i$.
We say that L is a vector field of type $(0,1)$ if $a_i = 0\ \forall\, i$.

Definition 11.1. *Let $\mathcal{L} = \{L_1, L_2..., L_k\}$ be a list of vector fields and let ω be a 1-form. We define for $k \geq 3$:*

$$\mathcal{L}\omega := L_1 L_2 ... L_{k-2}\omega([L_{k-1}, L_k])$$

When $k = 2$ we define:

$$\mathcal{L}\omega := \omega([L_1, L_2]$$

Finally when $k = 1$ we define:

$$\mathcal{L}\omega := \omega(L_1)$$

12. Calculating the Boundary System

We now reproduce the method Catlin used to calculate the boundary system and commutator multitype in his original paper.[3]

STEP 1:
We first let $r_1 = r$. We then pick L_1 to be any vector field such that $L_1 r = 1$ on all of $\partial\Omega$. We know such a vector field exists as $dr \neq 0$ on $\partial\Omega$. We then set $c_1 = 1$.

STEP 2:
Let p be the number of non-zero eigenvalues of the Levi form at z_0. Then set $c_2 = c_3 = ... = c_p = c_{p+1} = 2$ and pick $L_2, ..., L_{p+1}$ be vector fields of type $(1,0)$ such that $L_i r \equiv 0$ and the $p \times p$ Hermitian matrix $\partial\overline{\partial}r(L_i, \overline{L}_j)(z_0)$ is non singular for $2 \leq i, j \leq p+1$.

STEP 3:
Let $T^{1,0}_{p+1}$ denote the set of $(1,0)$-vector fields such that $L(r) = 0$ and $\partial\overline{\partial}r(L, \overline{L}_i) = 0$ for $2 \leq i \leq p+1$. We will set c_{p+2} to be the length of the smallest list $\mathcal{L} = \{L^1, ...L^l\}$ such that the following holds:

- $\exists\, L \in T^{1,0}_{p+1}$ *s.t* $\forall\, 1 \leq i \leq l$, *we have* $L^i = L$ *or* $L^i = \overline{L}$
- $\mathcal{L}\partial r(z_0) \neq 0$

If no such list exists we let $c_{p+2} = c_{p+3} = ... = c_n = \infty$ and we are finished. However if we can find lists with such properties, we then pick one of minimal length and denote it $\mathcal{L}_{p+2}$. We also set $L_{p+2} \in T^{1,0}_{p+1}$ be the vector field such that for each $L^i \in \mathcal{L}_{p+2}$ we have $L^i = L_{p+2}$ or $L^i = \overline{L}_{p+2}$.
Let $\mathcal{L}'_{p+2} = \{L^2, L^3, ..., L^n\}$.

We now define the functions:

$$f(z) = Re\{\mathcal{L}'_{p+2}\partial r\}$$

$$g(z) = Im\{\mathcal{L}'_{p+2}\partial r\}$$

Now we know $L^1(f+ig)(z_0) \neq 0$. Hence we can pick either $r_{p+2} = f$ or $r_{p+2} = g$ so that the condition $L^1 r_{p+2}(z_0) \neq 0$ is satisfied.

INDUCTIVE STEP:
Assume for some integer ν such that $p+2 \leq \nu \leq n$, we have constructed finite positive numbers $c_1, c_2 ... c_\nu$, functions $r_1, r_{p+2}, r_{p+3}, ..., r_\nu$ and vector fields $L_1, L_2 ... L_\nu$ such that for each k, $p+2 \leq k < \nu$ the following properties hold:

(1) $L_k(r_k)(z_0) \neq 0$
(2) $L_k \in T^{1,0}_k$

Where $T^{1,0}_k$ denotes the set of all vector fields L of type $(1,0)$ such that $\partial\overline{\partial} r(L, \overline{L}_j) \equiv 0$ when $2 \leq j \leq p+1$ and $L(r_i) = 0$ when $i = 1$ or $p+2 \leq i < k$.
We now consider lists of vector fields $\mathcal{L} = \{L^1, ..., L^l\}$ such that for each L^i we have $L^i \in \{L_{p+2}, \overline{L}_{p+2}, ..., L_\nu, \overline{L}_\nu, L, \overline{L}\}$ where $L \in T^{1,0}_{\nu+1}$. Let l_i denote the total number of occurrences of L_i and $\overline{L}_i$ in $\mathcal{L}$. Similarly we let $l_{\nu+1}$ denote the total number of occurrences of L and $\overline{L}$ in $\mathcal{L}$.
We will say $\mathcal{L}$ is ordered if:

- $L^1, L^2, ..., L^{l_{\nu+1}} \in \{L, \overline{L}\}$
- $L^{l_{\nu+1}+1}, L^{l_{\nu+1}+2}, ..., L^{l_{\nu+1}+l_\nu} \in \{L_\nu, \overline{L}_\nu\}$
- $L^{l_{\nu+1}+l_\nu+1}, L^{l_{\nu+1}+l_\nu+2}, ..., L^{l_{\nu+1}+l_\nu+l_{\nu-1}} \in \{L_{\nu-1}, \overline{L}_{\nu-1}\}$

$\vdots$

- $L^{l_{\nu+1}+l_\nu+...+l_2+1}, L^{l_{\nu+1}+l_\nu+...+l_2+2}, ..., L^{l_{\nu+1}+l_\nu+l_{\nu-1}+...l_2+l_1} \in \{L_1, \overline{L}_1\}$

That is, the first $l_{\nu+1}$ are either L or $\overline{L}$, the next l_ν vector fields are either L_ν or $\overline{L}_\nu$ and so on...

We define $\mathcal{L}$ to be $(\nu+1)$-*admissable* if:

(1) $l_{\nu+1} > 0$
(2) $\displaystyle\sum_{i=p+2}^{\nu} \frac{l_i}{c_i} < 1$

We must now make one more assumption on the vector fields $L_{p+2}, L_{p+3}, ..., L_\mu$: If $\mathcal{L}$ is an ordered list of the vector fields $L_{p+2}, \overline{L}_{p+2}, ..., L_\mu, \overline{L_\mu}$ and $\displaystyle\sum_{i=p+2}^{\nu} \frac{l_i}{c_i} < 1$ then $\mathcal{L}\partial r(z_0) = 0$.

We now set:

$$A_{\nu+1} := \{\mathcal{L} : \mathcal{L} \text{ is ordered and } (\nu+1)\text{-admissable and } \mathcal{L}\partial r(z_0) \neq 0\}$$

Given $\mathcal{L} \in A_{\nu+1}$ we will set $c(\mathcal{L})$ to be the solution of:

$$\sum_{i=p+2}^{\nu} \frac{l_i}{c_i} + \frac{l_{\nu+1}}{c(\mathcal{L})} = 1$$

We now have the machinery required to find $c_{\nu+1}$. If $A_{\nu+1} = \emptyset$ we set: $c_{\nu+1} = c_{\nu+2} = ... = c_n$ and we are finished our construction of the Catlin multitype. However if $A_{\nu+1}$ is non-empty we set

$$c_{\nu+1} := \inf\{c(\mathcal{L}) : \mathcal{L} \in A_{\nu+1}\}$$

Pick $\mathcal{L}_{\nu+1} = \{L^1, ..., L^l\}$ to be a list such that $\mathcal{L}_{\nu+1} \in A_{\nu+1}$, $\mathcal{L}_{\nu+1}\partial r(z_0) \neq 0$ and $c_{\nu+1} = c(\mathcal{L}_{\nu+1})$.
We set $L_{\nu+1} := L^1$ or $L_{\nu+1} := \overline{L}^1$ so that the condition $L_{\nu+1} \in T^{1,0}_{\nu+1}$ is satisfied.
Let $\mathcal{L}'_{\nu+1} := \{L^2, ..., L^l\}$ and set r_{p+2} to be either:

$$r_{\nu+1} := Re\{\mathcal{L}'_{\nu+1}\partial r\}$$

or

$$r_{\nu+1} := Im\{\mathcal{L}'_{\nu+1}\partial r\}$$

so that $L_{\nu+1}r_{\nu+1} \neq 0$.
We repeat this procedure until we have obtained all entries of the Catlin mutitype.
The set:

$$\mathcal{B}_\nu = \{r_1, r_{p+2}, r_{p+3}, ..., r_\nu; L_2, L_3, ..., L_\nu\}$$

is called the boundary system at the point z_0 of rank p and codimension $n - \nu$. We call $L_2, ..., L_\nu$ special vector fields.

Remark 12.1. *All the assumptions we made at the beginning of the inductive step will be satisfied if the proceeding steps are followed correctly.*

There were several choices made in the construction of the boundary system and there is no reason for us to expect that the Catlin multitype is independent of these choices. However the next theorem from Catlins paper will resolve this issue in the case that our domain is pseudoconvex.

Theorem 12.1 (3). *Let Ω be a domain given by the defining function r. Suppose Ω is pseudoconvex in a neighbourhood of a point z_0 and* *let $\mathcal{B}_\nu$ be a boundary system at the point z_0. If $\mathcal{C}^\nu(z_0)$ is the Catlin multitype corresponding to $\mathcal{B}_\nu$, then*

$$\mathcal{C}^\nu(z_0) = \mathcal{M}^\nu(z_0)$$

where $M^\nu(z_0)$ is the first ν entries of $M(z_0)$.

This theorem shows that if our domain is pseudoconvex in a neighbourhood of the point of interest, then the Catlin multitype is independent of the choices we make for our special vector fields $L_2, ..., L_\nu$ and our functions $r_{p+2}, ..., r_\nu$. The theorem also says that if the Catlin multitype is finite then $\mathcal{C}(z_0) = \mathcal{M}(z_0)$.

13. Examples of boundary systems

Example 13.1. *Let ρ be the defining function of the strongly pseudoconvex domain Ω. Then the rank of the Levi form is $n-1$ at any point $z_0 \in \partial\Omega$ and hence the multitype of the domain at any point $z_0 \in \Omega$ is:*

$$\mathcal{C}(z_0) = (1, 2, 2, ..., 2)$$

Let us consider a specific example of a strongly pseudoconvex domain and calculate its boundary system.

Example 13.2. *Let the domain $\Omega \subset \mathbb{C}^{n+1}$ be given by the defining function*

$$r(z_1, ..., z_{n+1}) = -2Re\{z_{n+1}\} + \sum_{i=1}^{n} |z_i|^2$$

STEP 1:
We first set $r_1 := r$.
It is then easy to see that if we set

$$L_1 = -\frac{\partial}{\partial z_{n+1}}$$

we will satisfy the condition $L_1 r_1 = 1$ on all of $\partial\Omega$.
STEP 2:
As Ω is strongly pseudoconvex we have that the rank of the Levi form will be n. We hence need to find $L_2, ..., L_{n+1}$ such that $L_i r \equiv 0$ on and $\partial\overline{\partial}r(L_i, \overline{L}_j)(z_0)$ is non singular for $2 \leq i, j \leq n+1$. It is easy to see that the simplest vector fields that satisfy $L_i r \equiv 0$ are

$$L_i = \frac{\partial}{\partial z_i} + \overline{z}_i \frac{\partial}{\partial z_{n+1}}$$

as

$$L_i r = \left(\frac{\partial}{\partial z_i} + \overline{z}_i \frac{\partial}{\partial z_{n+1}}\right)\left(-2Re\{z_{n+1}\} + \sum_{i=1}^{n} |z_i|^2\right) = \overline{z}_i + \overline{z}_i(-1) \equiv 0$$

We now need to check the second condition. We must first compute the $(1,1)$-form $\partial\overline{\partial}r$. We have:

$$\overline{\partial}r = \sum_{i=1}^{n+1} \frac{\partial r}{\partial \bar{z}_i} d\overline{z}_i = -d\overline{z}_{n+1} + \sum_{i=1}^{n} z_i d\overline{z}_i$$

hence

$$\partial\overline{\partial}r = \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

From this we can see that:

$$(\partial\overline{\partial}r(L_i, L_j))_{i,j} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

which is clearly non-singular.
Hence we may conclude that the boundary system of the domain is

$$\mathcal{B}(0) = \left\{ r, \frac{\partial}{\partial z_2} + \overline{z}_2 \frac{\partial}{\partial z_{n+1}}, \dots\dots, \frac{\partial}{\partial z_n} + \overline{z}_n \frac{\partial}{\partial z_{n+1}} \right\}$$

We now consider a more difficult example in order to fully illustrate each step of the computation for both the boundary system and the Catlin multitype. Step by step examples of the computation of boundary systems are seemingly quite rare in published literature, so this will hopefully serve as a novelty to any reader attempting to understand boundary systems for the first time. However first we state a lemma which will be useful in our calculations.

Lemma 13.1. *Let r be a defining function of $\Omega \subset \mathbb{C}^n$. Suppose X and Y are vector fields on $\mathbb{C}^n$ of types $(1,0)$ and $(0,1)$ respectively. Then if $X(r) = 0$ we have:*

$$\partial r([X, Y]) = \partial\overline{\partial}r(X, Y)$$

Proof. We will reduce both sides of the equality. In the standard coordinates of $\mathbb{C}^n$ we have

$$X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial z_j}$$

and

$$Y = \sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k}$$

then

$$[X,Y] = XY - YX = \sum_{j=1}^{n} X_j \frac{\partial}{\partial z_j} \sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k} - \sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k} \sum_{j=1}^{n} X_j \frac{\partial}{\partial z_j}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} X_j \frac{\partial Y_k}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} + X_j Y_k \frac{\partial^2}{\partial z_j \partial \overline{z}_k} - Y_k \frac{\partial X_j}{\partial \bar{z}_k} \frac{\partial}{\partial z_j} - Y_k X_j \frac{\partial^2}{\partial \overline{z}_k \partial z_j}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} X_j \frac{\partial Y_k}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} - Y_k \frac{\partial X_j}{\partial \bar{z}_k} \frac{\partial}{\partial z_j}$$

as the mixed partial derivatives commute. Hence we see that the right hand side of the statement becomes

$$\partial r([X,Y]) = -\sum_{j=1}^{n} \sum_{k=1}^{n} Y_k \frac{\partial X_j}{\partial \bar{z}_k} \frac{\partial r}{\partial z_j}$$

Turning our attention to the left hand side we have

$$\overline{\partial} r = \sum_{k=1}^{n} \frac{\partial r}{\partial \overline{z}_k} d\overline{z}_k = 0$$

hence

$$\partial \overline{\partial} r = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k$$

so the right hand side of the statement becomes

$$\partial \overline{\partial} r(X,Y) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} X_j Y_k$$

In order to finish the proof we will consider the difference:

$$\partial \overline{\partial} r(X,Y) - \partial r([X,Y]) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} X_j Y_k + Y_k \frac{\partial X_j}{\partial \bar{z}_k} \frac{\partial r}{\partial z_j}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} Y_k \left(X_j \frac{\partial^2 r}{\partial \overline{z}_k \partial z_j} + \frac{\partial X_j}{\partial \bar{z}_k} \frac{\partial r}{\partial z_j} \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k} \left(X_j \frac{\partial r}{\partial z_j} \right)$$

$$\sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k} \left(\sum_{j=1}^{n} X_j \frac{\partial r}{\partial z_j} \right) = \sum_{k=1}^{n} Y_k \frac{\partial}{\partial \bar{z}_k} X(r) = 0$$

as $X(r) = 0$. $\square$

Note that the lists of vector fields we shall be working with will often satisfy the conditions of the lemma. In this case we shall have

$$\mathcal{L}\partial r = L_1L_2...L_{k-2}\partial r([L_{k-1}, L_k]) = L_1L_2...L_{k-2}\partial\overline{\partial}r(L_{k-1}, L_k)$$

Example 13.3. *Let $\Omega \subset \mathbb{C}^{n+1}$ be the domain given by the defining function*

$$r(z_1, ..., z_{n+1}) = -2Re\{z_1\} + \sum_{k=2}^{n+1} |z_k|^{2(k-1)}$$

We aim to calculate the Catlin multitype at 0, $\mathcal{C}(0) = (c_1, ..., c_{n+1})$ and the corresponding boundary system.

STEP 1:
We first set $r_1 := r$ and

$$L_1 := -\frac{\partial}{\partial z_1}$$

as this will satisfy the condition $L_1 r_1 = 1$ on all of $\partial\Omega$. Finally we set $c_1 := 1$ as always.

STEP 2:
The Levi form of r is given by:

$$\left(\frac{\partial^2 r}{\partial z_i \partial\overline{z}_j}(0)\right)_{1\le i,j\le n+1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The Levi form has one non zero eigenvalue, we therefore need to find one vector field L_2 such that $L_2 r_1 \equiv 0$ and $\partial\overline{\partial}r(L_2, \overline{L}_2)(0) \neq 0$. We first need to calculate $\partial\overline{\partial}r$:

$$\overline{\partial}r = \sum_{j=1}^{n+1} \frac{\partial r}{\partial\overline{z}_j} d\overline{z}_j = -d\overline{z}_1 + \sum_{k=2}^{n+1} (k-1) z_k |z_k|^{2(k-2)} d\overline{z}_k$$

$$\partial\overline{\partial}r = \sum_{k=2}^{n+1} (k-1)^2 |z_k|^{2(k-2)} dz_k \wedge d\overline{z}_k$$

The simplest vector field satisfying the desired properties is

$$L_2 = \overline{z}_2 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$$

A rudimentary computation shows that:

$$L_2 r = \overline{z}_2(-1) + \overline{z}_2 = 0$$

and

$$\partial\overline{\partial}r(L_2,\overline{L}_2)(0)=1\neq 0$$

As the Levi form has just one eigenvalue, we only get $c_2=2$ in the multitype.

STEP 3:
Suppose $L\in T_3^{1,0}$. We may then write L as:

$$L=\sum_{j=1}^{n+1}v_j\frac{\partial}{\partial z_j}$$

where the v_j's are smooth functions of $z_1,...,z_{n+1}$.
The condition $L(r)=0$ then gives

$$-v_1+\sum_{k=2}^{n+1}(k-1)v_k\overline{z}_k|z_k|^{2(k-2)}=0$$

and the condition $\partial\overline{\partial}r(L,L_2)=0$ gives us

$$0=\partial\overline{\partial}r(L,\overline{L}_2)=\sum_{k=2}^{n+1}(k-1)^2|z_k|^{2(k-2)}dz_k\wedge d\overline{z}_k\left(\sum_{j=1}^{n+1}v_j\frac{\partial}{\partial z_j},\ z_2\frac{\partial}{\partial\bar{z}_1}+\frac{\partial}{\partial\bar{z}_2}\right)=v_2$$

We see that $L\in T_{1,0}^3\iff L$ is of the form

$$L=\left(\sum_{k=3}^{n+1}(k-1)v_k\overline{z}_k|z_k|^{2(k-2)}\right)\frac{\partial}{\partial z_1}+\sum_{j=3}^{n+1}v_j\frac{\partial}{\partial z_j}$$

We now need to consider lists of L and $\overline{L}$ acting on ∂r. First we try $\mathcal{L}=\{L,\overline{L}\}$:

$$\mathcal{L}\partial r=\partial r([L,\overline{L}])=\partial\overline{\partial}r(L,\overline{L})$$

$$=\sum_{k=2}^{n+1}(k-1)^2|z_k|^{2(k-2)}dz_k\wedge d\overline{z}_k\left(v_1\frac{\partial}{\partial z_1}+\sum_{j=3}^{n+1}v_j\frac{\partial}{\partial z_j},\overline{v}_1\frac{\partial}{\partial\bar{z}_1}+\sum_{j=3}^{n+1}\overline{v}_j\frac{\partial}{\partial\bar{z}_j}\right)$$

$$=\sum_{k=3}^{n+1}(k-1)^2|v_k|^2|z_k|^{2(k-2)}$$

We see that $\mathcal{L}\partial r(0)=0$. By the properties of the commutator it is obvious that for any list $\mathcal{L}$ of length two we have $\mathcal{L}\partial r=0$. We wish to find an extension to the list $\{L,\overline{L}\}$ such that $\mathcal{L}\partial r(0)\neq 0$. By inspection of the formula obtained for $\{L,\overline{L}\}\partial r$ we can see that the shortest list of vector fields $\mathcal{L}$, that will give $\mathcal{L}\partial r(0)\neq 0$ must differentiate $\partial\overline{\partial}r(L,\overline{L})$ once more w.r.t z_3 and once more w.r.t $\overline{z}_3$. Hence our desired list has *length at least four. We get the simplest* $L \in T_3^{1,0}$ *that differentiates w.r.t* z_3 *if we set* $v_3 = 1$ *and* $v_4 = v_5 = ... = v_{n+1} = 0$*, giving*

$$L = 2\overline{z}_3|z_3|^2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}$$

If we then set $\mathcal{L} = \{L, \overline{L}, L, \overline{L}\}$ *we see*

$$\mathcal{L}\partial r(0) = \left(2\overline{z}_3|z_3|^2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}\right)\left(2z_3|z_3|^2\frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_3}\right)\left(4|z_3|^2\right)\Big|^{z=0} = 4 \neq 0$$

We may hence set $L_3 := L$ *and conclude* $c_3 = 4$*. It remains to find* r_3*. We have* $\mathcal{L}' = \{\overline{L}, L, \overline{L}\}$*, so*

$$\mathcal{L}'\partial r = \left(2z_3|z_3|^2\frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_3}\right)\left(4|z_3|^2\right) = 4z_3 = 4x_3 + i4y_3$$

If we choose

$$r_3 = 4x_3 = 2z_3 + 2\overline{z}_3$$

we will satisfy the required condition

$$Lr_3 = 2 \neq 0$$

STEP 4:

Rather than doing out the tedious induction, it is more illustrative to calculate c_4 *and observe the obvious pattern.*
Suppose $L \in T_4^{1,0}$*. Note that* $T_4^{1,0} = \{L \in T_3^{1,0} : L(r_3) = 0\}$ *hence we may write*

$$L = \sum_{j=1}^{n+1} v_j \frac{\partial}{\partial z_j}$$

From the definition of $T_k^{1,0}$ *we can see that,* $L \in T_4^{1,0} \iff L \in T_3^{1,0}$ *and* $L(r_3) = 0$*.*
We know that:

$$L \in T_3^{1,0} \iff L = \left(\sum_{k=3}^{n+1}(k-1)v_k\overline{z}_k|z_k|^{2(k-2)}\right)\frac{\partial}{\partial z_1} + \sum_{j=3}^{n+1} v_j\frac{\partial}{\partial z_j}$$

for some smooth functions $v_3, v_2, ..., v_{n+1}$*.*
The additional assumption $L(r_3) = 0$ *gives*

$$0 = L(r_3) = \left(\sum_{j=1}^{n+1} v_j\frac{\partial}{\partial z_j}\right)(2z_3 + 2\overline{z}_3) = 2v_3 \iff v_3 = 0$$

We conclude that

$$L \in T_4^{1,0} \iff L = \left(\sum_{k=4}^{n+1}(k-1)v_k\overline{z}_k|z_k|^{2(k-2)}\right)\frac{\partial}{\partial z_1} + \sum_{j=4}^{n+1} v_j\frac{\partial}{\partial z_j}$$

We now need to study which lists are 4*-admissible, i.e., lists of the form* $\mathcal{L} = \{L^1, ..., L^{l_4}, L^{l_4+1}, ..., L^{l_4+l_3}\}$ *where:*

- $L^1, ..., L^{l_4} \in \{L, \overline{L}\}$
- $L^{l_4+1}, ..., L^{l_4+l_3} \in \{L_3, \overline{L}_3\}$
- $l_4 > 0$

However it is easy to see that if $\overline{L}$ *is not made purely of either* L *and* $\overline{L}$ *or* L_3 *and* $\overline{L}_3$ *then* $\mathcal{L}\partial r(0) = 0$*. This is because* L_3 *annihilates terms that don't depend on* z_3 *and* L *has no derivatives w.r.t* z_3*. We conclude that if* $\mathcal{L}\partial r(0) \neq 0$ *then either* $l_3 = 0$ *or* $l_4 = 0$ *but as* $l_4 > 0$ *it must be that* $l_3 = 0$*. Therefore if* $\mathcal{L} \in A_4$ *then* $l_3 = 0$*. We need to minimise the solution to*

$$\frac{l_3}{c_3} + \frac{l_4}{c(\mathcal{L})} = 1$$

for $\mathcal{L} \in A_4$*.*
However as $l_3 = 0$ *when* $\mathcal{L} \in A_4$ *we get:*

$$c(\mathcal{L}) = l_4$$

We can now see our task is simply to find the length of the smallest list in A_4*.*
We first find:

$$\partial\overline{\partial}r(L, \overline{L}) = \partial\overline{\partial}r\left(v_1\frac{\partial}{\partial z_1} + \sum_{j=4}^{n+1} v_j\frac{\partial}{\partial z_j}, \overline{v}_1\frac{\partial}{\partial \bar{z}_1} + \sum_{j=4}^{n+1} \overline{v}_j\frac{\partial}{\partial \bar{z}_j}\right)$$

$$= \sum_{k=4}^{n+1} (k-1)^2|v_k|^2|z_k|^{2(k-2)}$$

By inspection of the formula obtained for $\{L, \overline{L}\}\partial r$ *we can see that the shortest list of vector fields* $\mathcal{L}$*, that will give* $\mathcal{L}\partial r(0) \neq 0$ *must differentiate* $\partial\overline{\partial}r(L, \overline{L})$ *twice more w.r.t* z_4 *and twice more w.r.t* $\overline{z}_4$*. Hence our desired list has length at least six. We get the simplest* $L \in T_4^{0,1}$ *that differentiates w.r.t* z_4 *if we set* $v_4 = 1$ *and* $v_5 = v_6 = ... = v_{n+1} = 0$*, giving*

$$L = 3\overline{z}_4|z_4|^4\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_4}$$

If we let $\mathcal{L} = \{L, \overline{L}, L, \overline{L}, L, \overline{L}\}$*, a simple computation gives*

$$\mathcal{L}\partial r(0) = (4-1)^2(4-2)^2(4-3)^2 = 18 \neq 0$$

We conclude $\mathcal{L} \in A_4$ *and* $c(\mathcal{L}) = 6$*. Therefore we can set* $L_4 := L$ *and we see that* $c_4 = 6$*.*

It remains to calculate r_4. $\mathcal{L}' = \{\overline{L}_4, L_4, \overline{L}_4, L_4, \overline{L}_4\}$ *so*

$$\mathcal{L}'\partial r = \overline{L}_4 L_4 \overline{L}_4 \partial\overline{\partial} r(L_4, \overline{L}_4)$$

$$= \left(3z_4|z_4|^4 \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_4}\right)\left(3\overline{z}_4|z_4|^4 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_4}\right)\left(3z_4|z_4|^4 \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_4}\right)(9|z_4|^4)$$

$$= 36z_4 = 36x_4 + i36y_4$$

Hence we may set $r_4 := 36x_4 = 18z_4 + 18\overline{z}_4$ *as it will satisfy:*

$$L_4 r_4 = 18 \neq 0$$

STEP k:

If we keep repeating the inductive step it is clear we will similarly get

$$L_k = (k-1)\overline{z}_k|z_k|^{2(k-2)} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_k}$$

$$r_k = \frac{(k-1)!^2}{2} z_k + \frac{(k-1)!^2}{2} \overline{z}_k$$

$$c_k = 2(k-1)$$

We conclude that Catlin's commutator multitype at 0 *is given by*

$$\mathcal{C}(0) = (1, 2, 4, ..., 2n)$$

and the boundary system is

$$B(0) = \left\{ \begin{array}{c} r, r_3 = 2z_3 + 2\overline{z}_3, r_4 = 18z_4 + 18\overline{z}_4, ..., r_{n+1} = \frac{n!^2}{2} z_{n+1} + \frac{n!^2}{2}\overline{z}_{n+1}, \\ \\ \overline{z}_2 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}, 2\overline{z}_3|z_3|^2 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}, ..., n\overline{z}_{n+1}|z_{n+1}|^{2(n-1)} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_{n+1}} \end{array} \right\}$$

If we look at the Levi form of r *we see:*

$$\sum_{j,k} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k = \sum_{k=2}^{n+1} (k-1)^2 |P_k|^{2(k-2)} |w_k|^2 \geq 0$$

Hence we may conclude that Ω *is pseudoconvex and*

$$\mathcal{M}(0) = (1, 2, 4, ..., 2n)$$

14. Further theorems on the mutitype

Definition 14.1. *The D'Angelo* [8] *q-type of the point z_0, denoted $\Delta_q(\partial\Omega, z_0)$, is a measure of the maximum order of contact of $q-$dimensional complex analytic varieties with $\partial\Omega$. A $q-$dimensional complex analytic variety is the zero locus of $n-q$ holomorphic functions.*

Definition 14.2. *Let Ω be a domain with C^1 boundary and let $M \subset \partial\Omega$ be a manifold. The holomorphic dimension of M is defined to be the maximum dimension of the kernal of the Levi form considered over all $P \in M$*

Catlin used his commutator multitype to prove the following properties of the mulltitype on pseudoconvex domains.

Theorem 14.1. [3] *Let Ω be a pseudoconvex domain with smooth boundary. Then $\mathcal{M}(z)$ has the following properties:*

- *$\mathcal{M}(z)$ is upper semi-continuous w.r.t lexicographic ordering, i.e*

$$\forall z_0 \in \partial\Omega \;\; \exists\, U \in N(z_0) \;\, s.t \;\; z \in U \cap \partial\Omega \implies \mathcal{M}(z) \leq \mathcal{M}(z_0)$$

- *Suppose that $\mathcal{M}(z_0) = (m_1, m_2..., m_n)$ and that $m_{n-q} < \infty$. Then $\exists\, U \in N(z_0)$ and a submanifold $M \subset U \cap \partial\Omega$ of holomorphic dimension at most q such that $z_0 \in M$ and:*

$$\{z \in U \cap \partial\Omega : \mathcal{M}(z) = \mathcal{M}(z_0)\} \subset M$$

- *If $\mathcal{M}(z_0) = (m_1, ...m_n)$, then:*

$$m_{n+1-q} \leq \Delta_q(\partial\Omega, z_0)$$

where $\Delta_q(\partial\Omega, z_0)$ is the D'Angelo q-type of the point z_0.

.

This theorem shows how the multitype contains a lot of geometric information about the boundary of our domain. Specifically the last part essentially shows that the the multitype acts as a lower bound for how much our boundary "looks like" certain complex analytic varieties of various dimensions.

During our main example we were always able to chose the simple special vector fields. A natural question is if our special vector fields were more complicated, does there exeist a change of coordinates to make them simpler? The following theorem answers this question for some of our vector fields.

Theorem 14.2. [4] *Let M be a pseudoconvex smoth real hypersurface in $\mathbb{C}^n$ with $0 \in M$ and of Catlin multitype $\Lambda = (\lambda_1, ..., \lambda_n)$. Let p be* *the rank of the Levi form at* 0 *and let* $q > 0$ *be such that*

$$\lambda_{p+2} = \lambda_{p+3} = ... = \lambda_{p+q+1} < \lambda_{p+q+2} < \infty$$

Then given any boundary system

$$\mathcal{B}_n(0) = \{r_1, r_{p+2}, ..., r_n, L_2, ..., L_n\}$$

there exists a holomorphic change of coordinates such that the boundary system becomes

$$\tilde{\mathcal{B}}_n(0) = \{\tilde{r}_1, \tilde{r}_{p+2}, ..., \tilde{r}_n, \tilde{L}_2, ..., \tilde{L}_n\}$$

where

$$\tilde{r}_j = Re\{z_j\} + o(\lambda_j^{-1})$$
$$\tilde{L}_j = \frac{\partial}{\partial z_j} + o(\lambda_j^{-1})$$

Theorem 14.3. [4] *Let* M *be a pseudoconvex smoth real hypersurface in* $\mathbb{C}^n$ *with* $0 \in M$ *and of Catlin multitype* $\Lambda = (\lambda_1, ..., \lambda_n)$ *with* $\lambda_n < +\infty$.
Then there exists a holomorphic change of coordinates at 0 preserving the multitype so that the defining function for M in the new coordinates is given by

$$r = -2Re(z_1) + p(z_2, ..., z_n, \bar{z}_2, ...\bar{z}_n) + o_{\Lambda^{-1}}(1)$$

where $p(z_2, ..., z_n, \bar{z}_2, ...\bar{z}_n)$ *is a polynomial of weight* 1 *w.r.t* Λ^{-1} *which contains the sum of squares*

$$|z_2|^{2k_{22}} + |z_2|^{2k_{32}}|z_3|^{2k_{33}} + ... + |z_2|^{2k_{n2}}|z_3|^{2k_{n3}}...|z_n|^{2k_{nn}}$$

Note that $o_{\Lambda^{-1}}(1)$ *are terms of weight larger than* 1 *w.r.t* Λ^{-1}. *We also have that* $k_{jj} > 0$ *for all* j *and the (total) degree of* p *in each* $(z_j, \overline{z}_j)$ *is not greater than* k_{jj}.

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