Guide: Dr. Karuna Kalita
By:
Alok Singh, 08010304
Murukesh Mohanan, 08010329

True Modal Control

Overview

- Any physical system can be represented in a n-DOF spring-mass-damper form using the lumped parameter strategy.
- However, this yields a set of coupled dynamic equations, which are not amenable to control.
- Using Structure Preserving Equivalences (SPEs), we can decouple this system, albeit in a $2n$-dimensional matrix space.
- These SPEs are used to obtain stable filters which provide us with n decoupled equations, which are more suited for analysis.
- Since the stable filters and SPEs are real, they can be used in controllers which do not support complex arithmetic.

Modal Control

- The physical variables are linear combinations of independent quantities associated with the system, called *natural modes*, which are manifested in the decoupled equations.
- As such, the number of physical variables may be more than the degrees of freedom.
- Hence, controlling the physical variables themselves may be wasteful and counterproductive.
- Since the decoupled equations are easier to analyse, controlling the modes in an appropriate combination is more likely to obtain the desired state, at the same time revealing side-effects.

Modal Control

- Modal Control is not a particularly new idea, having its roots in research dating back the 1960s.
- It developed independently in Chemical Engineering and Structural Dynamics and the advances in the two fields merged in the late 80s and early 90s.
- So far, except in special cases, Modal Control is used in a limited form.
- Usually, higher modes are ignored altogether.
- Sometimes, traditional methods do not yield decoupled equations.

- The Lumped Parameter Strategy produces a set of ODEs of the form:

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{0}$$

- It is the damping matrix, C, that determines the whether traditional methods will decouple the system.
- Generally, the damping involved is approximated to be *classical*, so that a transformation which diagonalizes M and K diagonalizes C.
- The modes of the system are obtained from the associated eigenvaue problem.
- In Gyroscopic systems, the damping skew-symmteric, and is not diagonalized by traditional methods.
- Ignoring off-diagonal terms is tantamount to ignoring the gyroscopic nature of the system itself.

Structure Preserving Equivalences

- We linearize the set of n second-order ODEs to a set of 2n first order ODEs. First, we define the 2nx2n matrices A, B, D (called Lancaster Augmented Matrices or LAMs):

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix}$$

- Then we can write the ODEs as a system of first-orfer ODEs in different ways, one such representation being:

$$\mathbf{A} \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix} + \mathbf{B} \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{f}$$

- Where traditional methods consider the eigenvalues of K and M, we use A and B, to involve the full system.

- The matrices X and Y, composed of the right and left eigenvectors of the system diagonalize A and B:

$$\mathbf{Y}^T\mathbf{A}\mathbf{X} = \mathbf{\Lambda}$$
$$\mathbf{Y}^T\mathbf{B}\mathbf{X} = \mathbf{I}_{2n}$$

- However, we need transformations which diagonalize K, M and C. This can be done through and A and B, via SPEs. If Π_L and Π_R are SPEs, then

$$\mathbf{\Pi}_L^T\mathbf{A}\mathbf{\Pi}_R = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_D \end{bmatrix}$$
$$\mathbf{\Pi}_L^T\mathbf{B}\mathbf{\Pi}_R = \begin{bmatrix} \mathbf{0} & \mathbf{M}_D \\ \mathbf{M}_D & \mathbf{C}_D \end{bmatrix}$$

- The transformed matrices retain the block-structure of A and B, but the blocks can be made diagonal.

- The SPEs can be thought of as operating on A and B in steps, with each step consisting of an operation, such as converting the complex matrices to real ones.
- Each step consists of mutliplication by an appropriately constructed matrix.
- Then the SPEs are simply products of the matrices used in each step:

$$\mathbf{\Pi}_L = \mathbf{X}^{[1]}\mathbf{P}\mathbf{J}_L\mathbf{F}\mathbf{\Gamma}$$
$$\mathbf{\Pi}_R = \mathbf{X}^{[1]}\mathbf{P}\mathbf{J}_R\mathbf{F}\mathbf{\Gamma}$$

Here, we have:

- A Permutation Matrix

$$\mathbf{P} = \mathbf{I}_{2n}\left[\mathfrak{C}^{+}\mathfrak{T}^{+}\mathfrak{C}^{-}\mathfrak{T}^{-}\right]$$

- The Realization Matrices for the left and right eigenvectors

$$\mathbf{J}_L = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} & \frac{-\iota}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{+} & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} & \frac{\iota}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}^{-} \end{bmatrix}$$

$$\mathbf{J}_R = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} & \frac{-\iota}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{+} & \mathbf{0} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} & \frac{\iota}{\sqrt{2}}\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}^{-} \end{bmatrix}$$

- The Elimination Matrix

$$\mathbf{F} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} & \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}^{+} & \mathbf{0} & \mathbf{I}_{n-p} \\ \mathbf{I}_p & \mathbf{0} & \mathbf{\Phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-p} & \mathbf{0} & \mathbf{\Psi}^{-} \end{bmatrix}$$

- The Scaling Matrix

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta} \end{bmatrix}$$

Need and defintion of stable filter

Any dynamic system can be expressed by **lumped parameters** as:

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{C}_0 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_0 & 0 \\ 0 & -\mathbf{M}_0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$$

The Laplace transform of this equation gives:

$$\left\{ \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_0 \end{bmatrix} - s \begin{bmatrix} -\mathbf{C}_0 & -\mathbf{M}_0 \\ -\mathbf{M}_0 & \mathbf{0} \end{bmatrix} \right\} \begin{bmatrix} \underline{\mathbf{q}} \\ \underline{\mathbf{q}}s \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} \\ \mathbf{0} \end{bmatrix}$$

We define the following:

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_R & \mathbf{Q}_R \\ \mathbf{R}_R & \mathbf{S}_R \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}} \\ \underline{\mathbf{q}}s \end{bmatrix}$$

or

$$\begin{bmatrix} \underline{\mathbf{q}} \\ \underline{\mathbf{q}}s \end{bmatrix} = \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

more,

$$\begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} \\ \underline{\mathbf{f}} \end{bmatrix}$$

&

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{j} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{f}} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{W}_R$ etc are obtained from transformation, now when we apply these conditions on our equation system obtained after laplace-transform, we get,

$$\left\{\begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} - s\begin{bmatrix} -\mathbf{C}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix}\right\}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \\ \mathbf{j} \end{bmatrix}$$

Premulptiplying this equation with

$$\begin{bmatrix} \mathbf{I} & s\mathbf{I} \end{bmatrix}$$

we get,

$$\begin{bmatrix} \mathbf{K}_D & -s\mathbf{M}_D \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} s\mathbf{C}_D + s^2\mathbf{M}_D & s\mathbf{M}_D \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = [\mathbf{h} + s\mathbf{j}]$$

which gives;

$$\mathbf{K}_D\mathbf{u} + s\mathbf{C}_D\mathbf{u} + s^2\mathbf{M}_D\mathbf{u} = \mathbf{h} + s\mathbf{j}$$

$$\mathbf{h} + s\mathbf{j} = ({\mathbf{W}_L}^T + s{\mathbf{X}_L}^T)\underline{\mathbf{f}}$$

& hence;

$$(\mathbf{K}_D + s\mathbf{C}_D + s^2\mathbf{M}_D)(\mathbf{P}_R + s\mathbf{Q}_R)\underline{\mathbf{q}} = ({\mathbf{W}_L}^T + s{\mathbf{X}_L}^T)\underline{\mathbf{f}}$$

further more,

$$({\mathbf{W}_L}^T + s{\mathbf{X}_L}^T)^{-1}(\mathbf{K}_D + s\mathbf{C}_D + s^2\mathbf{M}_D)(\mathbf{P}_R + s\mathbf{Q}_R)\underline{\mathbf{q}} = (\mathbf{K}_0 + s\mathbf{C}_0 + s^2\mathbf{M}_0)\underline{\mathbf{q}}$$

which implies,

$$(\mathbf{K}_D + s\mathbf{C}_D + s^2\mathbf{M}_D)(\mathbf{P}_R + s\mathbf{Q}_R) = ({\mathbf{W}_L}^T + s{\mathbf{X}_L}^T)(\mathbf{K}_0 + s\mathbf{C}_0 + s^2\mathbf{M}_0)$$

assuming, $\mathbf{P}_R$ & $\mathbf{Q}_R$ are represented by $\mathbf{V}_0$ & $\mathbf{V}_1$ respectively, similarly ${\mathbf{W}_L}^T$ & ${\mathbf{X}_L}^T$ are represented by $\mathbf{U}_0$ & $\mathbf{U}_1$ respectively.Then,

$$(\mathbf{K}_D + s\mathbf{C}_D + s^2\mathbf{M}_D)(\mathbf{V}_0 + s\mathbf{V}_1) = (\mathbf{U}_0 + s\mathbf{U}_1)(\mathbf{K}_0 + s\mathbf{C}_0 + s^2\mathbf{M}_0)$$

where, $(\mathbf{U}_0 + \mathrm{s}\mathbf{U}_1)$ & $(\mathbf{V}_0 + \mathrm{s}\mathbf{V}_1)$ are the filters, & from this proof it is clear that these filters are obtained from Structure Preserving Equivalences (SPEs), which are not unique & since we need to find stable filters, so we need to check whether the poles of obtained filters are stable or not to check their stability.

As SPEs are not unique so filters defined on them are also not unique, so as to produce stable filters, in order to find stable diagonalized system, we use a method called Automorphic SPE.

Automorphic SPE

Here we use two diagonal n x n matrices F_L and G_L:

$$\begin{bmatrix} \mathbf{P}_L & \mathbf{Q}_L \\ \mathbf{R}_L & \mathbf{S}_L \end{bmatrix} = \begin{bmatrix} (\mathbf{F}_L - \frac{1}{2}\mathbf{G}_L\mathbf{C}_D) & -\mathbf{G}_L\mathbf{M}_D \\ \mathbf{G}_L\mathbf{K}_D & (\mathbf{F}_L + \frac{1}{2}\mathbf{G}_L\mathbf{C}_D) \end{bmatrix}$$

For automorphic SPEs, F_L and G_L must satisfy the condition

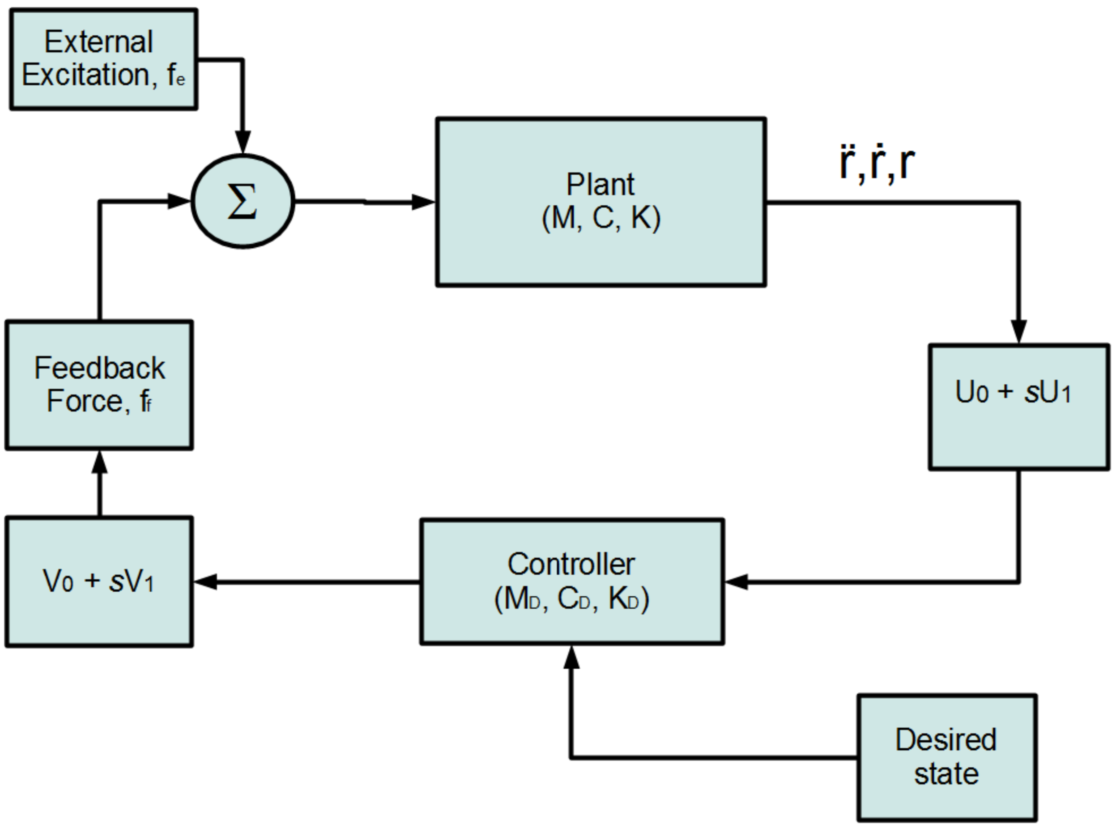
$$\mathbf{F}_L{\mathbf{G}_R}^T + \mathbf{G}_L{\mathbf{F}_R}^T = \mathbf{0}$$

For each diagonal element,

$$\begin{bmatrix} (f + \frac{1}{2}gd) & gm \\ -gk & (f - \frac{1}{2}gd) \end{bmatrix} \begin{bmatrix} (f - \frac{1}{2}gd) & -gm \\ gk & (f + \frac{1}{2}gd) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Stable filters: its use and application in modal control

After finding out diagonalized LAMs, using SPE, We need filters mainly to convert physical variables and force component into diagonalised form, so as to generate decoupled dynamic equation.



An example

$$K = \begin{bmatrix} -0.2111 & -0.6014 & -3.6084 & 1.2366 \\ 1.1902 & 0.5512 & 3.3072 & -0.6313 \\ -1.1162 & -1.0998 & -6.5988 & -2.3252 \\ 0.6353 & 0.086 & 0.516 & -1.2316 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.7621 & 0.4447 & 0.7382 & 0.9169 \\ 0.4565 & 0.6154 & 0.1763 & 0.4103 \\ 0.0185 & 0.7919 & 0.4057 & 0.8936 \\ 0.8214 & 0.9218 & 0.9355 & 0.0579 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.371 & -1.0226 & 0.3155 & 0.5045 \\ 0.7283 & 1.0378 & 1.5532 & 1.8645 \\ 2.1122 & -0.3898 & 0.7079 & -0.3398 \\ -1.3573 & -1.3813 & 1.9574 & -1.1398 \end{bmatrix}$$

For these K, M and C matrices the SPEs and stable filters have been determined.

Note that K is not a full-rank matrix, and hence an eigenvalue of the system is 0.

- The eigenvalues are 0, -0.63733 ± 0.81085ı, 0.72065 ± 1.6661ı, -2.2081, -2.6647 and 4.5719. Since K is rank deficient by 1, one eigenvalue is 0.
- Π_L and Π_R are as below:

	-4.5361	-4.7581	1.1273	1.7235	0.361	2.6338	0.54805	0.376
	-1.3873	4.4512	0.51076	0.74126	-2.1229	4.1793	-1.7778	-5.1083
	3.6224	4.4636	-0.1825	-2.4663	-7.9988	-9.8561	0.40299	5.4459
Pi_l =	-2.9268	-5.256	0.4947	2.592	8.071	11.164	1.6839	-6.1805
	-0.33939	-2.4761	-0.51525	-0.35349	-4.1035	-1.6019	1.7841	2.1741
	0.64427	-1.2683	0.53952	1.5503	-0.45874	2.6231	1.2884	2.9757
	0	0	0	0	0	0	0	0
	0.6625	0.91635	0.13822	-0.50732	-1.6633	-3.5083	0.75832	1.6245

	0.54109	-0.98012	0.20876	0.20962	-1.3144	0.77435	0	0.00054078
	-1.2813	-0.48042	-0.17383	-0.084144	1.2858	-1.0662	0	0.021006
	0.74982	-0.6004	-0.30476	-0.26981	0.8346	0.35366	0	0.035304
Pi_r =	0.66366	0.70695	-0.02413	-0.02239	0.57687	-0.11158	0	-0.017289
	1.398	-2.5516	-0.46097	0.0065882	2.2165	0.13595	0	0.21065
	-1.3676	3.5132	0.38383	0.2559	-2.9202	-2.0171	0	-0.044082
	-0.88774	-1.1654	0.67296	0.4301	-0.31402	-0.090663	0	-0.20248
	-0.6136	0.36767	0.053283	-0.21062	-0.071663	0.54613	0	-0.055363

$$U\_0 = \begin{matrix} 0.38973 & 0.13388 & 0.48205 & -0.40774 \\ -0.28175 & -0.16811 & -0.8201 & 0.20615 \\ 2.8082 & 71.308 & -65.927 & -59.113 \\ 149.81 & -427.29 & 1321 & 210.08 \end{matrix}$$

$$U\_1 = \begin{matrix} -0.32406 & 0.061639 & -0.10838 & -0.39636 \\ -0.20641 & -0.26196 & -0.30202 & -0.12607 \\ -205.21 & -138.02 & -97.962 & -309.11 \\ 1798.1 & 1385.7 & 658.98 & 2920.6 \end{matrix}$$

$$V\_0 = \begin{matrix} -0.056884 & -2.3347 & -0.62926 & -0.4960 \\ -1.4852 & 4.4839 & 0.27321 & 0.12621 \\ 112.54 & 138.67 & -5.6699 & -76.621 \\ -1215.5 & -1994 & 32.736 & 1021.6 \end{matrix}$$

$$V\_1 = \begin{matrix} 4.1627 & 1.7837 & -1.765 & -2.1693 \\ 0.5353 & -0.86994 & 0.62705 & 1.7242 \\ 0 & 0 & 0 & 0 \\ 265.85 & 436.14 & -7.1609 & -223.45 \end{matrix}$$

$$K\_D = \begin{matrix} -7.3548 & 0 & 0 & 0 \\ 0 & -22.337 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.5828 \end{matrix}$$

$$M\_D = \begin{matrix} -6.9146 & 0 & 0 & 0 \\ 0 & -6.7787 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.12993 \end{matrix}$$

$$C\_D = \begin{matrix} -8.8138 & 0 & 0 & 0 \\ 0 & 9.7702 & 0 & 0 \\ 0 & 0 & 0.45287 & 0 \\ 0 & 0 & 0 & 0.2478 \end{matrix}$$

- Using the stable filters U_0, U_1, V_0, V_1, we obtain the decoupled equaions represented by K_D, M_D and C_D.

Future Goals

Aside from applying True Modal Control to a real system, there are other avenues for academic research as well:

- The presence of zeros in M_D in systems with singular system matrices, and its implication of equations of motion without an acceleration term.
- The effect of defective systems coupled equations that the Jordan form represents. Investigate whether LAMs can be defective, and if so, explore the conditions when this may occur.
- The relation between stable filters and SPEs, that is, exploring the field of all stable filters that can be obtained from a given set of SPEs.

Sources

- S.D. Garvey, M.I. Friswell, and U. Prells. Co-ordinate Transformations for Second Order Systems. part I: General Transformations. *Journal of Sound and Vibration*, doi: 10.1006/jsvi.2002.5165.

- Moody T. Chu and Nicoletta Del Buono. Total decoupling of general quadratic pencils, part I: Theory. *Journal of Sound and Vibration*. doi: 10.1016/j.jsv.2007.05.058.

Thanks!

Any questions?