#3

$n = 100$

$$S = \sum_{i=1}^{100} X_i \qquad X \sim BN\,Comp\left(0.2,\ q = \frac{1}{1.2};\ f_B\right)$$

$$f_B(100j) = 0.4\ 0.6^{j-1} \qquad j = 1, 2, \ldots$$

$$X = \sum_{i=1}^{M} B_i \qquad \varphi_X(t) = P_M(\varphi_B(t)) = \left(\frac{1/1.2}{1 - 1/1.2 + \frac{1}{1.2}\varphi_B(t)}\right)^{0.2}$$

$$\varphi_S(t) = \varphi_{X_1}(t) \ldots \varphi_{X_{100}}(t) \quad \text{var (Code)}$$

$$f_S(k) = \begin{cases} 0.026 \\ 0.034 \\ 0.044 \\ 0.057 \end{cases}$$

#6

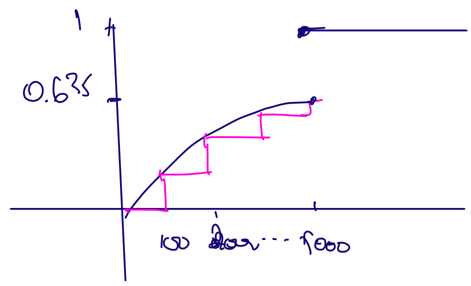
$$X \sim BN\,Comp\left(0.4,\ q = 2/3,\ f_B\right)$$

$$B = \min(C;\ 1000) \qquad C \sim Weibull\left(0.8,\ \frac{1}{1000}\right)$$

$\tilde{B}$: lower

$$F_C(x) = 1 - e^{-(\beta x)^{\tau}} \qquad F_C(1000) = 0.365$$

$$f_B(x) = \begin{cases} f_C(x) & x < 1000 \\ 0.365 & x \geq 1000 \end{cases}$$



$$f_B(x) = \begin{cases} 0 & 0 \\ f_C(1000) & 1000 \\ f_C(2000) - f_C(1000) & 2000 \\ f_C(3000) - f(2000) & 3000 \\ f_C(4000) - f(3000) & 4000 \\ f_C(5000) - f(4000) & 5000 \end{cases}$$

$$\varphi_X(t) = P_M(\varphi_B(t))$$

7

$$f_M(x) = (1-\theta) + \theta f_{M'}(x) \qquad \theta = 0.05$$

$$f_B(1000j) = 0.4 \quad 0.6^{j-1} \quad \mathbb{N}^+ \qquad M' \sim Poi(1)$$

$$p_M(0) = \theta p_{M'}(0) \qquad E(M) = 0.95 + 0.05 \cdot 1 = 1$$

$$E(X) = 1 \times 1/0.4 = 2.5$$

$$p_M(1) = (1-\theta) + \theta p_{M'}(1)$$

$$P_M(t) = \theta p_{M'}(0) + t\big((1-\theta) + \theta p_{M'}(1)\big) + \sum_{j=2}^{\infty} p_{M'}(j)\, t^j$$

$$P_M(t) = (1-\theta)t + \theta \sum_{j=0}^{\infty} p_{M'}(j)\, t^j$$

$$\varphi_X(t) = P_M(\varphi_B(t)) = (1-\theta) \quad + \quad \theta P_{M'}(\varphi_B(t))$$

#9

$W_1 \quad W_2$

$Z = v W_1 + v^2 W_2 \qquad W \sim \text{BN Comp}(2, \tfrac{1}{2}, F_B)$

$B \sim \text{Pareto}(3, 4000) \qquad \text{loan} \qquad P = 1000$

$v = 0.95 \qquad E[Z] = 4000(v + v^2) \quad . \; \$ \, 460$

$$M_W(t) = P_N(M_B(t)) = \left(1 - \frac{q}{(1-q)\,e_B(t)}\right)^r$$

$X = v W_1 \rightarrow W_1 = \frac{X}{v}$

$$f_W(x) = \frac{3 \cdot 4000^3}{(4000 + x)^4} \qquad w = \frac{x}{0.95}$$

$$f_X(x) = \frac{3 \cdot 4000^3}{\left(\frac{x}{0.95} + 4000\right)^4} \frac{1}{0.95} = \frac{3 \cdot 4000^3}{\left(\frac{1}{0.95}\right)^4 (x + 0.95 \cdot 4000)} \cancel{\frac{1}{0.95}}$$

$$= \frac{3\,(4000 \times 0.95)^3}{(x + 0.95 \cdot 4000)^4}$$

$X_1 \sim Pa(3, 3800)$

$X_2 \sim Pa(3, 3610) \qquad Z = \sum X + \sum X.$

#11

$$M_i \sim Poi(\lambda_i) \qquad \lambda_i = 0.06 - 0.01i$$

1000 i

$$S(t) = \sum_{k=1}^{N(t)} B_k \qquad B \in \{1000, 2000, \ldots, 5000\}$$

$$\begin{cases} M_1 \sim Poi(0.05) \\ M_2 \sim Poi(0.04) \\ M_3 \sim Poi(0.03) \\ M_4 \sim Poi(0.02) \\ M_5 \sim Poi(0.01) \end{cases} \qquad \begin{aligned} N_1(t) &= \sum_{i=1}^{M_1} B_i \\ N_2(t) &= \sum_{i=1}^{M_2} B_i \\ &\vdots \\ N_5(t) &= \sum_{i=1}^{M_5} B_i \end{aligned}$$

$$S(t) = \sum_{k=1}^{N(t)} B_k = \sum_{k=1}^{N_1(t)} B_k + \cdots + \sum_{k=1}^{N_5(t)} B_k$$

$$P_{S(t)}(t) = e^{\lambda_1 (P_B(t) - 1) + \cdots \lambda_5 (P_B(t) - 1)}$$

$$= e^{\lambda_N \left(\frac{\lambda_1}{\lambda_N} P_{B_1}(t) + \cdots + \frac{\lambda_5}{\lambda_N} P_{B_5}(t) - 1 \right)}$$

$$S(t) \sim CompPoi(\lambda_N t, f_B)$$

$$\mathbb{P}(C = c) = \frac{\lambda_1}{\lambda_N} P_B(c) + \cdots + \frac{\lambda_1}{\lambda_N} P_B(c)$$

$N = 0.15$

$$= \frac{5}{15} P_{B_1}(c) + \frac{4}{15} P_{B_2}(c) + \frac{3}{15} P_{B_3}(c) + \frac{2}{15} P_{B_4}(c) + \frac{1}{15} P_{B_5}(c)$$

Atelier

#1 X_1 et X_2

$S = X_1 + X_2$

$\varphi_{X_1}(k) = \mathbb{E}[X_1 | S = k]$

$\varphi_{X_2}(k) = \mathbb{E}[X_2 | S = k]$

a) $\varphi_{X_1}(k) + \varphi_{X_2}(k) = \mathbb{E}[X_1 | S = k] + \mathbb{E}[X_2 | S = k]$

$\mathbb{E}[X_1 + X_2 | S = k] = \mathbb{E}[S | S = k] = k$

b) $\mathbb{E}[X_1 \mathbb{1}_{\{S=k\}}]$

$\mathbb{E}[X_1 \mathbb{1}_{\{S=k\}}] = \sum_{j=0}^{k} j \, f_{X_1}(j) \, f_{X_2}(k-j)$

$X_1 + X_2 = k$

$X_1 = k - X_2$

$0 \to X_2 = k \to f_{X_1}(0) \, f_{X_2}(k)$

$1 \to X_2 = k-1 \to f_{X_1}(1) \, f_{X_2}(k-1)$

$\vdots$

$k \to X_2 = 0 \to f_{X_1}(k) \, f_{X_2}(0)$

$$\mathbb{E}[X_1 | S = k] = \frac{\mathbb{E}[X_1 \mathbb{1}_{\{S=k\}}]}{f_S(k)} = \frac{\sum_{j=0}^{k} j \, f_{X_1}(j) \, f_{X_2}(k-j)}{f_S(k)}$$

$$\mathbb{E}\Big[\mathbb{E}[X_1 | S = k]\Big] = \sum_{k=0}^{\infty} \varphi_X(k) \, f_S(k) = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X_1 \mathbb{1}_{\{S=k\}}]}{\cancel{f_S(k)}} \cdot \cancel{f_S(k)}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} j \, f_{X_1}(j) \, f_{X_2}(k-j)$$

$$= \mathbb{E}[X_1] \sum_{k=0}^{\infty} g_{X_1} * f_{X_2}(k) = \mathbb{E}[X_1]$$

#2

$$X_i \sim NBinom(r_i, q_i) \quad \begin{cases} r_1 = 0.5 & q_1 = 0.2 \\ r_2 = 8 & q_2 = 0.5 \end{cases}$$

$$\mathbb{E}[X_1] = 2 \quad \mathbb{E}[X_2] = 8 \quad \mathbb{E}[S] = 10$$

$$\mathbb{E}[X_1 1_{\{S=k\}}] = \sum_{j=0}^{k} \underbrace{j f_{X_1}(j)} f_{X_2}(k-j)$$

#3

$$\varphi_{X_1}(t) = \mathbb{E}[X_1 1_{\{S=k\}}] / f_S(k) \qquad X_i \sim Poi(\lambda_i)$$

$$\sum_{j=0}^{k} j\, e^{-\lambda_1} \frac{\lambda_1^j}{j!} e^{-\lambda_2} \frac{\lambda_2^{k-j}}{(k-j)!}$$

$$\frac{1}{k!} \sum_{j=0}^{k} e^{-(\lambda_1+\lambda_2)} j \lambda_1^j \lambda_2^{k-j} \frac{k!}{(k-j)!\, j!}$$

$$\frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} j \binom{k}{j} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^j \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-j} \sim Bin\left(k, \frac{\lambda_1}{\lambda_2+\lambda_1}\right)$$

$$(\lambda_1+\lambda_2)^k \frac{e^{-(\lambda_1+\lambda_2)}}{k!} k \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\varphi_{X_1}(k) = \mathbb{E}[X_1 1_{\{S=k\}}] / f_S(k) = \cancel{e^{(\lambda_1+\lambda_2)}} \frac{\cancel{k!}}{\cancel{(\lambda_1+\lambda_2)^k}} \cancel{(\lambda_1+\lambda_2)^k} \frac{\cancel{e^{-(\lambda_1+\lambda_2)}}}{\cancel{k!}} k \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\varphi_{X_1}(k) = k \frac{\lambda_1}{\lambda_1+\lambda_2}$$

b) $\varphi_{X_1}(s) = s\,\dfrac{\lambda_1}{\lambda_1+\lambda_2}$, $\varphi_{X_2}(s) = s\,\dfrac{\lambda_2}{\lambda_1+\lambda_2}$

c) $\mathbb{E}[\varphi_{X_i}(s)] = \mathbb{E}\left[s\,\dfrac{\lambda_i}{\lambda_1+\lambda_2}\right] = \lambda_i$ ✓

Distribution de frequence de fft

#1 $M = K_1 + K_2$ $\quad K_1 \sim Poi(\lambda)$ $\quad K_2 \sim NBinom(r, q)$

$$P_M(s) = \frac{q^r}{(1-(1-q)s)^r}\, e^{\lambda(s-1)}$$

- $\mathbb{E}[M] = \mathbb{E}[K_1] + \mathbb{E}[K_2] = \lambda + r\,\dfrac{1-q}{q}$
- $Var(M) = \lambda + r\,\dfrac{1-q}{q^2}$

$$P_M(s) = \mathbb{E}[s^M] = \mathbb{E}[s^{K_1+K_2}] \stackrel{\perp}{=} \mathbb{E}[s^{K_1}]\,\mathbb{E}[s^{K_2}] \quad ✓$$

$$P_M(s) = \left(\frac{q}{1-(1-q)s}\right)^r e^{\lambda(s-1)}$$

$$\mathbb{P}(M=k) = \sum_{j=0}^{k} f_{X_1}(j)\, f_{X_2}(k-j)$$

#2 $P_X(s) = P_M(P_B(s))$ $\quad P_M(s) = \left(\dfrac{q}{1-(1-q)s}\right)^r e^{\lambda(s-1)}$

$\lambda = 1.5$ $\quad r = 0.5,\ q = 1/4$

$$f_B(k) = \Phi\left(\frac{\ln(k)-\mu}{\sigma}\right) - \Phi\left(\frac{\ln(k-1)-\mu}{\sigma}\right) \quad k \in \mathbb{N}^+$$

$\mu = \ln(100) - \dfrac{0.64}{2}$ $\quad \sigma = 0.8$

$\mathbb{E}[M] = 3 \quad \mathbb{E}[B] = e^{\mu + \sigma^2/2} = e^{\ln(100) - \frac{0.64}{2} + \frac{0.64}{2}} = 100$

$\mathbb{E}[X] = 300$

$Var(M) = 7.5 \quad Var(B) = 8964.80$

$P_X(s) = P_M(P_B(t)) = P_{K_1}(P_B(t))\, P_{K_1}(P_B(t))$

$$X = \underbrace{\sum_{j=1}^{K_1} B_j}_{} + \underbrace{\sum_{j=1}^{K_2} B_j}_{}$$

#3

$P_X(s) = P_M(P_B(s))$

$$f_B(k) = \Phi\left(\frac{\ln(k) - \mu}{\sigma}\right) - \Phi\left(\frac{\ln(k-1) - \mu}{\sigma}\right)$$

$M \sim Poi\,IG(\lambda, \beta)$

$$\begin{cases} M \sim Poi(\lambda \Theta) \\ \Theta \sim IG(1, \beta) \end{cases}$$

$\mathbb{E}[M] = \mathbb{E}[\mathbb{E}[M|\Theta]]$

$= \mathbb{E}[\lambda\Theta] = \lambda$

$Var(M) = \mathbb{E}[Var(M|\Theta)] + Var(\mathbb{E}[M|\Theta])$

$= \mathbb{E}[\lambda\Theta] + Var(\Theta\lambda) = (\lambda + \lambda^2\beta)$

$\varphi_X(t) = P_M(\varphi_B(t))$