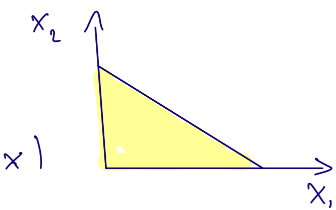
Soit $X_1, \dots, X_n$, $S = X_1 + \dots + X_n$

- On veut identifier / évaluer la distribution S
- Connaître f_S permet d'évaluer $\varphi(S)$ ou $\mathcal{Z}(S)$

Soit X_1 et X_2 , $S = X_1 + X_2$

$$f_S(x) = \int_0^x f_{X_1}(y)\, f_{X_2}(x-y)\, dy$$

$$F_S(x) = \int_0^x f_{X_1}(y)\, F_{X_2}(x-y)\, dy = \mathbb{P}(X_1 + X_2 \leq x)$$



Si on s'interesse à évaluer $f_S(2r)$, il faut parcourire toute la courbe

$$f_S(2r) = \int_0^{2r} f_{X_1}(x)\, f_{X_2}(2r - x)\, dx$$

Ca devient difficile avec plusieur v.a. donc:

↳ Approximation v.a. continue par discrète

↳ Convolution fmp de v.a. discrète

Soit X_1 et X_2 v.a. discrètes définies sur $0h, 1h, 2h, \dots$ h pas de discrétisation

$$f_{X_i}(kh) = \mathbb{P}(X_i = kh)$$

$$f_S(kh) = \sum_{j=0}^{k} f_{X_1}(jh) \cdot f_{X_2}((k-j)h) \leftarrow \text{produit de Convolution}$$

temps d'exécution est long $\in O(n^2)$

si $X_1 + X_2 + X_3 \to T(n) \in O(n^3)$ (...)

Ca devient trop long: il faut utiliser d'autre méthodes

Produit de Convolution discret

Soit $f(k), g(k), h(k) \geqslant 0$, $\{0, 1, \ldots, n-1\}$

$f * g = g * f$

$(f * g) * h = f * (g * h)$

$f * (g + h) = f * g + f * g$

$f * (c_1 g + c_2 h) = f * (c_1 g) + f * (c_2 h)$

en Actuariat :

$\sum f(j) = \sum g(j) = \sum h(i) = 1$

mélange : $f * (c_1 g + c_2 h)$

$= f_{X_1} * (c_1 f_{X_2} + c_2 f_{X_3})$

$c_1 + c_2 = 1$

Soit :

$$F(t) = f(0) + f(1) t + f(2) t^2 + \cdots + f(n-1) t^{n-1} = \sum_{j_1=0}^{n-1} f(j_1) t^{j_1}, \quad t\ (0,1)$$

$$G(t) = g(0) + g(1) t + \quad \cdots \quad + g(n-1) t^{n-1} = \sum_{j_2=0}^{n-1} g(j_2) t^{j_2}$$

$$H(t) = F(t) \times G(t)$$

$$H(t) = \left(\sum_{j_1=0}^{n-1} g(j_1) t^{j_1} \right) \left(\sum_{j_2=0}^{n-1} f(j_2) t^{j_2} \right)$$

ex :

$$\left(g(0) + g(1) t + g(2) t^2\right) \left(f(0) + f(1) t + f(2) t^2\right)$$

$$= g(0) f(0) + g(0) f(1) t + g(0) f(2) t^2$$
$$+ g(1) f(0) t + g(1) f(1) t^2 + g(1) f(2) t^3$$
$$+ g(2) f(0) t^2 + g(2) f(1) t^3 + g(2) f(2) t^4$$

$$= t^0 \left(g(0) f(0) \right) + t \left[g(0) f(1) + g(1) f(0) \right] + t^2 \left[g(0) f(2) + g(1) f(1) + g(2) f(0) \right]$$
$$+ t^3 \left[g(1) f(2) + g(2) f(1) \right] + t^4 \left(g(2) f(2) \right)$$

$$= \sum_{i=0}^{4} t^{i} \sum_{j=0}^{i} g(j)\, f(i-j) = \sum_{i=0}^{4} t^{i}\, f*g\,(i)$$

donc :

$$H(t) = F(t) \times G(t) = \sum_{i=0}^{2(n-1)} t^{i}\, f*g\,(i)$$

Fonction Génératrice :

Soit f fonction discrète

f. Génératrice : $F(t) = \sum_{j_1=0}^{\infty} f(j_1)\, t^{j_1} \qquad t \in (0,1)$

- $aF(t) + bG(t) = \sum (aF(t) + bG(t))\, t^{k}$
- $t^{m} F(t) = \sum_{k=0}^{\infty} f(k)\, t^{k+m}$
- $F(ct) = \sum_{k=0}^{\infty} f(k)\, c^{k} t^{k}$
- $F'(t) = \sum_{k=1}^{\infty} k\, f(k)\, t^{k-1}$
- $tF'(t) = \sum_{k=0}^{\infty} k\, f(k)\, t^{k}$

Fonctions génératrice de probabilité

$f_X(k) \geq 0, \quad \sum_{k} f_X(k) = 1$

Soit X défini sur $\mathbb{N}$: $P_X(t) = \sum_{k=0}^{\infty} f_X(k)\, t^{k}$

$$\begin{cases} P_X(0) = f_X(0) \\ P_X(1) = 1 \end{cases}$$

exemple :

$X \sim Ber(q) \quad P_X(t) = \bar{q} + tq$

$X \sim Poi(\lambda) \quad P_X(t) = e^{\lambda(t-1)}$

$X \sim Geom(q) \quad P_X(t) = \dfrac{q}{1-\bar{q}t} = q \dfrac{\frac{1}{\bar{q}}}{\frac{1}{\bar{q}} - t}$

$X \sim BinNeg(r,q) \quad P_X(t) = \left(q \dfrac{1/\bar{q}}{\frac{1}{\bar{q}} - t}\right)^r$

- soit f_X et f_Y et $P_X(t)$ et $P_Y(t)$

si $P_X(t) = P_Y(t) \ \forall t \iff f_X(k) = f_Y(k)$

Deux approches:

(1) $f_X(k) = \dfrac{1}{k!} \dfrac{\partial^k}{\partial t} P_X(t) \Big|_{t=0}$

(2) identifier coefficient t^k

Th 2

Soit $X_1, \dots, X_n$ positives indépendants , $S = X_1 + \dots + X_n$

$$P_S(t) = P_{X_1}(t) \cdot \ldots \cdot P_{X_n}(t) \quad , \quad t \in (0,1)$$
$$= \sum f_{X_1 + \dots + X_n}(k) \, t^k$$

. $X_1, \dots, X_n \quad X_i \sim Poi(\lambda_i) \qquad S = X_1 + \dots + X_n$

$$P_S(r) = P_{X_1}(r) \ldots P_{X_n}(r) = \underset{\sim Poi(\lambda_1 + \dots + \lambda_n)}{e^{\lambda_1(t-1)} \ldots e^{\lambda_n(t-1)}} = e^{(\lambda_1 + \dots + \lambda_n)(t-1)}$$

Exemple

$$P_{X_1}(t) = a_0 + a_2 t^2, \quad P_{X_2}(t) = b_0 + b_5 t^5, \quad P_{X_3}(t) = c_0 + c_7 t^7$$

$S = X_1 + \dots + X_n$

$$P_S(t) = (a_0 + a_2 t^2)(b_0 + b_5 t^5)(c_0 + c_7 t^7)$$

$$= (a_0 b_0 + a_0 b_5 t^5 + a_2 b_0 t^2 + a_2 b_5 t^7)(c_0 + c_7 t^7)$$

$$= a_0 b c_0 + a_0 b_0 c_7 t^7 + a_0 b_5 c_0 t^5 + a_0 b_5 c_7 t^{12} + a_2 b_0 c_0 t^2 + a_2 b_0 c_7 t^9$$
$$+ a_2 b_5 c_0 t^7 + a_2 b_5 c_7 t^{14}$$

$$f_S(0) = a_0 b_0 c_0, \dots, f_S(14) = a_2 b_5 c_7$$

Exemple

$X_1 \sim Geom(q_1) \quad X_2 \sim Geom(q_2)$ $\qquad q_1 = q_2 = q$

$S = X_1 + X_2$

$$P_S(t) = \left(\frac{q}{1 - \bar{q}t}\right)\left(\frac{q}{1 - \bar{q}t}\right) = \left(\frac{q}{1 - \bar{q}t}\right)^2$$

$S \sim BinNeg(2, q)$

Exemple

$X_1 \sim Geom(q_1) \quad X_2 \sim Geom(q_2)$

$$P_S(t) = \left(\frac{q_1}{1 - \bar{q}_1 t}\right)\left(\frac{q_2}{1 - \bar{q}_2 t}\right) = \left(q_1 \frac{1/\bar{q}_1}{1/\bar{q}_1 - t}\right)\left(q_2 \frac{1/\bar{q}_2}{1/\bar{q}_2 - t}\right)$$

$\beta_1 = 1/\bar{q}_1, \ \beta_2 = 1/\bar{q}_2$

$$= \left(\frac{\beta_2}{\beta_2 - \beta_1}\right) q_1 \frac{\beta_1}{\beta_1 - t} + \left(\frac{\beta_1}{\beta_1 - \beta_2}\right) q_2 \frac{\beta_2}{\beta_1 - t}$$

TP-3

$$X = \begin{cases} \sum_{k=1}^{M} B_i & M > 0 \qquad B_i \text{ iid} \\ 0 & M = 0 \qquad B = \{B_k, k \in \mathbb{N}^*\} \end{cases}$$

donc v.a X prend des valeurs sur $\mathbb{N}$

$$f_X(k) = \mathbb{P}(X = k)$$

$$P_X(t) = \mathbb{E}[t^X] = \mathbb{E}[t^{\sum B_i}] = \mathbb{E}\left[\mathbb{E}[t^{\sum B_i} \mid M] \mid M\right]$$

$$= \mathbb{E}_n\left[(P_B(t))^M\right]$$

$$P_X(t) = P_M(P_B(t))$$

Deux Cas:

$$\begin{cases} S : \text{ Somme fini de v.a. iid} \longrightarrow \text{Algo DePril} \\ S : \text{ loi Composée} \longrightarrow \text{Algo de Panjer} \end{cases}$$

Exemple:

$M \sim Bin(2, q)$ $\quad B = \{1, 2\}$ $\quad P_B(1) = b_1 \quad P_B(2) = b_2 \qquad b_1 + b_2 = 1$

$f_X(k) = ?$ $\qquad k = 0, 1, 2, 3, 4$

$$P_X(t) = P_M(P_B(t))$$

$$= \left(\bar{q} + q(b_1 t + b_2 t^2)\right)^2 = \bar{q}^2 + 2\bar{q}q(b_1 t + b_2 t^2) + q^2(b_1 t + b_2 t^2)^2$$

$$= \bar{q}^2 + 2\bar{q}q\, b_1 t + 2\bar{q}q b_2 t^2 + q^2 b_1^2 t^2 + 2q^2 b_1 b_2 t^3 + q^2 b_2^2 t^4$$

$$= \underbrace{\bar{q}^2}_{f_X(0)} + \underbrace{2\bar{q}q\, b_1 t}_{f_X(1)} + t^2 \underbrace{(2\bar{q}q b_2 + q^2 b_1^2)}_{f_X(2)} + t^3 \underbrace{(2q^2 b_1 b_2 t^3)}_{f_X(3)} + t^4 \underbrace{q^2 b_2^2}_{f_X(4)}$$

Que ferons-nous si $M \sim Poi(\lambda)$, $M \sim NBin(r, q) \longrightarrow$ Panjer

<u>Exemple</u>

$$M \sim Poi(\lambda) \qquad P_B(1) = b_1 \quad P_B(2) = b_2$$

$$P_X(t) = P_M(P_B(t)) = e^{\lambda(t b_1 + b_2 t^2 - 1)}$$

$$\frac{\partial}{\partial t} P_X(t) = P_B(t)' \, P_M'(P_B(t))$$

$$= P_B(t)' \, \lambda P_M(P_B(t))$$

$$f_X(1) = \lambda f_B(1) f_X(0)$$

$$\frac{\partial^2}{\partial t^2} P_X(t) = P_B^{(2)}(t) P_M'(P_B(t)) + \lambda P_B'(t) P_B'(t) P_M'(P_B(t))$$

$$= \lambda P_B^{(2)}(t) P_M(P_B(t)) + \lambda^2 P_B'(t) P_B'(t) P_M(P_B(t))$$

$$f_X(2) = \lambda f_B(2) f_X(0) + \frac{\lambda}{2} f_B(1) f_X(1)$$

$$\frac{\partial^3}{\partial t^3} P_X(t) = P_B^{(3)} P_M'(P_B(t)) + P_B^{(2)} P_M^{(2)}(P_B(t)) + \lambda\Big(2 P_B''(t) P_B'(t) P_M'(P_B(t))$$

$$+ P_B'(t) P_B'(t) P_M''(P_B(t))\Big)$$

$$= P_B^{(3)} P_M'(P_B(t)) + P_B^{(2)} P_M^{(2)}(P_B(t)) + \lambda^2 2 P_B''(t) P_B'(t) P_M(P_B(t))$$

$$+ \lambda^3 P_B'(t) P_B'(t) P_B'(t) P_M(P_B(t))$$

$$f_X(3) = \lambda f_B(3) f_X(0) + \frac{2\lambda^2}{3}\left(f_B(2)\right)^2 f_X(0) + \frac{2}{3}\lambda^2 f_B(2) f_B(1) f_X(0) + \frac{\lambda^3}{3!}\left(f_B(1)\right)^3 f_X(0)$$

$$= (\dots)$$

Somme v.a. iid et Algo de DePril

$$P_S(t) = \left(P_X(t)\right)^n$$

$$P_S'(t) = n\left(P_X(t)\right)^{n-1} P_X'(t)$$

$$P_X(t) P_S'(t) = n P_S(t) P_X'(t)$$

$$\left(\sum_{j=0}^{\infty} p_X(j) t^j\right)\left(\sum_{k=0}^{\infty} k\, p_S(k) t^{k-1}\right) = n\left(\sum_{k=0}^{\infty} p_S(k) t^k\right)\left(\sum_{j=0}^{\infty} j\, p_X(j)\, t^{j-1}\right)$$

$$\left(\sum_{j=0}^{\infty} p_X(j) t^j\right)\left(\sum_{k=0}^{\infty} k\, p_S\, t^k\right) = n\left(\sum_{k=0}^{\infty} p_S(k) t^k\right)\left(\sum_{j=0}^{\infty} j\, p_X(j) t^j\right)$$

$$\sum_{k=0}^{\infty} t^k \sum_{j=0}^{k} k\, p_X(k-j)\, p_S(j) = n \sum_{k=0}^{\infty} t^k \sum_{j=0}^{k} j\, p_S(k-j)\, p_X(j)$$

$$\sum_{k=0}^{\infty} t^k \underbrace{\sum_{j=0}^{k} (k-j)\, p_S(k-j)\, p_X(j)} = n \sum_{k=0}^{\infty} t^k \underbrace{\sum_{j=0}^{k} j\, p_S(k-j)\, p_X(j)}$$

$$\sum_{j=0}^{k} (k-j)\, p_S(k-j)\, p_X(j) = \sum_{j=0}^{k} n j\, p_S(k-j)\, p_X(j)$$

$$k\, p_S(k)\, p_X(0) + \sum_{1}^{k} (k-j)\, p_S(k-j)\, p_X(j) = \sum_{1}^{k} n j\, p_S(k-j)\, p_X(j)$$

$$k\, p_S(k)\, p_X(0) = \sum_{j=1}^{k} \left(j(n+1) - k\right) p_S(k-j)\, p_X(j)$$

$$p_S(k) = \frac{1}{f_X(0)} \sum_{j=1}^{k} \left(\frac{j}{k}(n+1) - 1\right) p_S(k-j)\, p_X(j)$$

$$f_S(k) = \frac{1}{f_X(0)} \sum_{j=1}^{k} \left(\frac{j}{k}(n+1) - 1 \right) f_S(k-j)\, f_X(j)$$

soit v.a. iid $X_1, \dots, X_n \in \{k_0 h, (k_0+1)h, \dots\}$ $(\text{si } f(x) = 0)$

$S = X_1 + \dots + X_n \in \{n k_0 h, (n k_0 + 1)h, \dots\}$

on défini $Y_i = X_i - k_0 h \in \{0h, 1h, 2h, \dots\}$ $i \in \{1, 2, \dots, n\}$

- $T = \sum_{1}^{n} Y_i$
- Valeurs de $f_T(kh)$ $k \in \mathbb{N}$ algo de DePril
- valeurs de $f_S(kh) = f_T((k - nk_0)h)$, $k \in \{k_0 h, k_0 n + 1, \dots\}$

Somme aléatoire de v.a. iid

$$X = \begin{cases} \sum_{1}^{M} B_i & , M > 0 \\ 0 & , M = 0 \end{cases}$$

- M est v.a. fréquence
- $\underline{B} = \{B_k, k \in \mathbb{N}^+\}$ $\quad B \perp M$

Par Conséquence $f_X(k) = \mathbb{P}(X = k)$, $k \in \mathbb{N}$

$$P_X(t) = P_M(P_B(t))$$

- l'objectif est $f_X(k)$

$$f_X(0) = f_M(0) + \sum_{j=1}^{\infty} f_M(j)\, f_B^{*j}(0)$$

$$= f_M(0) + \sum_{j=1}^{\infty} f_M(j) \left(f_B(0)\right)^j$$

$$f_X(0) = \sum_{j=0}^{\infty} f_M(j)\left(f_B(0)\right)^j$$

$$f_X(0) = P_M\left(f_B(0)\right)$$

$$f_X(kh) = \sum_{j=1}^{\infty} f_M(j)\, \underbrace{f_{B_1+\dots+B_j}(kh)}$$

Solution ? $\rightarrow$ Algo de Panjer

Condition : M fait partie de la famille (a, b, 0)

M fait partie de (a, b, 0) si :

$$f_M(k) = \left(a + \frac{b}{k}\right) f_M(k-1)$$

* Poisson (λ)

$$f_M(k-1) = e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$f_M(k) = e^{-\lambda} \frac{\lambda^k}{k!} \longrightarrow e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}\left(a + \frac{b}{k}\right) = e^{-\lambda} \frac{\lambda^{k-1}}{k!}$$

$a = 0 \quad b = \lambda$

* BinNeg (r, q)

$$f_M(k-1) = \frac{(k+r-2)!}{(k-1)!\,(r-1)!}\, q^r (1-q)^{k-1}$$

$$f_M(1) = \frac{r!}{(r-1)!}\, q^r (1-q) = \frac{(r-1)!}{(r-1)!}\, q^r \left(a + \frac{b}{1}\right)$$

$a = (1-q)$, $b = (1-q)(r-1)$

$$= \frac{(r-1)!}{(r-1)!}\, q^r (1-q) + \frac{(r-1)!}{(r-1)!}\, q^r (1-q)\, r$$

* Binomial(n, q)

$$f_M(0) = \frac{n!}{(n-0)!\,0!} q^0 (1-q)^{n-0}$$

$$f_M(1) = \frac{n!}{(n-1)!\,1!} q (1-q)^{n-1}$$

$$\frac{n!}{(n-0)!\;0!} q^0 (1-q)^{n-0} (a+b) = \frac{n!}{(n-1)!\,1!} q (1-q)^{n-1}$$

$$\frac{q}{1-q} + \frac{q}{1-q}$$

$$\frac{n!\, q (1-q)^{n-1}}{(n-0)!\; 0!} \left(-1 + \frac{n+1}{k} \right)$$

$$a = \frac{-q}{1-q} \qquad b = \frac{(n+1)q}{1-q}$$

$$P_M(t) = \sum_{j=0}^{\infty} f_M(j)\, t^j$$

$$P_M'(t) = \sum_{j=0}^{\infty} j\, f_M(j)\, t^{j-1}$$

$$= \sum_{j=1}^{\infty} j \left(a + \frac{b}{j}\right) f_M(j-1)\, t^{j-1}$$

$$= \sum_{j=1}^{\infty} j a\, f_M(j-1)\, t^{j-1} + \sum_{j=1}^{\infty} b\, f_M(j-1)\, t^{j-1}$$

$$= a \sum_{j=1}^{\infty} f_M(j-1)\,(j-1+1)\, t^{j-1} + b\, P_M(t)$$

$$= a \sum_{j=1}^{\infty} f_n(j-1)(j-1)t^{j-1} + a P_n(t) + b P_n(t)$$

$$= a \sum_{j=1}^{\infty} f_n(j-1)(j-1)t^{j-2}\, t + (a+b) P_n(t)$$

$$= at \sum_{j=1} f_n(j-1)(j-1)t^{j-2} + (a+b) P_n(t)$$

$$= at \sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_n(j-1)\, t^{j-1}$$

$$P_n'(t) = at P_n'(t) + (a+b) P_n(t)$$

<u>exemple - Poisson</u>

$$a = 0, \quad b = \lambda$$

$$P_n'(t) = \lambda P_n(t) = \lambda e^{\lambda(t-1)} \quad \checkmark$$

<u>*Binomial</u>

$$P_n'(t) = at P_n'(t) + (a+b) P_n(t)$$

$$\left\{ \begin{array}{l} a = \dfrac{-q}{1-q} \\ b = \dfrac{(n+1)q}{(1-q)} \end{array} \right.$$

$$P_n'(t) = \left(\frac{-q}{1-q}\right) t P_n'(t) + \left(\frac{(n+1)q - q}{1-q}\right) P_n(t)$$

$$= \left(\frac{-q}{1-q}\right) t P_n'(t) + \frac{nq}{1-q} P_n(t)$$

$$P_n'(t)\left[1 - \frac{-qt}{1-q}\right) = \frac{nq}{1-q} P_n(t)$$

$$P_n'(t)\left(\frac{1-q+qt}{\cancel{1-q}}\right) = \frac{nq}{\cancel{(1-q)}} P_n(t) = nq \frac{1}{1-q+qt}(1-q+qt)^n$$

$$= nq(1-q+qt)^{n-1} \quad \checkmark$$

Soit $S = X_1 + \dots + X_n$

$$P_S(t) = P_{\underline{X}}(t_1, \dots, t_n)$$

- $P_{\underline{X}}(0, \dots, 0) = f_{\underline{X}}(0, \dots, 0)$ et $P_{\underline{X}}(1, \dots, 1) = 1$
- $P_{\underline{X}}(t_1, \dots, t_n)\Big|_{t_j = 1,\ j=1,\dots,n\ j \neq i} = P_{X_i}(t_i)$

$$f_{\underline{X}}(k_1, \dots, k_n) = \frac{1}{k_1!} \cdots \frac{1}{k_n!} \frac{\partial^n}{\partial k_1 \dots \partial k_n} P_{\underline{X}}(t_1, \dots, t_n)\Big|_{t_1 = \dots = t_n = 0}$$

$$P_{\underline{X}}(t_1, \dots, t_n) = \sum_{k_1} \cdots \sum_{k_n} f_{\underline{X}}(k_1, \dots, k_n)\, t_1^{k_1} \cdots t_n^{k_n}$$

Algorithme de Panjer

$$X = \begin{cases} \sum_{i=1}^{M} B_i \\ 0 \end{cases}$$

- $M \in$ famille $(a, b, 0)$
- $\underline{B} = \{B_k,\ k \in \mathbb{N}^*\}$
- $\underline{B} \perp M$

$$P_X(t) = P_M(P_B(t))$$

$$P_X'(t) = P_M'(P_B(t))\, P_B'(t)$$

$$= \left(a P_B(t) P_M'(P_B(t)) + (a+b) P_M(P_B(t))\right) P_B'(t)$$

$$= a P_B(t) P_B' P_M'(P_B(t)) + (a+b) P_M(P_B(t)) P_B'(t)$$

$$= a P_B(t) P_X'(t) + (a+b) P_X(t) P_B'(t)$$

$$= a\left(\sum_{k=0}^{\infty} p_B(k)t^k\right)\left(\sum_{j=0}^{\infty} j\, p_X(j)\, t^{j-1}\right) + (a+b)\left(\sum_{j=0}^{\infty} p_X(j)\, t^j\right)\left(\sum_{k=0}^{\infty} k\, p_B(k)\, t^{k-1}\right)$$

$$tP_X'(t) = a\left(\sum_{k=0}^{\infty} p_B(k)t^k\right)\left(\sum_{j=0}^{\infty} j\, p_X(j)\, t^j\right) + (a+b)\left(\sum_{j=0}^{\infty} p_X(j)t^j\right)\left(\sum_{k=0}^{\infty} k\, p_B(k)t^k\right)$$

$$= a\sum_{k=0}^{\infty} t^k \sum_{j=0}^{k} (k-j)\, p_B(j)\, p_X(k-j) + (a+b)\sum_{k=0}^{\infty} t^k \sum_{j=0}^{k} j\, p_X(k-j)\, p_B(j)$$

$$\sum_{k=0}^{\infty} k\, p_X(k)\, t^k = \sum_{k=0}^{\infty} t^k\left(a\sum_{j=0}^{k}(k-j)\, p_B(j) p_X(k-j) + a\sum_{j=0}^{k} j\, p_X(k-j)\, p_B(j) + b\sum j\, p_X(k-j)\, p_B(j)\right)$$

$$= \sum_{k=0}^{\infty} t^k\left(a\sum_{j=0}^{k} k\, p_B(j) p_X(k-j) + b\sum_{j=0}^{k} j\, p_X(k-j)\, p_B(j)\right)$$

$$= \sum_{k=0}^{\infty} t^k\left(ak\sum_{j=0}^{k} p_B(j)\, p_X(k-j) + b\sum_{j=0}^{k} j\, p_X(k-j)\, p_B(j)\right)$$

$$k\, p_X(k) = ak\sum_{j=0}^{k} p_B(j)\, p_X(k-j) + b\sum_{j=0}^{k} j\, p_X(k-j)\, p_B(j)$$

$$k\, p_X(k) = ak\, p_X(k)\, p_B(0) + \sum_{j=1}^{k}\left(p_B(j)\, p_X(k-j)\right)(ak+bj)$$

$$k\, p_X(k)(1 - a\, p_B(0)) = (\ldots)$$

$$f_X(k) = \frac{1}{1-af_B(0)}\sum_{j=1}^{k}\left(a + b\frac{j}{k}\right) f_B(j)\, f_X(k-j)$$

$$P_X(t) = P_M(P_B(t)) \to f_X(0) = P_M(P_B(0))$$

$$f_X(kh) = \frac{1}{1-af_B(0)}\sum_{j=1}^{k}\left(a + b\frac{j}{k}\right) f_B(jh)\, f_X((k-j)h)$$

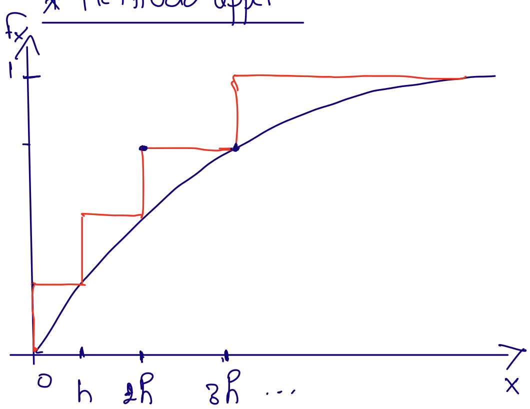
Méthodes de discrétisation

Objectif : approximer v.a. X continue par $\tilde{X}$ discrète

support $A_h = \{0, 1h, 2h, 3h, \dots\}$ $h>0$

$$f_{\tilde{X}}(kh) = \mathbb{P}(\tilde{X} = kh), \quad k \in \mathbb{N}$$

* Méthode upper



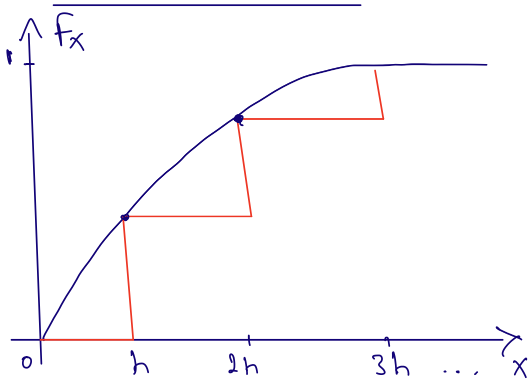
$$f_{\tilde{X}}(0) = F_X(h)$$

$$f_{\tilde{X}}(kh) = \mathbb{P}((k-1)h \leq X < kh)$$

$$f_{\tilde{X}}(kh) = F_X((k+1)h) - F_X(kh)$$

$$F_{\tilde{X}}(x) = \begin{cases} F_X(h) & 0 \leq x < h \\ F_X(2h) & h \leq x < 2h \\ F_X(3h) & 2h \leq x < 3h \\ (\dots) & \end{cases}$$

* Méthode lower



$$f_{\tilde{X}}(0) = 0$$

$$f_{\tilde{X}}(kh) = F_X(kh) - F_X((k-1)h)$$

$$F_{\tilde{X}}(kh) = \begin{cases} 0 & \leq x < h \\ F(h) & h \leq x < 2h \\ F_X(2h) & 2h \leq x < 3h \\ (\dots) & \end{cases}$$

Ordre de dominance stochastique

$h < h'$

$$\tilde{X}^{h} \underset{sd}{\lesssim} \tilde{X}^{h'} \underset{sd}{\lesssim} X \underset{sd}{\lesssim} \tilde{X}^{h'} \underset{sd}{\lesssim} \tilde{X}^{h}$$

donc:

$$\tilde{S}^{h} \lesssim \tilde{S}^{h'} \underset{sd}{\lesssim} S \underset{sd}{\lesssim} \tilde{S}^{h'} \underset{sd}{\lesssim} \tilde{S}^{h}$$

illustration : méthode de discrétisation

* upper:

$$f_{\tilde{Y}(u,h)}(0) = F_Y(h)$$

$$f_{\tilde{Y}(u,h)}(kh) = F_Y((k+1)h) - F_Y(kh)$$

selon cette méthode $\rightarrow F_Y(x) \lesssim F_{\tilde{Y}(u,h)}(x)$

* lower:

$$f_{\tilde{Y}(l,h)}(0) = 0$$

$$f_{\tilde{Y}(u,h)}(kh) = F_Y(kh) - F_Y((k-1)h)$$

$$F_Y(x) \geqslant F_{\tilde{Y}(l,h)}(x)$$

. soit $h_2 < h_1$

$$F_{Y(l,h_1)}(x) \lesssim F_{Y(l,h_2)}(x) \lesssim F_Y(x) \lesssim F_{Y(u,h_2)}(x) \lesssim F_{Y(u,h_1)}(x)$$

$Y^{(u,h)} \xrightarrow{D} Y$ i.e. $\lim\limits_{h\to 0} F_{Y(u,h)}(x) = F_Y(x)$

$Y^{(l,h)} \xrightarrow{D} Y$ i.e. $\lim\limits_{h\to 0} F_{Y(l,h)}(x) = F_Y(x)$

Produit de Convolution et discrétisation

X_1 et X_2

$S = X_1 + X_2$

On défini X_1, X_2, S par les v.a. $\tilde{X}_1^{(\text{mét}, h)}$ $\tilde{X}_2^{(\text{mét}, h)}$ et $\tilde{S}^{(\text{mét}, h)}$ où mét = "u", "l" $A_h = \{0, 1h, 2h, 3h, \dots\}$

$$f_S(x) = f_{X_1+X_2}(x) = \int_0^x f_{X_1}(x) f_{X_2}(x-y)\,dy \qquad F_S(x) = \int_0^x f_S(s)\,ds$$

$$f_{\tilde{S}^{(\text{mét},h)}}(x) = \sum_{j=0}^{x} f_{\tilde{X}_1^{(\text{mét},h)}}(hj)\, f_{\tilde{X}_2^{(\text{mét},h)}}((x-j)h)$$

$$F_{\tilde{S}^{(\text{mét},h)}}(x) = \sum_{l=0}^{x} f_{\tilde{S}^{(\text{mét},h)}}(lh)$$

Somme aléatoire et discrétisation

$\underline{X} = \{X_i,\ i \in \mathbb{N}^+\}$

$$S = \sum_{i=1}^{M} X_i \qquad \sum_{i=1}^{0} a_i = 0$$

On défini $\tilde{X}^{(\text{mét}, h)}$ et $\tilde{S}^{(\text{mét}, h)}$

$$A_h = \{0, 1h, 2h, \dots\}$$

$$f_S(x) = f_M(0) + \sum_{i=1}^{\infty} f_M(i) \underbrace{f_{X_1+\dots+X_i}(x)}_{\text{(souvent pas fermé)}}$$

$$f_{\tilde{S}^{(\text{mét},h)}}(kh) = f_M(0) + \sum_{i=1}^{\infty} f_M(i)\, f_{\tilde{X}_1+\dots+\tilde{X}_i}^{(\text{mét},h)}(kh)$$

Algorithme de Panjer : illustrations

$$X = \sum_{i=1}^{M} B_i$$

$$\mathbb{P}(B = hj) = f_B(hj)$$

$$\mathcal{L}_X(t) = P_\Pi(\mathcal{L}_B(t))$$

$$B \sim LN(\mu, \sigma) \qquad \mu = \ln(10) - 0.32 \qquad \sigma = 0.8$$

$$P_\Pi(r) = \alpha e^{\lambda_1(r-1)} + (1-\alpha) e^{\lambda_2(r-1)}$$

$$\mathcal{L}_X(t) = \alpha e^{\lambda_1(\mathcal{L}_B(t)-1)} + (1-\alpha) e^{\lambda_2(\mathcal{L}_B(t)-1)}$$

$$\mathcal{L}_X(t) = \alpha P_{M_1}(\mathcal{L}_B(t)) + (1-\alpha) P_{M_2}(\mathcal{L}_B(t))$$

$$= \alpha \mathcal{L}_{Y_1}(t) + (1-\alpha)\mathcal{L}_{Y_2}(t)$$

$$F_X(x) = \alpha F_{Y_1}(x) + (1-\alpha) F_{Y_2}(x)$$

$$f_X(x) = \alpha f_{Y_1}(x) + (1-\alpha) f_{Y_2}(x)$$

$$Y_1 = \sum_{i=1}^{M_1} B_i \qquad Y_2 = \sum_{i=1}^{M_2} B_i \qquad M_i \sim Poi(\lambda_i)$$

après on utilise Panjer pour $\sum_{i=1}^{n_1} \tilde{B}_i$ $\qquad f_{\tilde{B}^u}(kh) = F_B((k+1)h) - F_B(kh)$

$\hookrightarrow \quad f_{\tilde{Y}_i}(kh) \qquad$ après : $f_{\tilde{X}^u}(kh) = \alpha f_{\tilde{Y}_1^u}(kh) + f_{\tilde{Y}_2^u}(kh)\,(1-\alpha)$

* $S = X_1 + X_2$

$$\mathcal{L}_{X_i}(t) = P_{M_i}(\mathcal{L}_{B_i}(t))$$

$B_i \sim Exp(\beta_i) \qquad P_{M_i}(t) = e^{\lambda_i(r-1)}$

$$\mathcal{L}_S(t) = \mathbb{E}[e^{-tX_1}]\,\mathbb{E}[e^{-tX_2}]$$

$$= \mathcal{L}_{X_1}(t)\,\mathcal{L}_{X_2}(t) = P_{M_1}(\mathcal{L}_{B_1}(t))\,P_{M_2}(\mathcal{L}_{B_2}(t))$$

$$\mathcal{L}_S(t) = e^{(\lambda_1+\lambda_2)\left(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}\mathcal{L}_{B_1}(t) + \frac{\lambda_2}{\lambda_1+\lambda_2}\mathcal{L}_{B_2}(t)}\, - 1\right)}$$

$$\mathcal{L}_C(t) = \alpha\frac{\beta_1}{\beta_1+t} + (1-\alpha)\frac{\beta_2}{\beta_2+t}$$

lemma:

$$\frac{\beta_1}{\beta_1+t} = q\,\frac{\beta_2}{\beta_2+t}\,\frac{1}{1-(1-q)\left(\frac{\beta_2}{\beta_2+t}\right)}$$

$$\mathcal{L}_C(t) = \alpha\frac{\beta_2}{\beta_2+t}\,\frac{q}{1-(1-q)\frac{\beta_2}{\beta_2+t}} + (1-\alpha)\frac{\beta_2}{\beta_2+t}$$

$P_D(r) = \alpha r P_J(r) + (1-\alpha) r \qquad J \sim Geom(q)$

$$\mathbb{P}(D=k) = \begin{cases} 0 & k=0 \\ \alpha q + (1-\alpha) & k=1 \\ \alpha q(1-q)^{k-1} & k\geq 2 \end{cases}$$

$$P_D(r) = (\alpha q + (1-\alpha))r + \sum_{2}^{\infty} p_D(k)\, r^k$$

$$\mathcal{L}_C(t) = \mathcal{P}_D(\mathcal{L}_B(t))$$

$$\mathcal{L}_S(t) = \mathcal{P}_N(\mathcal{P}_D(\mathcal{L}_B(t)))$$

puisque $N \sim Poi(\lambda_1 + \lambda_2)$

$$\mathcal{P}_X(t) = \mathcal{P}_M(\mathcal{P}_B(t))$$

$$\sum_{k=}^{\infty} \underbrace{p_X(k)}\, t^k$$

On pose $\mathcal{P}_L(r) = \mathcal{P}_N(\mathcal{P}_D(r)) = \sum_{k=0}^{\infty} r_k r^k$

$\mathbb{P}(L=k) = r_k \leftarrow$ on le fait avec de Pail

$$\mathcal{P}_L\left(\frac{\beta_1}{\beta_2+t}\right) = \sum_{k=0}^{\infty} r_k \left(\frac{\beta_2}{\beta_2+t}\right)^k$$

$$\mathcal{L}_S(t) = \sum_{k=0}^{\infty} r_k \left(\frac{\beta_2}{\beta_2+t}\right)^k$$

$$f_S(x) = \sum_{k=0}^{\infty} r_k\, H(x, k, \beta_2).$$

ou

$$\mathcal{L}_S(t) = \mathcal{L}_{X_1}(t)\, \mathcal{L}_{X_2}(t)$$

$$\mathcal{L}_X(t) = \mathcal{P}_{M_1}(\mathcal{L}_{B_2}(t)\, \mathcal{P}_D(\mathcal{L}_{B_2}(t)))$$

$$\mathcal{L}_B(t)\, \mathcal{P}_D(\mathcal{L}_{B_2}(t))$$

$$\mathcal{P}_V(r) = r\, \mathcal{P}_D(r)$$

$$\mathbb{P}(V=k) = \begin{cases} 0 & k=0 \\ q(1-q)^{k-1} & k=1 \end{cases}$$

$$\mathcal{L}_X(t) = \mathcal{P}_{M_1}\left(\mathcal{P}_V\left(\frac{\beta_2}{\beta_2+t}\right)\right)$$

Soit $P_L(r) = P_{M_1}(P_V(r)) = \sum r_x r^x$

$$\mathcal{L}_X(t) = P_L\left(\frac{\beta_2}{\beta_2+t}\right)$$

$$\mathcal{L}_S(t) = P_L\left(\frac{\beta_2}{\beta_2+t}\right) P_{M_1}\left(\frac{\beta_2}{\beta_2+t}\right)$$

$$\mathcal{L}_S(t) = \sum_{k=0}^{\infty} \left(\frac{\beta_2}{\beta_2+t}\right)^k \sum_{j=0}^{k} f_L(j)\, f_{M_1}(k-j)$$

$$\mathcal{L}_S(t) = \sum_{k=0}^{\infty} r_k \left(\frac{\beta_2}{\beta_2+t}\right)^k$$

$X_1 \sim Gamma(\alpha_1, \beta_1)$ $\quad X_2 \sim Gamma(\alpha_2, \beta_2)$

$$S = X_1 + X_2$$

$$\mathcal{L}_S(t) = \mathcal{L}_{X_1}(t)\,\mathcal{L}_{X_2}(t) = \left(\frac{\beta_1}{\beta_1+t}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2+t}\right)^{\alpha_2}$$

$$\left(\frac{\beta_1}{\beta_1+t}\right) = \left(\frac{\beta_2}{\beta_2+t}\right)\left(\frac{q}{1-(1-q)\frac{\beta_2}{\beta_2+t}}\right)$$

$$\mathcal{L}_S(t) = \left(\frac{\beta_2}{\beta_2+t}\right)^{\alpha_1+\alpha_2}\left(\frac{q}{1-(1-q)\frac{\beta_2}{\beta_2+t}}\right)^{\alpha_1}$$

$$\mathcal{L}_S(t) = \left(\frac{\beta_2}{\beta_2+t}\right)^{\alpha_1+\alpha_2} P_J\left(\frac{\beta_2}{\beta_2+t}\right)$$

$$= \sum_{k=0}^{\infty} P_J(k)\left(\frac{\beta_2}{\beta_2+t}\right)^{k+\alpha_1+\alpha_2} \qquad J \sim BinNeg(\alpha_1, q)$$

$$f_S(x) = \sum_{k=0}^{\infty} P_J(k)\ H(x, \alpha_1 + \alpha_2 + k, \beta_2)$$

$$X_i \sim \text{Gamma}(\alpha_i, \beta_i)$$

$$\mathcal{L}_S(t) = \mathcal{P}_{X_1}(t)\, \mathcal{P}_{X_2}(t)\, \mathcal{P}_{X_3}(t)$$

$$= \left(\frac{\beta_1}{\beta_1 + t}\right)^{\alpha_1} \left(\frac{\beta_2}{\beta_2 + t}\right)^{\alpha_2} \left(\frac{\beta_3}{\beta_3 + t}\right)^{\alpha_3}$$

$$= \quad (\ldots)$$