Résumé intro 2

* Distribution obtenu par un mélange

$$f_X(x) = \int f_{X|\Lambda}(x)\, f_\Lambda(\lambda)\, d\lambda$$

. le But Généraliser $Y \sim Poi(\lambda)$ $\begin{cases} \mathbb{E}[Y] = \lambda \\ Var(Y) = \lambda \\ P_Y(t) = e^{\lambda(t-1)} \end{cases}$

$$\lambda \to \Lambda = \lambda \cdot \Theta$$

Soit $\begin{cases} \mathbb{E}[\Theta] = 1 \\ Var(\Theta) < \infty \\ P_\Theta(t) < \infty \end{cases}$

X tel que $X|\Theta \sim Poi(\lambda \cdot \Theta)$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[\Theta\lambda] = \lambda$$

$$Var(X) = \mathbb{E}_\Theta[Var(X|\Theta)] + Var_\Theta(\mathbb{E}[X|\Theta])$$

$$Var(X) = \lambda Var(\Theta) + \lambda^2 Var(\Theta)$$

$$P_X(t) = \mathbb{E}_\Theta[\mathbb{E}[t^X|\Theta]] = \mathbb{E}_\Theta[e^{\lambda\Theta(t-1)}]$$

$$P_X(t) = M_\Theta(\lambda(t-1))$$

Example :

$$\Theta \sim Gamma(r, r)$$

$$\mathbb{E}[X] = \lambda \qquad Var(X) = \lambda + \frac{\lambda^2}{r} = \lambda\left(1 + \frac{1}{r}\right)$$

$$P_X(t) = M_\Theta(\lambda(t-1)) = \left(\frac{r}{r - \lambda(t-1)}\right)^r = \left(\frac{1}{1 - \frac{\lambda}{r}(t-1)}\right)^r$$

$$\frac{\lambda}{r} = \frac{1-q}{q} \to q = \frac{1}{1+\frac{\lambda}{r}} \qquad = \left(\frac{q}{q - (1-q)t}\right)^r$$

Qd $r \to \infty$ $P_X(t) = e^{\lambda(t-1)}$

* light et heavy tailed

- **Absolu :**

light : tous $\mathbb{E}[X^k]$ $\exists$

heavy : $\mathbb{E}[X^k]$ $\exists$, pour $k \leq n$

↳ si FGM $\exists$, alors tous $\mathbb{E}[X^k]$ $\exists$

↳ si tout $\mathbb{E}[X^k]$ $\exists$ $\xrightarrow{X}$ FGM $\exists$

- **Relatif :**

$$\lim_{x \to \infty} \frac{\bar{F}_X(x)}{\bar{F}_Y(x)} \underset{R.H}{\overset{0/0}{=}} \lim_{x \to \infty} \frac{f_X(x)}{f_Y(x)} \begin{cases} 0 & Y \text{ plus lourd que } X \\ \infty & X \text{ plus lourd que } Y \end{cases}$$

- $$\lim_{x \to \infty} e^{rx} \bar{F}_X(x) = \lim_{x \to \infty} \frac{\bar{F}_X(x)}{e^{-rx}} = \lim_{x \to \infty} \frac{\bar{F}_X(x)}{\bar{F}_Z(x)} = 0 \text{ pour } Z \sim Exp(r > r_0)$$

↳ light

- $$\lim_{x \to \infty} \frac{\bar{F}_X(x)}{e^{-rx}} = \lim_{x \to \infty} \frac{\bar{F}_X(x)}{\bar{F}_Z(x)} = \infty \longrightarrow \text{Heavy}$$

* fonction de mortalité

- $$\mu_X = \frac{f_X(x)}{\bar{F}_X(x)} \longrightarrow \mu_X\, dx = \frac{f_X(x)\,dx}{\bar{F}_X(x)} = \frac{\mathbb{P}(x < X \leq x+dx)}{\mathbb{P}(X > x)}$$

$$= \mathbb{P}(x < X \leq x+dx \mid X > x)$$

$$\mu_X = \frac{f_X(x)}{\bar{F}_X(x)} = -\frac{\partial}{\partial x} \ln\left(\bar{F}_X(x)\right)$$

$$\bar{F}_X(x) = e^{-\int_0^x \mu_X(t)\,dt}$$

$$f_X(x) = -\left(\bar{F}_X(x)\right)' = -\left[e^{-\int_0^x \mu_X(t)\,dt}\right]' = -\left[\left(-\int_0^x \mu_X(t)\,dt\right)' e^{-\int_0^x \mu_X(t)\,dx}\right.$$

$$= -\left[-\mu_X(x)\, e^{-\int_0^x \mu_X(t)\,dt}\right.$$

$$f_X(x) = \mu_X(x)\, e^{-\int_0^x \mu_X(t)\,dt}$$

- X est « light » μ_X croissante
- X est « heavy » μ_X décroissante

$$e_X(x) = \frac{\pi_X(x)}{\bar{F}_X(x)} \longrightarrow \begin{cases} X \text{ light} & \text{si } e_X(\cdot) \text{ décroissante} \\ X \text{ heavy} & \text{si } e_X(\cdot) \text{ croissante} \end{cases}$$

$$e_X(x) = \frac{\int_x^\infty \bar{F}_X(x)\,dx}{\bar{F}_X(x)\,dx} = \frac{-\bar{F}_X(x)}{-f_X(x)} = \frac{1}{\mu_X(x)}$$

	$\mathbb{E}[X^k]$	$\mu_X(x)$	$e_X(x)$
heavy	$\exists\ k \leq n$	décroissante	croissante
light	$\exists\ \forall k$	croissante	décroissante

$\mu_X(x)$ Croissante (décroissante) $\longrightarrow$ $e_X(x)$ décroissante (Croissante)

$e_X(x)$ décroissante (Croissante) $\not\longrightarrow$ $\mu_X(x)$ Croissante (décroissante)
(pas nécessairement)

•

$$\bar{F}_X(x) = \frac{e_X(0)}{e_X(x)}\, e^{-\int_0^x \frac{1}{e_X(t)}\, dt}$$

Si on a $e_X(\cdot)$ on peut retrouver $\bar{F}_X(x)$

$$f_{X_e}(x) = \frac{\bar{F}_X(x)}{\mathbb{E}[X]} \qquad x \geq 0$$

$$\bar{F}_{X_e}(x) = \frac{\int_x^\infty \bar{F}_X(y)\, dy}{\mathbb{E}[X]}$$

$$\mu_{X_e}(x) = \frac{\bar{F}_X(x)}{\mathbb{E}[X]} \cdot \frac{\mathbb{E}[X]}{\int_x^\infty \bar{F}_X(y)\, dy} = \frac{1}{e_X(x)}$$

$$\bar{F}_X(x) = \mathbb{E}[X] \cdot f_{X_e}(x) \,, \qquad f_{X_e}(x) = -\left(e^{-\int_0^x \frac{1}{e_X(t)}\, dt} \right)'$$

$$\bar{F}_X(x) = \frac{e_X(0)}{e_X(x)}\, e^{-\int_0^x \frac{1}{e_X(t)}\, dt} \qquad f_{X_e}(x) = \frac{1}{e_X(x)}\, e^{-\int_0^x \frac{1}{e_X(t)}\, dt}$$

* ditribution montants ds sinistre

- $X \sim Exp(\lambda)$

  $\{X - a \mid X > a\} \to X$ Sans mémoire

  Sum iid $\sim$ Erlong (n, λ)

  pour simuler $X^{(j)} = -\frac{1}{\lambda} \ln\left(1 - U^{(j)}\right)$

- $X \sim Erlang(n, \lambda)$

  $Y = X_1 + \dots + X_n \qquad X_i \sim Exp(\lambda)$

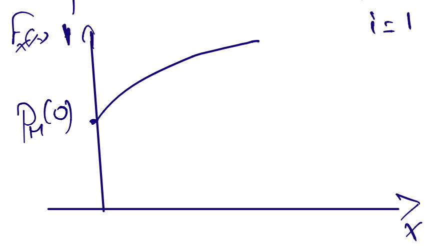
  $X^{(j)} = -\frac{1}{\lambda} \ln\left(1 - U_k^{(j)}\right)$

  $Y^{(j)} = X_1^{(j)} + X_2^{(j)} + \dots + X_n^{(j)} = \frac{1}{\lambda} \ln\left(\prod_{i=1}^{n} (1 - U^{(j)})\right)$

  « light tail » , μ_X croissante

* le cas particulier de somme composé : $X = \sum_{i=1}^{M} B_i$ où $M \sim Ber(q)$

$$M = \begin{cases} 1 & q \\ 0 & 1-q \end{cases}$$



$$F_X(x) = P_M(0) + P_M(1) F_B(x) = u$$

$$\frac{u - (1-q)}{q} = F_B(x)$$

$$VaR_p(X) = \begin{cases} 0 & 1 < k \leq 1-q \\ VaR_{\frac{k-(1-q)}{q}}(B) & k > 1-q \end{cases}$$

$$TVaR_p(X) = \frac{1}{1-p}\left[\mathbb{E}\left[X \, 1_{\{X > VaR_p(X)\}} \right] + VaR_p(X) \left(F_X(VaR_p(X)) - p \right) \right]$$

$$= \frac{1}{1-k}\left[\mathbb{E}[B \mathbb{1}_{\{B > VaR_{\frac{k-(1-q)}{q}}(B)\}}] + VaR_{\frac{k-(1-q)}{q}}(B) \left(F_X(VaR(B)) - p \right) \right)$$

$$TVaR_k(X) = \begin{cases} \frac{1}{1-k} \mathbb{E}[B] & \text{Si} \quad k \leq 1-q \\ \frac{1}{1-k} \mathbb{E}[B \mathbb{1}_{\{B > VaR_k(X)\}}] \end{cases}$$

* Somme n Poi Comp

$$S = X_1 + X_2 + \dots + X_n \qquad X_i \sim Pcomp(\lambda_i, f_{B_i})$$

$$S \sim Pcomp(\Sigma \lambda_i, f_c) \qquad \lambda_s = \sum_{i=1}^{n} \lambda_i$$

$$f_c(x) = \frac{\lambda_1}{\lambda_s} f_{B_1}(x) + \dots + \frac{\lambda_n}{\lambda_s} f_{B_n}(x)$$

* les Somme Composée

$$X = \sum_{1}^{M} B_i$$

$$\mathbb{E}[X] = \mathbb{E}[M] \cdot \mathbb{E}[B]$$

$$Var(X) = \mathbb{E}[Var(B|M)] + Var(\mathbb{E}[B|M])$$

$$Var(X) = \mathbb{E}[M] Var(B) + \mathbb{E}^2[B] Var(M)$$

$$M_X(t) = \mathbb{E}_M[(M_B(t))^M] = P_M(M_B(t))$$

* Somme de n Composée

- Somme $P\text{Comp}(\lambda_i, f_{B_i})$

$$S \sim P\text{comp}(\lambda_S, f_C)$$

- $X_i \sim \text{BiNeg}(r_i, q, f_B)$ $\qquad B_1 \sim B_2 \sim \ldots (B)$

$$S \sim \text{BiNeg}\left(\sum r_i, q, f_B\right) \qquad M = M_1 + M_2 + \ldots + M_n$$

- $X_i \sim B\,\text{Comp}(n_i, q, f_B)$

$$S \sim B\text{comp}\left(\sum n_i, q, f_B\right)$$

* Approximation de $S = X_1 + \ldots + X_n$

1) Normale

$$\mathbb{E}[T] = \mathbb{E}[S], \quad \text{Var}(S) = \text{Var}(T)$$

$$\frac{S_n - \mathbb{E}[S_n]}{\sigma_{S_n}} \longrightarrow Z \quad N(0,1)$$

$$T = \mu_S + \sigma_S Z \qquad f_S(x) \approx f_T(x)$$

$$\text{VaR}_p(X) \approx \text{VaR}_p(T)$$

$$\text{VaR}_p(T) = \mu_S + \sigma_S \text{VaR}_p(Z) \qquad \text{TVaR}_p(X) \approx \text{TVaR}_p(T)$$

$$\text{TVaR}_p(T) = \mu_S + \sigma_S \text{TVaR}_p(Z)$$

(2) LogNormale

$$\mathbb{E}[S] = \mathbb{E}[T] = e^{\mu + \frac{\sigma^2}{2}}$$

$$\mathbb{E}[S^2] = \mathbb{E}[T^2] = e^{2\mu + 2\sigma^2}$$

$$\mathbb{P}(S > x) \simeq \mathbb{P}(T > x) = 1 - \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right)$$

(3) Gamma translaté $\quad T \sim \text{Gamma}(\alpha, \beta)$

$$x_0 + T$$

Si masse à zero pas élevé

$$\mathbb{E}[S] = x_0 + \mathbb{E}[T] \qquad \gamma(T) = \frac{\mathbb{E}[(T - \mathbb{E}[T])^3]}{(\text{Var}(T))^{3/2}}$$

$$\text{Var}(S) = \text{Var}(T)$$

$$\gamma(S) = \gamma(T)$$

$$\mathbb{E}[(T - \mathbb{E}[T])^3] = \mathbb{E}\left[\sum_{i=0}^{3} \binom{3}{i} T^i (-\mathbb{E}[T])^{3-i}\right]$$

$$= \mathbb{E}\left[-\left(\alpha/\beta\right)^3 + 3\left(\frac{\alpha}{\beta}\right)^2 T - 3T^2\left(\frac{\alpha}{\beta}\right) + T^3\right]$$

$$= -\frac{\alpha^3}{\beta^3} + \frac{3\alpha^3}{\beta^3} - \frac{3\alpha^2(\alpha+1)}{\beta^3} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$$

$$= \frac{-\cancel{\alpha^3} + \cancel{3\alpha^3} - \cancel{3\alpha^3} + \cancel{3\alpha^2} + \cancel{\alpha^3} - \cancel{3\alpha^2} + 2\alpha}{\beta^3} = \frac{2\alpha}{\beta^3}$$

$$\gamma(T) = \frac{2\alpha}{\beta^3} \cdot \left(\frac{\beta^2}{\alpha}\right)^{3/2} = \frac{2\alpha}{\cancel{\beta^3}} \frac{\cancel{\beta^3}}{\alpha^3} = \frac{2}{\alpha^2}$$

$$\alpha = \frac{4}{(\gamma(S))^2}, \quad \beta = \frac{2}{\gamma(S)\sqrt{\text{Var}(S)}}, \quad x_0 = \mathbb{E}[S] - \frac{2\sqrt{\text{Var}(S)}}{\gamma(S)}$$

Si on a une masse élevé à zero: $\quad S' = \{S \mid S > 0\}$

$$q = \mathbb{P}(S > 0) \quad \text{et} \quad 1 - q = \mathbb{P}(S = 0)$$

$$\mathbb{P}(S \leq x) = \mathbb{P}(S = 0) + \mathbb{P}(S > 0)\,\mathbb{P}(S \leq x \mid S > 0)$$

$$f_S(x) = 1-q + q\, f_{S'}(x) \simeq 1-q + q\, f_{x_0+T}(x)$$

$$\alpha = \frac{4}{(\gamma(S'))^2}, \quad \beta = \frac{2}{\gamma(S')\sqrt{Var(S')}}, \quad x_0 = \mathbb{E}[S'] - \frac{2\sqrt{Var(S')}}{\gamma(S')}$$

* Propriétés de prime

- (1) $\rho(X) \leq \sup\{X\}$ — Sur charge
- (2) $\rho(X) \geq \mathbb{E}[X]$ = prime pure + marge — non-négative
- (3) $\rho(X+c) = \rho(X) + c$ — Invariance par translation
- (4) $\rho(c) = c$ — Constante
- (5) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ — Sous-additive
- (6) $\rho(X+Y) = \rho(X) + \rho(Y)$ — additive
- (7) $\rho(Xc) = c\,\rho(X)$ — Homogénéité
- (8) $\mathbb{P}(X \leq Y) = 1 \rightarrow \rho(X) \leq \rho(Y)$ — Monotonicité
- $X =_d Y \rightarrow \rho(X) = \rho(Y)$ — Invariance

** Definition :

Cohérente :

$$\begin{cases} \cdot\ \rho(X+c) = \rho(X) + c & \text{Invariance par translation} \\ \cdot\ \rho(Xc) = c\,\rho(X) & \text{Homogénéité} \\ \cdot\ \rho(X+Y) \leq \rho(X) + \rho(Y) & \text{Sous-additive} \\ \cdot\ \forall\ X \leq Y \text{ avec prob } 1 \rightarrow \rho(X) \leq \rho(Y) & \text{Monotonicité} \end{cases}$$

Convexe : $\rho(X+c) = \rho(X) + c$ + $X \leq Y$ avec prob 1 $\rightarrow \rho(X) \leq \rho(Y)$

+ $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$

Si ρ est convexe et positive homogène $\longrightarrow$ cohérente

* Prime et propriétés

- $\rho_1(X) = (1+\theta)\,\mathbb{E}[X]$
- $\rho_2(X) = \mathbb{E}[X] + \theta\,\mathrm{Var}(X)$
- $\rho_3(X) = \mathbb{E}[X] + \theta\,\sigma_X$
- $\rho_4(X) = VaR_p(X)$
- $\rho_5(X) = TVaR_p(X)$
- $\rho_6(X) = \frac{1}{a}\ln\left(M_X(a)\right)$
- $\rho_8(X) = TVaR_p(W_n) = TVaR_p\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right)$

$$\frac{1}{n} TVaR_p(X_1 + X_2 + \dots + X_n) \leq \frac{1}{n}\left[TVaR_p(X_1) + \dots + TVaR_p(X_n)\right]$$

$$\frac{1}{n} TVaR_p(X_1 + \dots + X_n) \leq TVaR_p(X)$$