EE 451 Project 2 Report

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09/11/2023

1. DH Parameters

As shown in Figure 1, the coordinate frames of each joint are marked. Since l_0 is constant, its frame is not taken into consideration for DH parameter calculation.

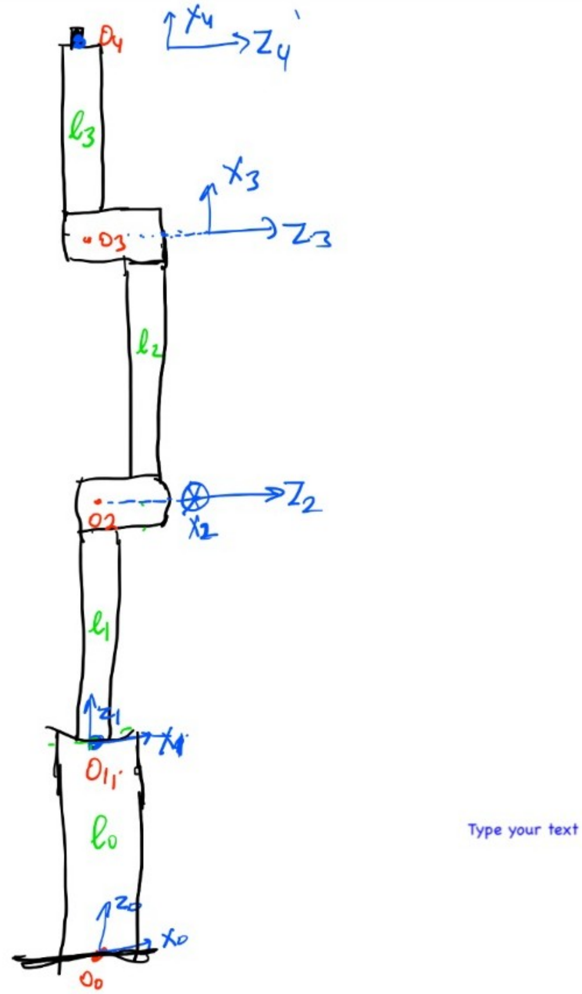

FIGURE 1. Coordinate frames of each joint

The DH parameters of this robot are below.

i	θ_i	d_i	a_i	α_i
1	$\theta_1 + \frac{\pi}{2}$	l_1	0	$\frac{\pi}{2}$
2	$\theta_2 + \frac{\pi}{2}$	0	l_2	0
3	θ_3	0	l_3	0

TABLE 1. DH Parameters

With this table, we can derive the homogenous transformation matrix A. A_1^0 is simply rotation around z axis then translation along z axis by l_1 and the rotation around x axes.
$A_1^0 = R_{z,\theta_1+\frac{\pi}{2}} \times T_{z,l_1} \times R_{x,\frac{\pi}{2}}$.

$$A_1^0 = \begin{bmatrix} -\sin(\theta_1) & 0 & \cos(\theta_1) & 0 \\ \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A_2^1 is simply rotation around z axis then translation along x axis by l_2.

$$A_2^1 = \begin{bmatrix} -\sin(\theta_2) & -\cos(\theta_2) & 0 & -l_2\sin(\theta_2) \\ \cos(\theta_2) & -\sin(\theta_2) & 0 & l_2\cos(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A_3^2 is simply rotation around z axis then translation along x axis by l_3.

$$A_3^2 = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & l_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & l_3\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The kinematic map A is the multiplication of these three matrices.
A $= A_1^0 \times A_2^1 \times A_3^2$

$$A_3^0 = \begin{bmatrix} \sin(\theta_1)\sin(\theta_2+\theta_3) & \sin(\theta_1)\cos(\theta_2+\theta_3) & \cos(\theta_1) & l_3\sin(\theta_1)\sin(\theta_2+\theta_3) + l_2\sin(\theta_1)\sin(\theta_2) \\ -\cos(\theta_1)\sin(\theta_2+\theta_3 & -\cos(\theta_1)\cos(\theta_2+\theta_3) & \sin(\theta_1 & -l_3\cos(\theta_1)\sin(\theta_2+\theta_3) - l_2\cos(\theta_1)\sin(\theta_2) \\ \cos(\theta_2+\theta_3) & -\sin(\theta_2+\theta_3) & 0 & l_3\cos(\theta_2+\theta_3) + l_2\cos(\theta_2) + l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The workspace of the robot is defined by the coordinates:

$$\begin{aligned} x &= l_3\sin(\theta_1)\sin(\theta_2+\theta_3) + l_2\sin(\theta_1)\sin(\theta_2), \\ y &= -l_3\cos(\theta_1)\sin(\theta_2+\theta_3) - l_2\cos(\theta_1)\sin(\theta_2), \\ z &= l_3\cos(\theta_2+\theta_3) + l_2\cos(\theta_2) + l_1 + l_0 \end{aligned}$$

The workspace of the robot is the last column of the kinematic map A but since this robot has a base link, this should also considered. It's clear that the base link only contributes to the z axes. By considering the link lengths and θ limits, the exact workspace of the robot can defined.

2. Forward Kinematics

Forward kinematics in robotics is a mathematical and computational technique used to determine the position and orientation of a robot's end-effector (e.g., its gripper or tool) in the robot's workspace based on the joint angles or joint variables.
Let's have a point in the frame of the end effector, $\mathbf{p}^3$, given by

$$\mathbf{p}^3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Its homogeneous representation is

$$\tilde{\mathbf{p}}^3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ 1 \end{bmatrix}$$

We can find the equivalent of this point with respect to the base frame $\mathbf{p}^0$ using the kinematic map A_3^0:

$$\tilde{\mathbf{p}}^0 = A_3^0 \cdot \tilde{\mathbf{p}}^3$$

Where $\tilde{\mathbf{p}}^0$ is the homonegeous representation of point p^0.

3. Inverse Kinematics

Inverse kinematics is a field of robotics and computer graphics that deals with calculating the joint parameters needed to place the end effector of a robotic arm at a desired position and orientation. This is the inverse problem of forward kinematics, where the pose (position and orientation) of the end effector is determined based on given joint parameters.

Let's assume we want the end effector to reach a point $\mathbf{p} = [x, y, z]$. Given the kinematic map of the robot, we can calculate the necessary joint movements by setting the last column of the kinematic map equal to the homogeneous representation of the point $\tilde{\mathbf{p}}^0$, which is $[x, y, z, 1]^T$. By solving the resulting equations, we can find the necessary joint angles for rotary joints and the distances for prismatic joints. The equations can be set up as follows:

$$A_n^0 = \begin{bmatrix} \cdots & x \\ \cdots & y \\ \cdots & z \\ \cdots & 1 \end{bmatrix}$$

where the last column of the matrix A_n^0 is equal to $[x, y, z, 1]^T$.

In this case, this equation becomes;

$$\begin{bmatrix} l_3 \sin(\theta_1) \sin(\theta_2 + \theta_3) + l_2 \sin(\theta_1) \sin(\theta_2) \\ -l_3 \cos(\theta_1) \sin(\theta_2 + \theta_3) - l_2 \cos(\theta_1) \sin(\theta_2) \\ l_3 \cos(\theta_2 + \theta_3) + l_2 \cos(\theta_2) + l_1 + l_0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Solving this system of equations will yield the joint parameters required to achieve the desired end effector position.

By solving the equations we get the below inverse kinematic equations;

$$\theta_1 = -\arctan\left(\frac{x}{y}\right)$$

$$\theta_3 = \arccos\left(\frac{(2z-1)^2}{2} + \frac{x^2 + y^2}{2l^2}\right)$$

$$\theta_2 = \arctan\left(\frac{2\sqrt{x^2 + y^2}}{2z - 1}\right) - \frac{\theta_3}{2}$$

By putting the x, y, z values, we can get the necessary joint angles.