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COMPUTABILITY OF GLOBAL SOLUTIONS TO FACTORABLE NONCONVEX PROGRAMS: PART I – CONVEX UNDERESTIMATING PROBLEMS ★

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For nonlinear programming problems which are factorable, a computable procedure for obtaining tight underestimating convex programs is presented. This is used to exclude from consideration regions where the global minimizer cannot exist.

1. Introduction

Methods for finding global solutions of not-necessarily-convex programming problems have been historically mainly of academic interest since their implementation usually involves aspects which are not computable (such as the supplying of the convex hull of a constraint set given by algebraic inequalities) or are computationally horrendous (such as the computation of the problem functions at all points in an n-dimensional lattice). A history of efforts in this area is contained in [4]. The few implementable approaches ([1, 2 and 5]) are for separable programming problems and employ branch-and-bound methods in conjunc tion with underestimating convex problems. Since it is possible to convert just about any nonlinear programming problem into an "equivalent" separable programming problem [3], it is in principle possible to obtain global solutions to the general problem. The price paid for this is an increase in dimensionality and in the number of constraints. One element missing from the usual branch-and-bound approach when applied to the

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solution of problems where the solution variables are infinitely divisible is that convergence is established only in the limit. Advantage is not taken of the knowledge of a point that is probably the solution. Part II of this paper will expand on this point.

A general method for obtaining a global solution to a nonlinear programming problem is outlined in Section 2. The definition of factorable functions (to which class the problem functions are assumed to belong) and the procedure for obtaining underestimating convex and overestimating concave functions for them are discussed in Section 3. The algorithmic details on how to compute the convex envelopes of functions of a single variable are contained in Section 4. Section 5 contains a rule for successively dividing up the regions over which the convex underestimating problems are computed. With this rule a theorem is proved showing the "tightness" of the convex underestimating problems. The final section contains a summary and discussion of the paper.

2. Outline of the global solution method

Let v^* denote the solution value of

$$\text{(A)} \qquad \begin{array}{ll} \text{minimize} & f(x)\,, \\ \text{subject to} & x \in \mathrm{R} \subseteq \mathrm{E}^n\,. \end{array}$$

Let $\epsilon > 0$ be the acceptable amount by which an estimate of v^* can differ from it. Set $k = 1$, $R^{1,1} = R$.

Iteration k $(k \geqslant 1)$

It is assumed that at iteration k there are k regions $\{R^{k,l}\}$ (for $l = 1, \dots, k$) defining k programs (for $l = 1, \dots, k$)

$$(\mathrm{A}^{k,l}) \qquad \begin{array}{ll} \text{minimize} & f(x)\,, \\ \text{subject to} & x \in R^{k,l} \subseteq \mathrm{E}^n\,. \end{array}$$

each with minimum value $v_*^{k,l}$ with the property that

$$v^* = \min_{l} \{v_*^{k,l}\}\,. \tag{1}$$

Step 1. Find $x^{k,l} \in R^{k,l}$ (possibly available from previous iterations), an estimate of a solution point of $(A^{k,l})$ (for $l = 1, ..., k$).

Step 2. Find $v^{k,l}$ such that $v^{k,l}$ is a lower bound on $v_*^{k,l}$, for $l = 1, ..., k$. (Such a value may be already available.) Let $K = K(k)$ be an integer such that

$$v^{k,K} = \min [v^{k,1}, ..., v^{k,k}] \ .$$

Convergence condition

If $f(x^{k,K}) \leqslant v^{k,K} + \epsilon$, then the algorithm terminates and $x^{k,K}$ is considered the global solution point for (A).

If the convergence condition is not met, then iteration $k + 1$ is performed where for $l = 1, ..., k$, $l \neq K$, programs $(\mathrm{A}^{k+1,l})$ are identical with $(\mathrm{A}^{k,l})$, program $(\mathrm{A}^{k+1,K})$ has the constraint set $R^{k+1,K}$, and program $(\mathrm{A}^{k+1,k+1})$ has the constraint set $R^{k+1,k+1}$, where these two sets are a partition of $R^{k,K}$ such that

$$\text{(S)} \qquad R^{k+1,K} \cup R^{k+1,k+1} = R^{k,K} \ .$$

Comments

A simple inductive argument shows that the algorithm is consistent, i.e., that property (1) holds for all k. This is because it holds for $k = 1$, and the partition requirement (S) implies that if it holds for k it must hold for $k + 1$.

The convergence condition is a natural one, since it follows directly from the assumption on the $v^{k,l}$'s that

$$f(x^{k,K}) \leqslant v^{k,K} + \epsilon \leqslant v^* + \epsilon \ .$$

In order for any specific realization of this general method to work, it must be possible to execute both steps 1 and 2. In this paper only step 2 is considered, the computation of a lower bound on the minimum value of problem $A^{k,l}$. For a class of programming problems called "factorable," a sequence of convex underestimating programs is generated. Solution of these convex underestimating problems yields the numbers $\{v^{k,l}\}$. Rules for partitioning $R^{k,K}$ as required in (S) are given so that if the convergence condition were never employed, the limiting sequence of values $v^{k,K(k)}$ would be v^*. Care has to be taken that this be the case since the algorithm would never terminate without this property.

The details on the implementation of step 1 and another method for computing the $v^{k,l}$'s will be considered in Part II.

3. Procedure for creating underestimating convex programs

In this section are presented the details for the generation of convex underestimating programming problems. Basic to this is the notion of a convex underestimating function. Suppose a function has the form

$$T[t(x)] + U[u(x)] \cdot V[v(x)] \tag{2}$$

(where $t(x)$, $u(x)$ and $v(x)$ are continuous functions of n variables and $T(\cdot)$, $U(\cdot)$ and $V(\cdot)$ are continuous functions of a single variable) and it is desired to find a convex function which bounds it below for all x in some convex set S when $t(x)$, $u(x)$ and $v(x)$ are restricted in range.

Assume that there are available convex functions $c_t(x)$, $c_u(x)$ and $c_v(x)$, concave functions $C_t(x)$, $C_u(x)$ and $C_v(x)$, and some numbers a_t, a_u, a_v, b_t, b_u and b_v for which it is known that for x in S,

$$c_t(x) \leqslant t(x) \leqslant C_t(x)\,,$$
$$c_u(x) \leqslant u(x) \leqslant C_u(x)\,,$$
$$c_v(x) \leqslant v(x) \leqslant C_v(x)\,,$$

and for which it is required that

$$a_t \leqslant t(x) \leqslant b_t\,,$$
$$a_u \leqslant u(x) \leqslant b_u\,,$$

and

$$a_v \leqslant v(x) \leqslant b_v\,.$$

A fundamental assumption of this paper is that it is possible to provide, for any function of a single variable on an interval, its convex and concave envelopes. The convex envelope of a function over a closed convex set is the highest convex function which everywhere underestimates the function, and the concave envelope is the lowest concave function which everywhere overestimates the function. The convex envelope of $T(\cdot)$ in the interval $[a_t, b_t]$ is denoted by $e_T(\cdot)$ and the concave envelope by $E_T(\cdot)$. The envelopes are functions of the interval as well as the function $T(\cdot)$ but for simplicity of notation this dependence will not be made explicit. Similar notation is used for the other functions.

Compute a point at which each function achieves its minimum on its interval domain, and a point at which each function achieves its maximum. (Note that the ability to do this is implied by the availability of the convex and concave envelopes.) That is, let $z^T_{\min}$, $z^T_{\max}$, A_T, B_T be the values where

$$\begin{aligned} A_T &= T(z^T_{\min}) = \inf_{a_t \leqslant z \leqslant b_t} T(z)\,, \\ B_T &= T(z^T_{\max}) = \sup_{a_t \leqslant z \leqslant b_t} T(z)\,. \end{aligned} \tag{3}$$

Similar notation is used for the other problem functions.

The lower bounding convex function of $T[t(x)]$ for $x \in S \cap \{x \mid a_t \leqslant t(x) \leqslant b_t\}$ is

$$T[t(x)] \geqslant e_T[\mathrm{mid}\{c_t(x), C_t(x), z^T_{\min}\}]\ , \tag{4}$$

where $\mathrm{mid}(z_1, z_2, z_3)$ is the function which selects the middle value of three scalars.

The proof that the function in (4) is convex and that it underestimates $T[t(x)]$ in the region of interest is left to the reader. It is important to notice two other properties of this function. It is a "tight" underestimating function in that if the interval $[a_t, b_t]$ were shrunk, the underestimating function would be higher; and if $a_t = b_t$, equality would hold in (4). Furthermore, if $T(\cdot)$ is continuously differentiable on $[a_t, b_t]$ and if $c_t(x)$ and $C_t(x)$ are also continuously differentiable, then so is the convex underestimating function for $T[t(x)]$ in (4).

The concave overestimating function for $T[t(x)]$ is

$$T[t(x)] \leqslant E_T[\mathrm{mid}\{c_t(x), C_t(x), z^T_{\max}\}]$$

for $x \in S \cap \{x \mid a_t \leqslant t(x) \leqslant b_t\}$.

In Table 1 are presented some convex and concave envelopes and

Table 1
Pertinent information relating to the convex and concave envelopes of three different functions of s single variable

Function	Interval	Convex envelope and minimizing point	Concave envelope and maximizing point
$T(z)$	$[a, b]$	$e(z)$, $z_{\min}$	$E(z)$, $z_{\max}$
z^2	$[a, b]$	$e(z) = z^2$	$E(z) = (a+b)z - ab$
		$z_{\min} = a$ if $a \geqslant 0$, $z_{\min} = b$ if $b \leqslant 0$, $z_{\min} = 0$ otherwise	$z_{\max} = a$ if $a^2 \geqslant b^2$, $z_{\max} = b$ if $b^2 > a^2$
e^z	$[a, b]$	$e(z) = e^z$, $z_{\min} = a$	$E(z) = \dfrac{e^b - e^a}{b-a} z + \dfrac{be^a - ae^b}{b-a}$ $z_{\max} = b$
$\Phi(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} e^{-t^2/2}\, dt$	$[-3.45, 1.55]$	$e(z) = \begin{cases} \Phi(z) & \text{when } -3.45 \leqslant z \leqslant -0.71 \\ 0.3101z + 0.4590 & \text{when } -0.71 < z \leqslant 1.55 \end{cases}$ $z_{\min} = -3.45$	$E(z) = \begin{cases} 0.1903z + 0.6567 & \text{when } -3.45 \leqslant z \leqslant 1.217 \\ \Phi(z) & \text{when } 1.217 < z \leqslant 1.55 \end{cases}$ $z_{\max} = 1.55$

their minimizing and maximizing points. An example of the previous development is given below.

Example 1. Suppose $T(z) = z^2$, $t(x) = e^{x_1} + x_2$, $S = \{x \mid 0 \leqslant x_1 \leqslant 1, -20 \leqslant x_2 \leqslant 30\}$, $a_t = -1$, and $b_t = 2$. In $[-1,2]$ the convex envelope of z^2 is z^2, i.e., $e_T(z) = z^2$ and $z^T_{\min} = 0$. The convex underestimating function for $e^{x_1} + x_2$ in S is $c_t(x) = e^{x_1} + x_2$ and its concave overestimate is $C_t(x) = 1 + (1.718)x_1 + x_2$.

Thus a convex underestimating function for $[e^{x_1} + x_2]^2$ in $S \cap \{x \mid -1 \leqslant e^{x_1} + x_2 \leqslant 2\}$ is

$$[\mathrm{mid}\{1.718x_1 + 1 + x_2, e^{x_1} + x_2, 0\}]^2 .$$

Finding the underestimating function for the product term is more complicated. First note that the convex envelope of the function $U \cdot V$ in the rectangle $A_U \leqslant U \leqslant B_u$, $A_V \leqslant V \leqslant B_V$ is

$$\max[B_V U + B_U V - B_U B_V, A_V U + A_U V - A_U A_V] .$$

Assume without loss of generality that $A_V \geqslant 0$, $A_U < 0$, and $B_U \geqslant 0$. It then follows that for x in

$$S \cap \{x \mid a_u \leqslant u(x) \leqslant b_u, a_v \leqslant v(x) \leqslant b_v\} ,$$

$$\begin{aligned} U[u(x)] \cdot V[v(x)] \geqslant \max\{&B_V e_U[\mathrm{mid}\{c_u(x), C_u(x), z^U_{\min}\}] \\ &+ B_U e_V[\mathrm{mid}\{c_v(x), C_v(x), z^V_{\min}\}] - B_U B_V , \\ &(A_V e_U[\mathrm{mid}\{c_u(x), C_u(x), z^U_{\min}\}] \\ &+ A_U E_V[\mathrm{mid}\{c_v(x), C_v(x), z^V_{\max}\}] - A_U A_V\} . \end{aligned} \tag{5}$$

Since the maximum of two convex functions is convex, the complete convex underestimating function for the product term has been obtained.

This underestimating function is not everywhere differentiable. There are several ways of handling this by altering the programming problem to create an "equivalent" one which has all continuously differentiable functions and is a convex programming problem. The simplest way involves the addition of at most one more variable to the problem and more inequality constraints. Suppose the functions $f(x)$ and $\{f_i(x)\}$ for

$i = 1, ..., m$ are convex, but not necessarily continuously differentiable. It is well known that the problem

$$\underset{x \in \mathrm{E}^n}{\text{minimize}} \quad f(x) ,$$

$$\text{subject to} \quad f_i(x) \leqslant 0 \quad \text{for } i = 1, ..., m$$

is equivalent to the problem

$$\underset{x \in \mathrm{E}^{n+1}}{\text{minimize}} \quad x_{n+1} ,$$

$$\text{subject to} \quad f_i(x) \leqslant 0 \quad \text{for } i = 0, 1, ..., m \ ,$$

where $f_0(x) = f(x) - x_{n+1}$. Suppose further that all the problem functions are continuously differentiable except the I-th one which has the form

$$f_I(x) = F_I(x) + \max[\alpha(x), \beta(x)] \ ,$$

where $F_I(x)$, $\alpha(x)$, and $\beta(x)$ are convex and continuously differentiable. Obviously the problem

$$\underset{x \in \mathrm{E}^{n+1}}{\text{minimize}} \quad x_{n+1} ,$$

$$\text{subject to} \quad f_i(x) \leqslant 0 \quad \text{for } i = 0, 1, ..., m \ , \quad i \neq I \ ,$$

$$F_I(x) + \alpha(x) \leqslant 0, \qquad F_I(x) + \beta(x) \leqslant 0$$

is equivalent to the two problems above and is a convex programming problem involving all continuously differentiable functions.

When the original problem involves continuously differentiable functions, then the method of generating convex underestimating functions yields a convex underestimating program which can be converted using the technique indicated above as needed to a continuously differentiable equivalent convex problem.

The convex underestimating function for (2) is the sum of (4) and (5). Minimizing this sum over S yields a lower bound to (2). A concave upper bounding function for (2) can be obtained in the same way as the above, and maximizing the resulting function over S results in an upper bound.

Fortunately, it is possible to apply the foregoing techniques recursively to obtain an underestimating convex programming problem for any problem which is a factorable programming problem.

Factorable programming problems

A factorable programming problem is one which is written in the following way:

$$\underset{x \in \mathrm{E}^n}{\text{minimize}} \quad X^N(x)\,,$$

$$\text{subject to} \quad -\infty < a_i \leqslant X^i(x) \leqslant b_i < +\infty \quad \text{for } i = 1, ..., N-1\,,$$

where $X^i(x) = x_i$ for $i = 1, ..., n$, and the remainder of the $X^i(x)$'s are defined recursively as follows:

Given $X^p(x)$ for $p = 1, ..., i-1$, then for $i = n+1, ..., N$,

$$X^i(x) = \sum_{p=1}^{i-1} T_p^i[X^p(x)] + \sum_{p=1}^{i-1} \sum_{q=1}^{p} V_{q,p}^i[X^p(x)] \cdot U_{p,q}^i[X^q(x)]\,, \tag{6}$$

where T's, U's, and V's are functions of a *single* variable.

This is the factorable programming representation of problem (A) of Section 2, where $f(x)$ corresponds to $X^N(x)$ and R to $\{x | a_i \leqslant X^i(x) \leqslant b_i,$ for $i = 1, ..., N-1\}$. Just about any function used for computational purposes can be put into a factorable programming problem. An example of a function which cannot is the gamma distribution function

$$\int_0^a \frac{(x_1 x_2)^{x_2}}{\Gamma(x_2)} t^{x_2 - 1}\, \mathrm{e}^{-x_1 x_2 t}\, \mathrm{d}t\ .$$

For the $X^i(x)$'s of factorable programming problems it is possible to use the techniques just described to compute recursively convex underestimating and concave overestimating functions. Some notation needs to be established first.

Let $\mathrm{e}_{T,p}^i(\cdot)/E_{T,p}^i(\cdot)$ denote the convex/concave envelope of $T_p^i(\cdot)$ on the interval $[a_p, b_p]$. Let $z_{\min,p}^{T,i}/z_{\max,p}^{T,i}$ denote a minimizer/maximizer of $T_p^i(\cdot)$ on $[a_p, b_p]$ and let

$$A_{T,p}^i = T_p^i(z_{\min,p}^{T,i}), \qquad B_{T,p}^i = T_p^i(z_{\max,p}^{T,i}) .$$

Similar notation holds for the factors of the product terms. There V replaces T above in one case, and U and q replace T and p in the other.

Let $c^i(x)/C^i(x)$ denote the convex/concave functions it is desired to compute which underestimate/overestimate $X^i(x)$. Now

$$c^i(x) = x_i = C^i(x) \quad \text{for } i = 1, ..., n .$$

Assume that $c^p(x)$, $C^p(x)$ are known for $p = 1, ..., i-1$. Then for any x satisfying $a_p \leqslant X^p(x) \leqslant b_p$ for $p = 1, ..., i-1$,

$$\begin{aligned} X^i(x) &\geqslant c^i(x) \\ &= \sum_{p=1}^{i-1} e_{T,p}^i [\operatorname{mid}\{c^p(x), C^p(x), z_{\min,p}^{T,i}\}] \\ &+ \sum_{p=1}^{i-1} \sum_{q=1}^{p} \max\{B_{V,p}^i \cdot e_{U,q}^i [\operatorname{mid}\{c^q(x), C^q(x), z_{\min,q}^{U,i}\}] \\ &\qquad + B_{U,q}^i \cdot e_{V,p}^i [\operatorname{mid}\{c^p(x), C^p(x), z_{\min,p}^{V,i}\}] \\ &\qquad - B_{U,q}^i \cdot B_{V,q}^i, A_{V,p}^i \cdot e_{U,q}^i [\operatorname{mid}\{c^q(x), \\ &\qquad C^q(x), z_{\min,q}^{U,i}\}] + A_{U,q}^i \cdot E_{V,p}^i [\operatorname{mid}\{c^p(x), \\ &\qquad C^p(x), z_{\max,p}^{V,i}\}] - A_{U,q}^i \cdot A_{V,p}^i\} . \end{aligned} \tag{7}$$

(Here, without loss of generality, it has been assumed that $A_{U,q}^i < 0$, all other bounds are nonnegative.) The upper bounding concave function is determined in an analogous way.

Several comments are in order. In general, every convex/concave function need not be computed. It is not necessary, for example, to compute $C^N(x)$. If a constraint never fails to be satisfied, its convex (concave) under (over) estimating function need not be computed unless needed for an $X^i(x)$ which contains that constraint in its definition.

For most problems, the a_i's and b_i's are given or known from natural conditions on the problem. If some of them are not given and are needed (for some $i > n$), they can be computed recursively from the bounds assumed available on the first n X^i's (which are, of course, the original problem variables).

The underestimating convex programming problem is then:

$$\begin{aligned} &\text{minimize} && c^N(x)\,, \\ &\text{subject to} && c^i(x) \leqslant b_i \quad \text{for } i = 1, ..., N-1\,, \\ &&& C^i(x) \geqslant a_i \quad \text{for } i = 1, ..., N-1\,. \end{aligned}$$

Clearly, any point feasible to problem (A) is feasible to this one, and the objective function value is no greater for the convex underestimating problem.

Example 2. To illustrate the preceding general rules, consider the problem of defining the constraint region given by the set of points satisfying

$$(\mathrm{e}^{x_1} - x_2^2)(x_1 x_2) \leqslant 0$$

in the rectangle defined by $-1 \leqslant x_1 \leqslant 3$, $-2 \leqslant x_2 \leqslant 3$. The region of feasible points is shown in Fig. 1. There are essentially four different regions of points feasible to the above constraint. The best that can be hoped for as a tight underestimating convex function is one which would define the convex hull of the feasible region. The convex hull is indicated in Fig. 1. The set of points defined as feasible by using the rules indicated previously are to be compared with the convex hull.

The above function can be put into the following canonical factorable form. For notational convenience, X^j is used instead of $X^j(x)$. Let

$$X^1 = x_1, \qquad X^2 = x_2\,, \tag{8}$$

$$X^3 = (X^1 \mathrm{e}^{X^1})(X^2), \qquad X^4 = X^3 + (-X^1)(X^2)^3\,. \tag{9}$$

This form is not unique.

Now

$$-0.3679 = -1\mathrm{e}^{-1} \leqslant x_1 \mathrm{e}^{x^1} \leqslant 3\mathrm{e}^3 = 60.257$$

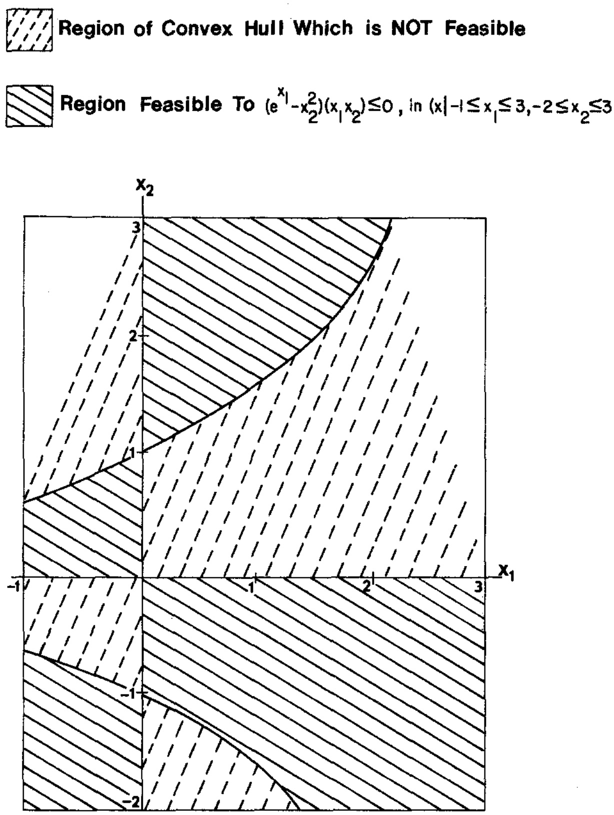

Fig. 1. Points feasible to inequality, with those in convex hull but not feasible.

and

$$-2 \leqslant x_2 \leqslant 3 .$$

The two lower bounding function deriving from this are

$$(x_1 e^{x_1})x_2 \geqslant -2x_1 e^{x_1} - 0.3679x_2 - 0.7358$$

and

$$(x_1 e^{x_1})x_2 \geqslant 3x_1 e^{x_1} + 60.257x_2 - 180.771 .$$

Since $-2x_1 e^{x_1}$ is concave in the interval $[-1,3]$, its convex envelope is the line connecting $(-1,2e^{-1})$, $(3,-6e^3)$. There thus obtains

$$-2x_1 e^{x_1} \geqslant -30.3122x_1 - 29.5765 \ .$$

The factors in the second term in the definition of X^4 have bounds

$$-3 \leqslant -x_1 \leqslant 1, \qquad -8 \leqslant x_2^3 \leqslant 27 \ .$$

Two inequalities obtain:

$$(-x_1)(x_2^3) \geqslant 8x_1 - 3x_2^3 - 24$$

$$(-x_1)(x_2^3) \geqslant -27x_1 + x_2^3 - 27 \ .$$

The convex envelope of x_2^3 in $[-2,3]$ is $\xi(x_2)$, where

$$\xi(x_2) = \begin{cases} 3x_2 - 2 & \text{when } -2 \leqslant x_2 \leqslant 1 \ , \\ x_2^3 & \text{when } \ 1 < x_2 \leqslant 3 \ . \end{cases}$$

Notice that $\xi(x_2)$ is once but not twice continuously differentiable at $x_2 = 1$. The computation of a convex envelope with separate functional forms can be made an automatic computerized procedure.

The convex envelope of $-3x_2^3$ in $[-2,3]$ is $\delta(x_2)$, where

$$\delta(x_2) = \begin{cases} -3x_2^3 & \text{when } -2 \leqslant x_2 \leqslant -1.5 \ , \\ -20.25x_2 - 20.25 & \text{when } -1.5 < x_2 \leqslant 3 \ . \end{cases}$$

This is shown in Fig. 2.

Combining all the previous information yields four convex lower bounding functions.

$$\text{I} \quad X^4 \geqslant -30.3122x_1 - 0.3679x_2 - 0.7358 - 29.5765 + 8x_1 + \delta(x_2) - 24$$
$$= -22.3122x_1 - 0.3679x_2 - 54.3122 + \delta(x_2) \ ,$$

$$\text{II} \quad X^4 \geqslant -30.3122x_1 - 0.3679x_2 - 0.7358 - 29.5765 - 27x_1 + \xi(x_2) - 27$$
$$= -57.3122x_1 - 0.3679x_2 - 57.3122 + \xi(x_2) \ ,$$

$$\delta(x_2) = \begin{cases} -3x_2^3 & \text{when} \quad -2 \leq x_2 \leq -1.5 \\ -20.25x_2 - 20.25 & \text{when} \quad -1.5 < x_2 \leq 3 \end{cases}$$

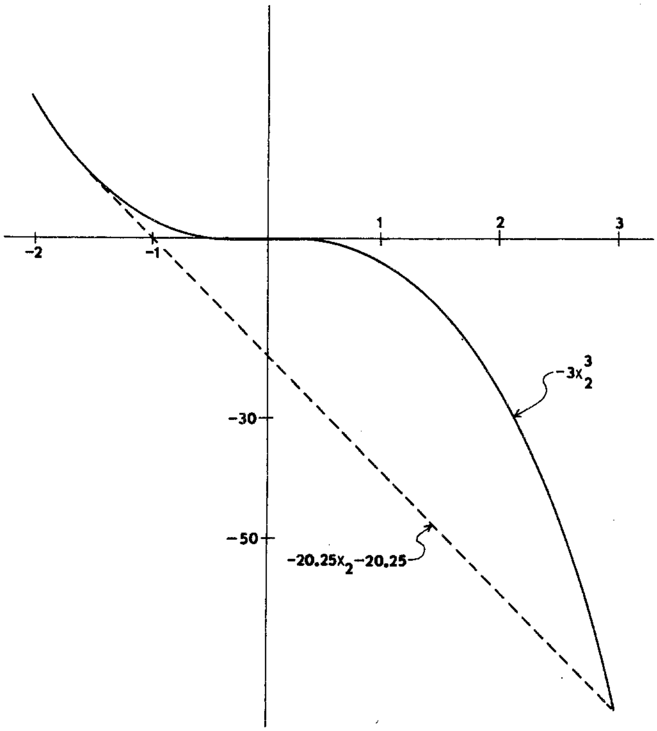

Fig. 2. Convex envelope ($\delta(x_2)$) of $-3x_2^3$ in $[-2, 3]$.

$$\text{III} \quad X^4 \geqslant 3x_1 e^{x_1} + 60.257x_2 - 180.771 + 8x_1 + \delta(x_2) - 24$$
$$= 3x_1 e^{x_1} + 8x_1 + 60.257x_2 - 204.771 + \delta(x_2)$$

$$\text{IV} \quad X_4^4 \geqslant 3x_1 e^{x_1} + 60.257x_2 - 180.771 - 27x_1 + \xi(x_2) - 27$$
$$= 3x_1 e^{x_1} - 27x_1 + 60.257x_2 - 207.771 + \xi(x_2) \; .$$

The relevant boundary of the feasible region to the convex underestimating functions is shown in Fig. 3.

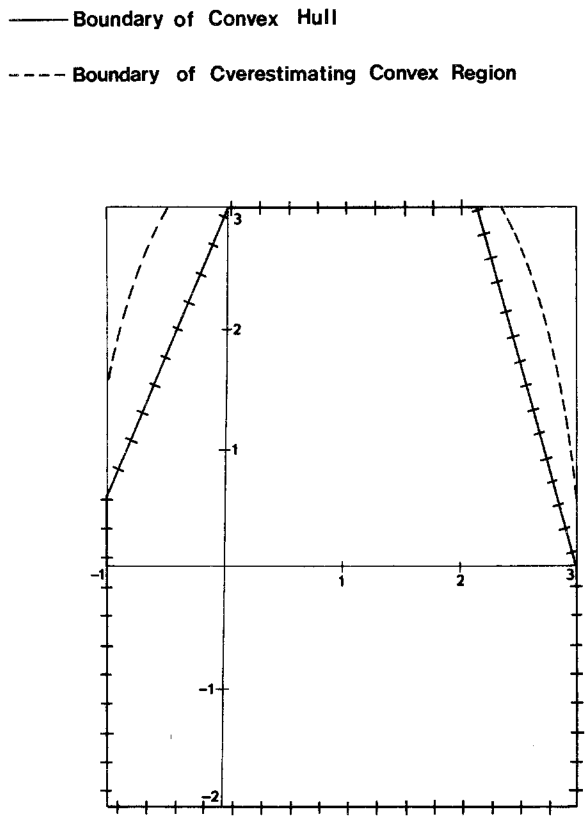

Fig. 3. Comparison of convex hull of example inequality with overestimating convex region.
—— Boundary of convex hull.
- - - - Boundary of overestimating convex region.

4. On computing convex envelopes of functions of a single variable

In order for the method of computing underestimating convex programs to be implementable, it must be possible to compute the convex envelopes of functions of a single variable. In most cases this is very easy to do. The general approach is to decide on the number of different intervals on which the representation of the convex envelope takes a different form. For example, the sin(z) may have as many as five different representations (see Example 3). When the function has convex portions, sometimes the convex envelope is the original function itself for an interval. It is a line for an adjacent one, may revert to the function itself for the next, and so on. The determination of how many different representations are required depends upon the size of the interval and also on the function. A computer program which does this for the sin function has been coded. The trigonometric functions are the most complicated ones.

The next step is to determine the exact point at which the convex envelope changes representation from a line to follow the curve of the function (or vice versa). Presumably bounds on the interval in which this occurs are given by the first step. This is simply the problem of finding a point where the slope of the curve equals the slope of the line connecting the point with a given one. If $T(z)$ is the function of a single variable, and a is a given point at the start of the interval, then the problem is to find a point $t \in [a, b]$ such that

$$\frac{T(t) - T(a)}{t-a} = \frac{\mathrm{d}T(t)}{\mathrm{d}t} = T'(t) \ . \tag{10}$$

This is equivalent to solving the equation

$$T(t) - T(a) - (t-a)T'(t) = 0 \ .$$

Newton's method for this has the form

$$t^{k+1} = t^k + \frac{1}{T''(t^k)} \left\{ \frac{T(t^k) - T(a)}{t^k - a} - T'(t^k) \right\} \ . \tag{11}$$

The literature on solving systems of equations of one variable is extensive and any number of methods can be applied to this problem. The important thing is that for the functions of a single variable, global information is known in an interval, and can be computerized. The traditional methods, with safeguards, can be guaranteed to solve (10) under these conditions. The form of Newton's method given in (11) seems to work very well in the cases tried so far. In Table 1 are some examples of the convex and concave envelopes of a single variable.

Next an example is given of the use of the convex envelope of a single variable and also used to illustrate the general method for obtaining global solutions.

Example 3. The programming problem is (see Fig. 4)

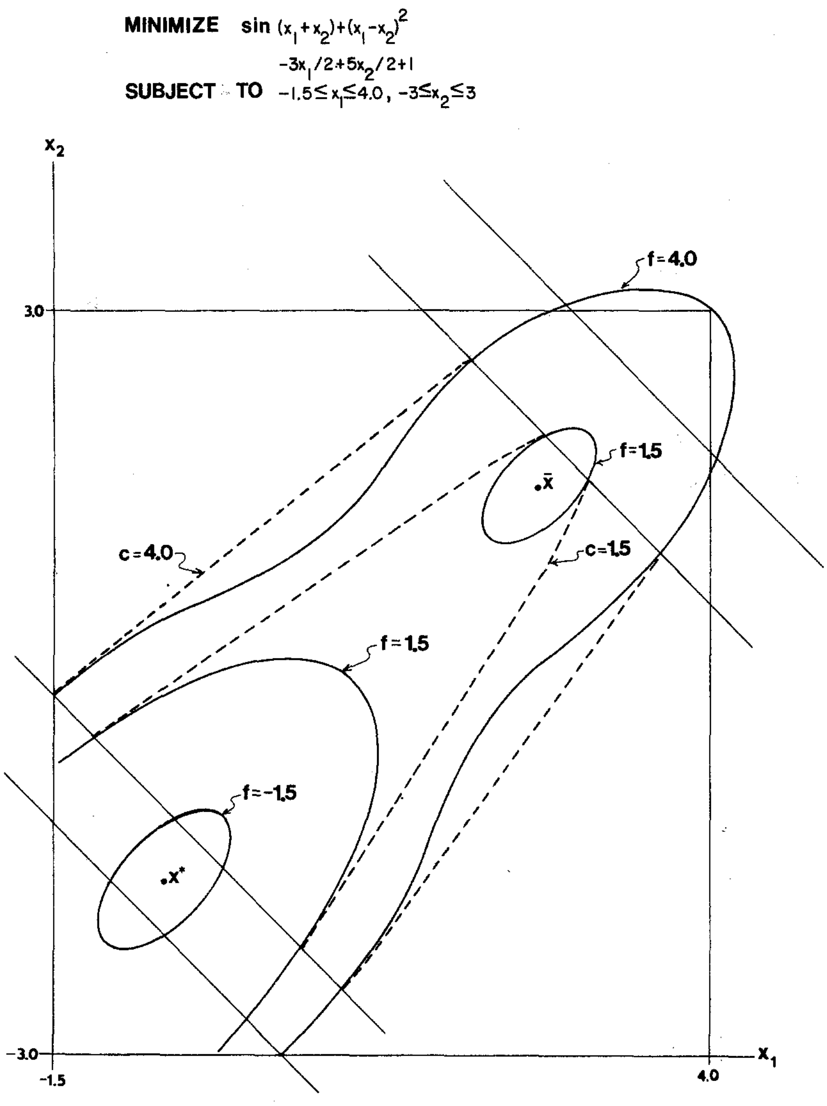

Fig. 4. Solution of problem with two local minimizers.

$$\underset{x \in E^2}{\text{minimize}} \quad \sin(x_1 + x_2) + (x_1 - x_2)^2 - 3x_1/2 + 5x_2/2 + 1\ ,$$

$$\text{subject to} \quad -1.5 \leqslant x_1 \leqslant 4.0, \qquad -3.0 \leqslant x_2 \leqslant 3.0\ .$$

There are two local minimizers to this problem (both unconstrained), one at approximately (–.5472, –1.5472) (which is the global minimizer), the other at approximately (2.5944, 1.5944). In factorable programming form this is

$$\underset{x\in E^2}{\text{minimize}}\ X^5(x) \quad \text{where } X^j(x)=x_j \quad \text{for } j=1,2\ ,$$

$$-4.5 \leqslant X^3(x) = X^1(x) + X^2(x) \leqslant 7.0\ ,$$

$$-4.5 \leqslant X^4(x) = X^1(x) - X^2(x) \leqslant 7.0\ ,$$

and

$$X^5(x) = \sin[X^3(x)] + (X^4(x))^2 - 3X^1(x)/2 + 5X^2(x)/2 + 1\ .$$

Only the convex envelope of the $\sin(X^3)$ for $X^3 \in [-4.5, 7.0]$ is needed. This is (see also Fig. 5)

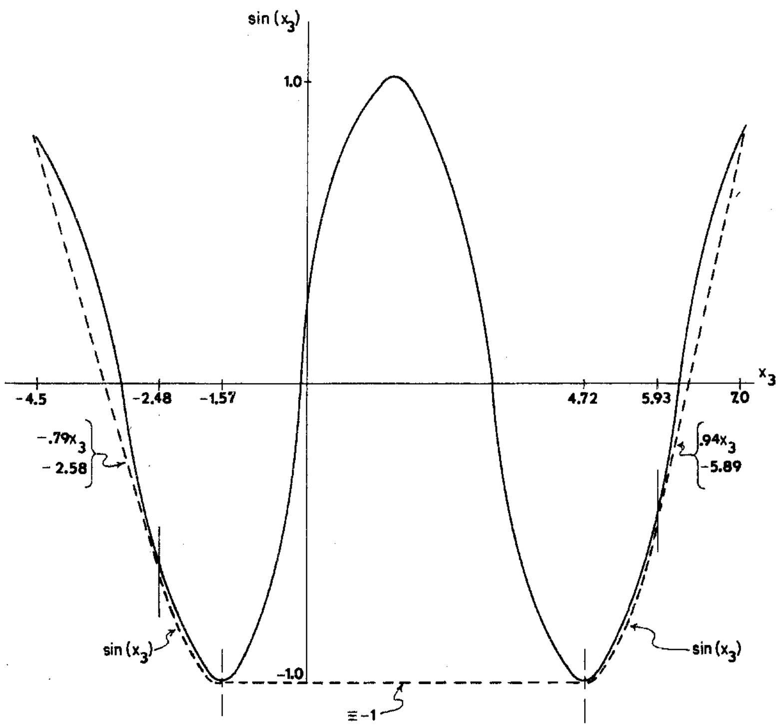

Fig. 5. Convex envelope of $\sin(X^3)$ for $-4.5 \leqslant X^3 \leqslant 7$.

$$
\begin{array}{rl}
-0.79X^3 - 2.58 & \text{for } -4.5 \leqslant X^3 < -2.48\,, \\
\sin(X^3) & \text{for } -2.48 \leqslant X^3 < -1.57\,, \\
-1.0 & \text{for } -1.57 \leqslant X^3 < \;\; 4.72\,, \\
\sin(X^3) & \text{for } \;\; 4.72 \leqslant X^3 < \;\; 5.93\,, \\
0.94X^3 - 5.89 & \text{for } \;\; 5.93 \leqslant X^3 \leqslant \;\; 7.0\,.
\end{array}
$$

Applying the recursive techniques to obtain the convex envelope of the function $X^5(x)$ can easily be done. To simplify the remainder of the presentation, the convex underestimating function can be written simply as a function of the original problem variables as

$$c(x) = Q(x) + (x_1 - x_2)^2 - 3x_1/2 + 5x_2/2 + 1\,,$$

where

$$
Q(x) = \begin{cases}
-0.79(x_1+x_2) - 2.58 & \text{for } -4.5 \leqslant x_1+x_2 < -2.48\,, \\
\sin(x_1+x_2) & \text{for } -2.48 \leqslant x_1+x_2 < -1.57\,, \\
-1.0 & \text{for } -1.57 \leqslant x_1+x_2 < \;\; 4.72\,, \\
\sin(x_1+x_2) & \text{for } \;\; 4.72 \leqslant x_1+x_2 < \;\; 5.93\,, \\
0.94(x_1+x_2) - 5.89 & \text{for } \;\; 5.93 \leqslant x_1+x_2 \leqslant \;\; 7.0\,.
\end{cases}
$$

The solution to $\min_{x \in \mathbf{R}} c(x)$ (which is obtainable by just about any nonlinear programming algorithm since $c(x)$ is convex and continuously differentiable, although it is not twice continuously differentiable everywhere) is $x^* = (-0.5472, -1.5472)$. Since the convex underestimating function agrees with the original function at x^*, we are assured that that point is a global solution to the original problem.

Since there was only one direction in which the function was nonconvex, this problem was solved without the need for further subdivision of the original feasible rectangle into smaller ones. In most problems, of course, the solution will not be obtained so easily.

5. Convergence of the underestimating convex programs to a global solution of the factorable programming problem

As indicated in Section 2, a concept of tightness needs to be established for the convex underestimating scheme. In particular, it is desirable to show that if the rules for splitting the regions to create new programming problems are done properly, the lower bound on the global solution value would tend to the global solution value in the limit.

First, notation needs to be established. The factorable programming representation of Problem ($A^{k,l}$) of Section 2, for $l = 1, ..., k$ is written

$$(A^{k,l}) \qquad \begin{array}{ll} \text{minimize} & X^N(x)\,, \\ \text{subject to} & a_i^{k,l} \leqslant X^i(x) \leqslant b_i^{k,l} \quad \text{for } i = 1, ..., N-1\,, \end{array}$$

where the defining equations (6) still hold.

Using the method of Section 4, a convex underestimating program for each ($A^{k,l}$) is formed. This is called Problem ($P^{k,l}$), and is written

$$(P^{k,l}) \qquad \begin{array}{ll} \text{minimize} & c^{N,k,l}(x)\,, \\ \text{subject to} & a_i^{k,l} \leqslant C^{i,k,l}(x)\,, \quad c^{i,k,l}(x) \leqslant b_i^{k,l} \\ & \text{for } i = 1, ..., N-1\,, \end{array}$$

where the defining equations for $c^{i,k,l}(x)$ are as in (7), except that where necessary the superscripts k, l must be appended there.

Let $y^{k,l}$ denote a solution point for ($P^{k,l}$) and let $v^{k,l}$ denote the solution value. Clearly, since each $P^{k,l}$ is a convex underestimating problem for $A^{k,l}$, then $v^{k,l}$ is lower bound on the solution value of $A^{k,l}$, $v_*^{k,l}$. This is consistent with the general description of the global solution method described in Section 2. It remains to show how the partition, or splitting (S) is done when the problems are written in canonical factorable form.

Consider the following differences:

$$T_p^i[X^p(y^{k,K})] - e_{T,p}^{i,k,K}[\mathrm{mid}\{c^{p,k,K}(y^{k,K}),\, C^{p,k,K}(y^{k,K}),\, z_{\min,p}^{T,i,k,K}\}]$$
$$i = n+1, ..., N; \quad p = 1, ..., i-1; \tag{12}$$

$$U_{p,q}^i[X^q(y^{k,K})] \cdot V_{q,p}^i[X^p(y^{k,K})] +$$
$$-\max\{B_{V,p}^{i,k,K} \cdot e_{U,q}^{i,k,K}[\mathrm{mid}\{c^{q,k,K}(y^{k,K}),\, C^{q,k,K}(y^{k,K}),\, z_{\min,q}^{U,i,k,K}\}]$$
$$+ B_{U,q}^{i,k,K} \cdot e_{V,p}^{i,k,K}[\mathrm{mid}\{c^{p,k,K}(y^{k,K}),\, C^{p,k,K}(y^{k,K}),\, z_{\min,p}^{V,i,k,K}\}]$$
$$- B_{U,q}^{i,k,K} \cdot B_{V,p}^{i,k,K}\,, \tag{13}$$
$$A_{V,p}^{i,k,K} \cdot e_{U,q}^{i,k,K}[\mathrm{mid}\{c^{q,k,K}(y^{k,K}),\, C^{q,k,K}(y^{k,K}),\, z_{\min,q}^{U,i,k,K}\}]$$

$$+A_{U,q}^{i,k,K} \cdot E_{V,p}^{i,k,K} [\text{mid}\{c^{p,k,K}(y^{k,K}), C^{p,k,K}(y^{k,K}), z_{\max,p}^{V,i,k,K}\}]$$

$$-A_{U,q}^{i,k,K} \cdot A_{V,p}^{i,k,K}\}, \quad i = n+1, ..., N, \quad p = 1, ..., i-1; \quad q = 1, ..., p;$$

$$E_{T,p}^{i,k,K} [\text{mid}\{c^{p,k,K}(y^{k,K}), C^{p,k,K}(y^{k,K}), z_{\max,p}^{T,i,k,K}\}] - T_p^i[X^p(y^{k,K})]$$

$$i = n+1, ..., N-1; \quad p = 1, ..., i-1; \tag{14}$$

$$\begin{aligned}
\min\{&B_{U,q}^{i,k,K} \cdot E_{V,p}^{i,k,K} [\text{mid}\{c^{p,k,K}(y^{k,K}), C^{p,k,K}(y^{k,K}), z_{\max,p}^{V,i,k,K}\}] \\
&+A_{V,p}^{i,k,K} \cdot E_{U,q}^{i,k,K} [\text{mid}\{c^{q,k,K}(y^{k,K}), C^{q,k,K}(y^{k,K}), z_{\max,q}^{U,i,K,k}\}] \\
&-A_{V,p}^{i,k,K} \cdot B_{U,q}^{i,k,K}, \\
&B_{V,p}^{i,k,K} \cdot E_{U,q}^{i,k,K} [\text{mid}\{c^{q,k,K}(y^{k,K}), C^{q,k,K}(y^{k,K}), z_{\max,q}^{U,i,k,K}\}] \\
&+A_{U,q}^{i,k,K} \cdot e_{V,p}^{i,k,K} [\text{mid}\{c^{p,k,K}(y^{k,K}), C^{p,k,K}(y^{k,K}), z_{\min,p}^{V,i,k,K}\}] \\
&-A_{U,q}^{i,k,K} \cdot B_{V,p}^{i,k,K}\} \\
-U_{p,q}^i&[X^q(y^{k,K})] \cdot V_{q,p}^i[X^p(y^{k,K})], \quad i = n+1, ..., N-1; \quad p = 1, ..., i-1; \\
q = 1&, ..., p.
\end{aligned} \tag{15}$$

It has been assumed, without loss of generality, that in all cases of the A's and B's, $A_{U,q}^{i,k,K}$ is less than zero.

Consider also the constraints for Problem $\mathrm{A}^{k,K}$:

$$X^i(x) \geqslant a_i^{k,K} \quad \text{for } i = 1, ..., N-1, \tag{16}$$

$$X^i(x) \leqslant b_i^{k,k} \quad \text{for } i = 1, ..., N-1. \tag{17}$$

Because the constraints (16) and (17) are not necessarily satisfied at $y^{k,K}$, the differences in (12) through (15) may not all be nonnegative. In fact, the T's, U's and V's may not be defined outside their ranges, in which case the differences may not exist. These will not be used. Consider only those differences in (12) and (13) where the associated i does not have the property that (17) is satisfied at $y^{k,K}$. Consider only those differences in (14) and (15) where the associated i does not have the property that (16) is satisfied at $y^{k,K}$. If $y^{k,K}$ is not a global solution to the factorable programming problem (A), then at least one of the remaining differences must be positive, and the associated $X^p(y^{k,K})$ (or associated $X^q(y^{k,K})$ when differences of the form (13) and (15) are considered) must be within its/their bounds, i.e.,

$$a_p^{k,K} \leqslant X^p(y^{k,K}) \leqslant b_p^{k,K} \tag{18}$$

or (18) *and*

$$a_q^{k,K} \leqslant X^q(y^{k,K}) \leqslant b_q^{k,K} \tag{19}$$

must hold.

The set of x's feasible to the underestimating problem is larger than the feasible region of the original problem. The problem functions may not be defined outside their original ranges. In this case, it may not be possible to determine which constraint is not "satisfied" for purposes of applying the splitting rule. To ensure this, an additional assumption will be made, that the problem functions are continuous in the region feasible to the underestimating problem. Formally, define, for $p = 1, ..., N-1$,

$$a_p^* = \inf\{X^p(x) \mid c_q(x) \leqslant b_q,\ C_q(x) \geqslant a_q \quad \text{for } q = 1, ..., p-1\}\ ,$$
$$b_p^* = \sup\{X^p(x) \mid c_q(x) \leqslant b_q,\ C_q(x) \geqslant a_q \quad \text{for } q = 1, ..., p-1\}.$$

The statement preceding (18) can now be proved.

Theorem 1. *If the $T_p^i(\cdot)$'s are continuous on $[a_p^*, b_p^*]$ for $i = 1, ..., N; p = 1, ..., i-1$, and if the U's and V's are continuous on their corresponding intervals, and if $y^{k,K}$ is not a global solution to the factorable programming problem, then at least one of the differences* (12) *through* (15) *is positive and the corresponding values of the $X^p(y^{k,K})$, (or $X^p(y^{k,K})$ and $X^q(y^{k,K})$, depending upon the case) lie within their required intervals, i.e., satisfy* (18)/(18) *and* (19).

Proof. Obviously, $a_j^{k,K} \leqslant X^j(y^{k,K}) \leqslant b_j^{k,K}$ for $j = 1, ..., n$, since the convex and concave underestimating/overestimating functions are just the problem variables themselves. Let i be any index greater than or equal to $n + 1$ and less than or equal to N for which the differences (12) through (15) are zero, and for which the X^p's and X^q's involved satisfy

(18) and (19). It follows by summing that

$$c^{i,k,K}(y^{k,K}) = X^i(y^{k,K}) = C^{i,k,K}(y^{k,K}) .$$

Since $c^{i,k,K}(y^{k,K}) \leqslant b_i^{k,K}$, and $C^{i,k,K}(y^{k,K}) \geqslant a_i^{k,K}$, it follows that

$$a_i^{k,K} \leqslant X^i(y^{k,K}) \leqslant b_i^{k,K} .$$

By induction it follows, therefore, that eventually a positive difference has to occur. Otherwise, $y^{k,K}$ is a point which is feasible, and which has an objective function value equal to a lower bound on the global solution to the problem, indicating that it is indeed a global minimizer.

The splitting rule is then: Find the difference in (12) through (15) which is maximal with the following restrictions: do not compute (12) and (13) for any i where $X^i(y^{k,K})$ satisfies (17); do not compute (14) and (15) for any i where $X^i(y^{k,K})$ satisfies (16); do not compute any differences in (12) and (14) where the $X^p(y^{k,K})$ does not satisfy (18); do not compute any differences in (13) and (15) where either $X^p(y^{k,K})$ does not satisfy (18) or $X^q(y^{k,K})$ does not satisfy (19).

Let I and P denote the superscript and subscript associated with the maximum difference if it is of the form (12). (The case when the maximum difference is of the form (14) is analogous and will not be considered.) For iteration $k+1$ problems $(\mathrm{A}^{k+1,l})$ for $l = 1, ..., k$, $l \neq K$ are identical with problems $(\mathrm{A}^{k,l})$. Problem $(\mathrm{A}^{k+1,K})$ is the same as problem $(\mathrm{A}^{k,K})$ except that the constraint $X^P(x) \geqslant a_p^{k,K}$ is replaced by

$$X^P(x) \geqslant [h_{T,I}^{P,k,K} + X^P(y^{k,K})]/2 \tag{20}$$

where

$$h_{T,I}^{P,k,K} = \mathrm{mid}\{c^{P,k,K}(y^{k,K}), C^{P,k,K}(y^{k,K}), z_{\min,P}^{T,I,k,K}\} ,$$

and problem $(\mathrm{A}^{k+1,k+1})$ is the same as problem $(\mathrm{A}^{k,K})$, except that the constraint $X^P(x) \leqslant b_P^{k,K}$ is replaced by

$$X^P(x) \leqslant [h_{T,I}^{P,k,K} + X^P(y^{k,K})]/2 . \tag{21}$$

If the maximum difference is of the form (13) or (15) (the case of (13) only is considered here), let I, P, and Q denote the associated indices. Depending upon the signs of the A's and B's involved, and which of the two functions in the $\max\{\cdot, \cdot\}$ is higher, the difference portion associated with the index P can be evaluated at any one of four values: $c^{P,k,K}(y^{k,K})$, $C^{P,k,K}(y^{k,K})$, $z_{\min,P}^{V,I,k,K}$, or $z_{\max,P}^{V,I,k,K}$. A similar statement holds for quantities associated with the index Q. Denote the value at which the difference is evaluated in the first case by $h_{V,I}^{P,k,K}$ and in the second by $h_{U,I}^{Q,k,K}$. If

$$X^P(y^{k,K})-h_{V,I}^{P,k,K} \geqslant X^Q(y^{k,K})-h_{U,I}^{Q,k,K} \quad , \tag{22}$$

divide on the range of X^P, otherwise divide on the range of X^Q.

The exact formulas for doing this are those used in (20) and (21). It needs, of course, to be shown that these rules are in fact a partitioning. The only way $[h_{T,I}^{P,k,K} + X^P(y^{k,K})]/2$ could fail to be interior to $[a_P^{k,K}, b_P^{k,K}]$ in the case when (12) contains the maximum difference, is if both quantities equal one of the endpoints. (Recall both $X^P(y^{k,K})$ and $h_{T,I}^{P,k,K}$ are in the interval.) But in this case, since the convex envelope $e_{T,P}^{I,k,K}$ agrees with the function at its endpoints, no difference would occur.

If a difference in (13) is maximum, then the only way the splitting rule can fail to divide the range of the X^P (or X^Q) involved is if $X^P(y^{k,K})$ and $h_{V,I}^{P,k,K}$ (or the corresponding terms for Q) are both at the same endpoint of the interval $[a_P^{k,K}, b_P^{k,K}]$. This implies that both differences in (22) are zero and that the other factor has its two values at the same endpoint. Since the convex envelopes agree there, no differences could have occurred in (13).

These rules for splitting reduce to those proposed by Falk and Soland [2], and Soland [5] when the problem functions are separable. The intuitive motivation for them is that in some sense the convex envelopes (and concave envelopes if that is the case) computed for the new feasible regions will be "brought up" closer to the value of the function at $X^P(y^{k,K})$ using this splitting rule than using any other. For the separable case this rule has been successful and should prove so in the factorable case.

In the following $[\bar{a}_i, \bar{b}_i]$ is a limiting interval of $[a_i^{k,K}, b_i^{k,K}]$, and $\bar{c}^N(\cdot)$ is a limiting convex envelope of $c^{N,k,K}(\cdot)$. Since without loss of generality the intervals $[a_i^{k,K}, b_i^{k,K}]$ can be considered nested, appropriate subsequences can be taken so that these limiting values exist.

The following continuity assumptions will be in force for the next three lemmas and theorem.

The functions $\{T_p^i(\cdot)\}$, for $i = 1, ..., N; p = 1, ..., i-1$, are continuous on $[a_p^*, b_p^*]$ and the $U_{p,q}^i(\cdot)$'s and $V_{q,p}^i(\cdot)$'s are continuous on their respective domains of definition.

Further, without loss of generality it can be assumed that $\bar{y}$ is the sole point of $y^{k,K}$.

Lemma 1. *Consider all the differences in* (12) *and* (13) *for which the index i involved does not correspond to a constraint of the form* (17) *which is satisfied an infinite number of times at* $y^{k,K}$. *In addition, consider all those differences in* (14) *and* (15) *for which the index i involved does not correspond to a constraint of the form* (16) *which is satisfied an infinite number of times at* $y^{k,K}$. *Consider also those terms in* (12) *and* (13) *for which the index* $i = N$. *From these terms, consider only those for which the* $X^p(y^{k,K})$ *and* $X^q(y^{k,K})$ *involved (if necessary) satisfy*

$$a^{k,K} \leqslant X^p(y^{k,K}) \leqslant b^{k,K}$$

and

$$a_q^{k,K} \leqslant X^q(y^{k,K}) \leqslant b_q^{k,K}$$

an infinite number of times. Then the limiting maximum of the value differences restricted as above is zero.

Proof. Assume the contrary, i.e., that the limiting maximum difference is not zero. Assume without loss of generality that the term which initiates the splitting rule an infinite number of times is of the form (12). Let I and P be indices associated with the term which initiates the splitting rule an infinite number of times and which tends to the maximum difference. One limiting interval can be assumed (again, without loss of generality) to have the form

$$[\bar{a}_p, (X^p(\bar{y}) + \bar{h}_{T,I}^P)/2] .$$

Since both $X^P(\bar{y})$ and $\bar{h}_{T,I}^P$ are in the interval, it follows that they are equal. Then, since the convex envelope agrees with the function at its endpoints,

$$T_P^I[X^P(\bar{y})] - \bar{e}_{T,P}^I[\bar{h}_{T,I}^P] = 0 ,$$

which is a contradiction to the assumption that the maximum difference tended to a value greater than zero.

Lemma 2. *Let i, p be two indices corresponding to a difference in* (12), *with the following restrictions. Assume that* $X^i(y^{k,K})$ *fails to satisfy* (17) *an infinite number of times and* $X^p(y^{k,K})$ *fails to satisfy*

$$X^p(y^{k,K}) \leqslant b_p^{k,K} \tag{23}$$

an infinite number of times. If

$$c^{p,k,K}(y^{k,K}) - X^p(y^{k,K}) \tag{24}$$

tends to zero, then

$$T_p^i[X^p(\bar{y})] - \bar{e}^i_{T,p}[\bar{h}^p_{T,i}] = 0\,. \tag{25}$$

Proof. In this case the $\mathrm{mid}\{c^{p,k,K}(y^{k,K}),\ C^{p,k,K}(y^{k,K}),\ z^{T,i,k,K}_{\min,p}\}$ is either $c^{p,k,K}(y^{k,K})$ or $z^{T,i,k,K}_{\min,p}$. In either case,

$$c^{p,k,K}(y^{k,K}) \leqslant b_p^{k,K} < x^p(y^{k,K})\ ,$$

and from (22) it follows that

$$\bar{c}^p(\bar{y}) = \bar{b}_p = X^p(\bar{y})\,.$$

By the property of a convex envelope agreeing with the function at the endpoints of its definition, (25) follows.

Lemma 3. *Let i, p be two indices corresponding to a difference in* (14) *with the following restrictions. Assume that* $X^i(y^{k,K})$ *fails to satisfy* (16) *infinitely often and* $X^p(y^{k,K})$ *fails to satisfy*

$$a_p^{k,K} \leqslant X^p(y^{k,K}) \tag{26}$$

an infinite number of times. If

$$C^{p,k,K}(y^{k,K}) - X^p(y^{k,K}) \tag{27}$$

tends to zero, then

$$\bar{E}^{i}_{T,p}[\bar{h}^{p}_{T,i}] - T^{i}_{p}[X^{p}(\bar{y})] = 0 \; . \tag{28}$$

Proof. The proof follows the proof of Lemma 2 and will not be given.

All the possibilities have not been covered in Lemmas 2 and 3. In Lemma 2 the conclusion follows if (26) is substituted for (23), and (27) for (24). In Lemma 3 the conclusion holds if (23) is substituted for (26), and (24) for (27). The lemmas required for the differences in (13) and (15) also follow from the same line of reasoning used in Lemma 2 and will not be stated explicitly.

Lemma 4. *The following equality and inequalities are satisfied at $\bar{y}$:*

$$X^{N}(\bar{y}) - \bar{c}^{N}(\bar{y}) = 0 \; , \tag{29}$$

$$X^{i}(\bar{y}) - \bar{a}_{i} \geqslant 0 \quad \text{for } i = 1, ..., N-1 \; , \tag{30}$$

and

$$\bar{b}_{i} - X^{i}(\bar{y}) \geqslant 0 \quad \text{for } i - 1, ..., N-1. \tag{31}$$

Proof. Let θ be the set of indices for which (30) and (31) hold. Let θ_1 be the set of indices for which both (16) and (17) are satisfied an infinite number of times. Let θ_2 be the set of indices not in θ_1, for which

$$X^{i}(y^{k,K}) \leqslant b^{k,K}_{i} \tag{32}$$

fails to be satisfied an infinite number of times, and for which

$$c^{i,k,K}(y^{k,K}) - X^{i}(y^{k,K}) \tag{33}$$

tends to zero, or for which

$$a^{k,K}_{i} \leqslant X^{i}(y^{k,K}) \tag{34}$$

fails to be satisfied an infinite number of times and for which

$$X^{i}(y^{k,K}) - C^{i,k,K}(y^{k,K}) \tag{35}$$

tends to zero.

Clearly the indices 1, ..., n are in θ and in θ_1. Consider the index $n + 1$. Because of the factorable representation, it is defined only in terms of the original problem variables, which are always within their bounds. If $X^{n+1}(y^{k,K})$ satisfies (16) and (17) infinitely often, obviously $n + 1 \in \theta$ and $n + 1 \in \theta_1$. Otherwise, all its terms are those characterized by Lemma 1. By summing appropriately it follows (without loss of generality) that $\bar{c}^i(\bar{y}) = X^i(\bar{y})$, and thus $n + 1$ is characterized by (32) and (33). It is therefore in θ_2 (and in θ as well). (If $n + 1 = N$, obviously, (29) is satisfied.)

The inductive step is as follows. Suppose the indices $1, ..., i-1$ are in the set θ. Some of these indices are in the set θ_1 and the remainder are in θ_2 and are characterized by (32) and (33) or (34) and (35). If the index i satisfies (16) and (17) infinitely often, it is obviously in the sets θ and θ_1. Otherwise, because of the factorable representation, it is defined only in terms whose indices are in the sets θ_1 and θ_2. Lemmas 1–3 imply that the difference between any term in the definition of X^i and its associated convex underestimating function tends to zero as $k \to \infty$. Appropriately summing yields satisfaction of (30) and (31) [or (29)] at $\bar{y}$. Thus, $i \in \theta_2$ as well as θ. Eventually all the indices $1, ..., N-1$ are in θ and (29) is satisfied.

Theorem 2. *Every accumulation point of $y^{k,K}$ is a global minimizer for problem* (A).

Proof. Let $\bar{y}$ be a point of accumulation. From Lemma 4, $\bar{y}$ is feasible by virtue of (30) and (31). Hence

$$X^N(\bar{y}) \geqslant v^* .$$

Because of the convex underestimating properties

$$v^* \geqslant \bar{c}^N(\bar{y}) .$$

From (29) it then follows that

$$X^N(\bar{y}) = v^* .$$

6. Summary

In this part it has been demonstrated how convex underestimating problems can be generated for nonlinear programming problems which are factorable. As part of a general algorithm for computing global solutions to nonconvex programs, this is helpful in eliminating regions which cannot contain a global minimizer. The method is computable and entails the computation of the convex envelopes of the functions of a single variable. The efficacy of the method depends upon the tightness of the underestimating problems. Computational work is underway to implement the ideas contained in this paper.

The second part of this paper will contain a method of verifying that a local solution is a global solution in a region.

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References

[1] E.M.L. Beale and J.A. Tomlin, "Special facilities in a general mathematical programming system for nonconvex problems using ordered sets of variables", in: J. Laurence, ed., *Proceedings of the fifth international conference on operational research* (Tavistock Publications, London, 1970) pp. 447–454.
[2] J.E. Falk and R.M. Soland, "An algorithm for separable nonconvex programming problems", *Management Science* 15(9) (1969) 550–569.
[3] G.P. McCormick, "Converting general nonlinear programming problems to separable nonlinear programming problems", Technical Paper Serial T-267, Program in Logistics, The George Washington University, Washington, D.C. (1972).
[4] G.P. McCormick, "Attempts to calculate global solutions of problems that may have local minima", in: F.A. Lootsma, ed., *Numerical methods for nonlinear optimization* (Academic Press, New York, 1972) pp. 209–221.
[5] R.M. Soland, "An algorithm for separable nonconvex programming problems II: nonconvex constraints", *Management Science* 17(11) (1971) 759–773.