Module 2

Elastic properties of materials

Elasticity:

When the deforming forces applied to a body is removed the body tends to recover its original condition i.e. the body will recover its original shape or size. This property of material body to recover its original condition when deforming forces are removed is called elasticity.

Elastic Bodies and Plastic bodies:

The bodies which recover its original condition completely on the removal of deforming force are called perfectly elastic.

Ex. Steel, Quartz fibre, Phospor bronze, Rubber etc

The bodies which do not show any tendency to recover their original condition on the removal of deforming forces are called perfectly plastic body.

Ex. Clay, Wax, Putty.

Deforming Force:

Consider a body which is not free to move and is acted upon by external forces. Due to the action of external forces the body changes its shape or sizes changes and now body is said to be deformed. Thus the applied external force which cause deformation is called deforming force.

Restoring force:

When deforming force is applied to a body then molecules of body tend to displace from their equilibrium position. As a result of this a reaction force developed within the body which tries to bring the molecule to its equilibrium position. This reaction force which is developed in the body is called internal force or elastic force or restoring force.

Rigid body:

A rigid body can be defined as one which does not undergo any deformation under the action of various deforming forces. When forces are applied on a rigid body the distance between any two particles of the body will remain unchanged, however large the force may be. In actual practice no material body is perfectly rigid. For practical purposes solid bodies are under the influence of weak forces are taken as rigid bodies. The nearest approach to a rigid is diamond and carborundum.

Load :

It is the combination of external forces acting on a body. The effect of load is to change the form or the dimensions of the body. It is thus essentially a deforming force.

Stress:

The restoring force per unit area set up inside the body is called stress. The restoring force is equal in magnitude but opposite that of the applied force. Therefore **stress is given by the ratio of the applied force to the area. Unit of stress is Nm^{-2}.**

Strain:

It is defined as t**he ratio of change in dimension of the body to its original dimension.**

It is not having any unit.

Tensile stress (Longitudinal stress):

The stress which brings about change in length of the object is called as Longitudinal stress. It is applied normal to the body.

Ex. Load suspended normally from the wire due to which the wire undergoes change in length.

If 'F' is the force applied and 'a' is the area of cross section.

Longitudinal stress = $\frac{F}{a}$

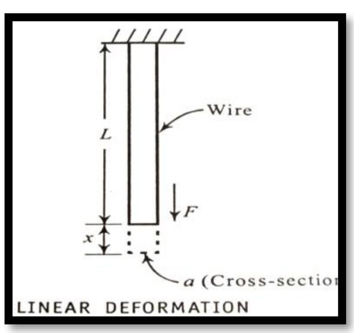
Longitudinal or Tensile strain:

If 'x' is the change in length produced due to the applied stress for an original length of 'L' then,

$$\textbf{Longitudinal} = \frac{change\ in\ length}{original\ length} = \frac{x}{L}$$

Tangential stress or Shear stress.

The stress which brings about change in shape is called as tangential stress. It is applied parallel to the surface.



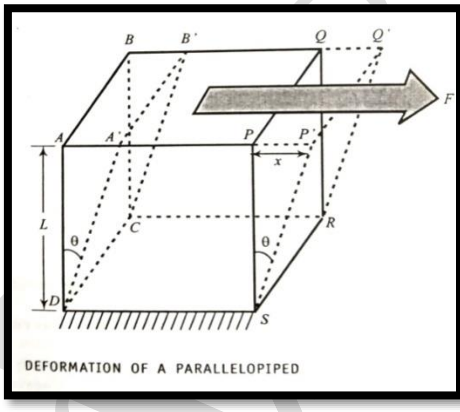
If ‘F’ is the force applied parallel to the surface and ‘a’ is the surface area.

Shear stress = $\frac{F}{a}$

Shear strain:

Shear strain is defined as the ratio of change in the shape of the object to original shape of the object.

If a force is applied tangentially to a free portion of the body whose other part is fixed then its layers slide one over the other; the body experiences a turning effect and changes its shape. This is called **shearing** and the angle through which the turning takes place is called **shearing angle (θ).**



DEFORMATION OF A PARALLELOPIPED

Within elastic limit it is measured by the ratio of relative displacement of one plane to its distance from fixed plane. It is also measured by the angle through which a line originally perpendicular to fixed plane is turned.

Shearing strain= $\theta = tan\ \theta = \frac{x}{L}$

Compressive stress or volume stress :

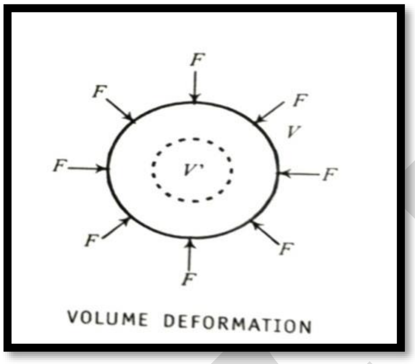
The compressive stress is restoring force developed within the body when body compressed under the action of deforming force. It brings about change in the volume of the object.

If F is the force applied uniformly and normally on a surface area ‘a’ then.

Compressive stress is = $\frac{F}{a}$

Volume strain:

If a uniform force is applied all over the surface of a body then the body undergoes a change in its volume (however the shape is retained in case of solid bodies). If *v* is the change in volume to an original volume *V* of the body then,



Volume strain $= \dfrac{\text{change in volume}}{\text{original volume}} = \dfrac{v}{V}$

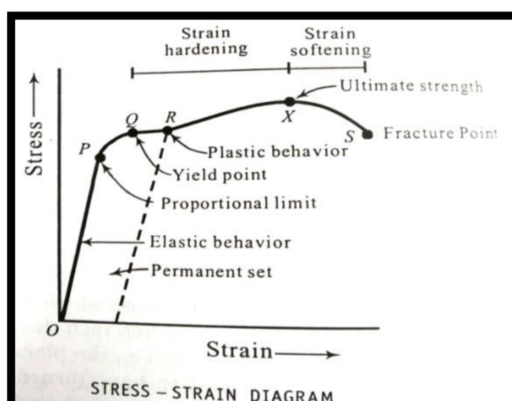
Hooke's law:

The fundamental law of elasticity was given by Robert Hook. It states that **"Stress produced in a body is directly proportional to the strain within the elastic limit".** Thus in such a case the ratio of stress to strain is a constant and it is called the modulus of elasticity or coefficient of elasticity. i.e., stress α strain,

or, $\dfrac{stress}{strain} = \text{a constant (E)}$

Stress – strain Graph when a wire is stressed

The relationship between stress and strain is studied by plotting a graph for various values of stress and the accompanying strain. This graph is obtained by plotting various values of stress and the accompanying strain of simple case of a bar or wire subjected to increasing tension. The graph obtained is in the general form as shown in the figure and is known as stress-strain diagram.

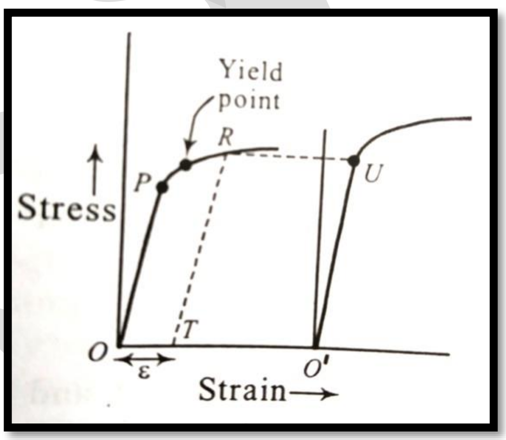


- In the graph the straight and slopping part OP of the curve shows that the strain produced is directly proportional to the stress or the Hook's law is obeyed perfectly up to P. In this region the material will recover its original condition of zero strain, on the removal of the stress. **The point 'P' is called as proportionality limit.**
- If the material is stressed between The region P and Q. the material will regain its original shape i.e it exhibits elasticity but it will not obey Hookes law. **The point Q is called as Yield point or elastic limit.**
- If the material is stressed between the region Q and S. it will not regain its original shape and size. it will not trace the original path instead it will trace the RT. i.e it has undergone a permanent deformation or plastic deformation.

  The point 'x' is called as the ultimate strength. It is the maximum stress that amaterial can withstand beyond which the material breaks.
- **The point 's' is called as the Breaking point.**The region 'QX' is called as **strain hardening.** The region XS is called as **Strain softening.**

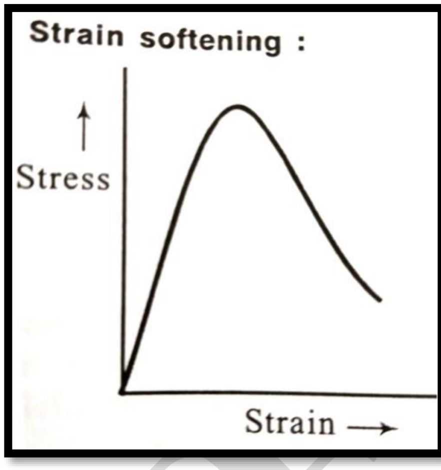
<u>**Strain Hardening and strain softening:**</u>

Certain materials that are plastically deformed earlier are stressed again, shows an **increased** yield point. This effect is called as **strain hardening.**



It is one of the process of Making a material harder by plastic deformation. It is also known as 'work hardening' or ' cold working'.

Certain materials like concrete or soil, are stressed their stress strain graph shows negative slope soon after the elastic region. The negative slope indicates the that there is softening effect of the material. This effect is called as **'strain softening'.**

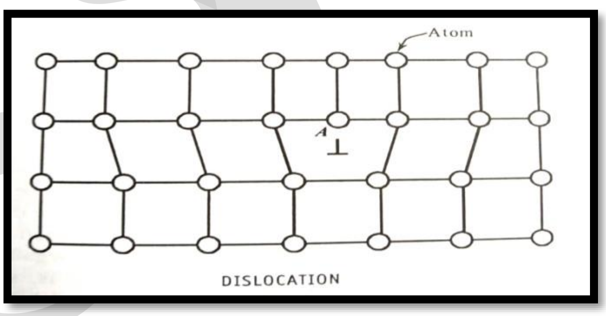


Effect of continuous stress, temperature, annealing and Impurities on the elastic properties of the body.

Effect of continuous stress.

When certain elastic materials are stressed continuously effect of creep comes in to play

Creep is the property due to which a material under steady stress undergo slow plastic deformation even below the proportionality limit.



It occurs due to the deformation caused by slip occurring along the crystallographic directions in the metal.

Ex. The warping of a shelf over time when heavy object is placed on it.

Effect of temperature:

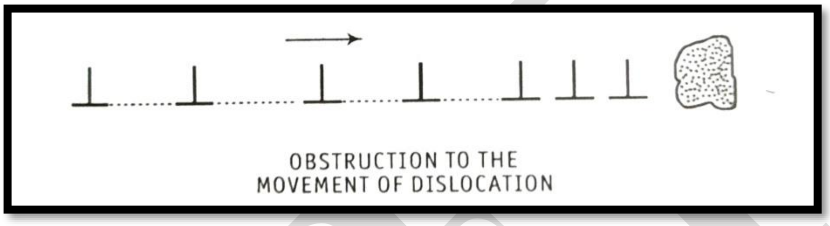
When the material is subjected stress at high temperatures. The effect of creep dominates. It is an important factor to be considered in the design of boilers , turbines , jet engines etc.

Annealing.

It is a type of heat treatment used increase the strength, hardness and toughness to meet the requirement of good machinibility, forging and casting. It improves the elasticity and increases the ductility.

In annealing the material is heated to very high temperature first to make the metal soft and then it gradually cooled down.

Effect of impurities:



Addition of impurities to metal results in either increase or decrease of elasticity depending on the type of impurity added.

If the impurity added obstructs the motion of dislocation in the lattice it increases the elastic modulus and hence the yield strength.

If the impurity added enables the movement of dislocation it causes cracks, inclusions and reduces the strength.

Types of elasticity

Corresponding to the three types of strain, we have three types of elasticity

a) Linear Elasticity or Elasticity of length or Young's modulus
b) Elasticity of volume or Bulk modulus
c) Modulus of Rigidity (corresponding to shear strain)

a) Young's modulus (Y).

The ratio of longitudinal stress to linear strain within elastic limit is called the coefficient of direct elasticity or Young's modulus and is denoted by Y .

If F is the force applied normally, to a cross-sectional area a, then the stress is F/a. If L is original length and x is change in length due to the applied force, the strain is given by x/L.

$$\mathbf{Y} = \frac{\text{Normal stress}}{\text{Longitudinal strain}} = \frac{F/a}{x/L} = \frac{FL}{ax} \ \mathbf{N/m^2}$$

b) Elasticity of volume or Bulk modulus (K):

The ratio of normal stress or pressure to the volume strain without change in shape of the body within the elastic limits is called Bulk modulus.

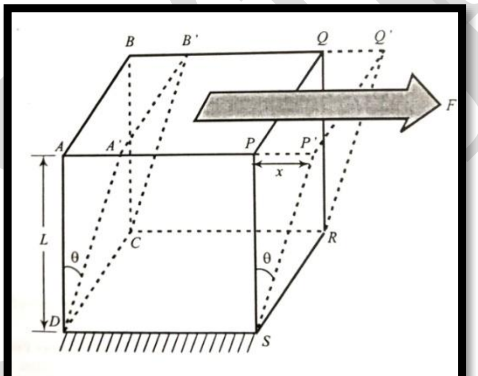
If ‘F’ is the force applied uniformly and normally on a surface area ‘a’ the stress or pressure is ‘F/a’ or P and if ‘*v*’ is the change in volume produced in an original volume V, the strain is given by ‘*v*/V’ and therefore

$$\mathbf{K} = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{F/a}{v/V} = \frac{FV}{av} = \frac{PV}{v}\ \mathbf{N/m^2}$$

Bulk modulus is referred to as incompressibility and hence its reciprocal is called **compressibility** (strain per unit stress).

c) Modulus of Rigidity (corresponding to shear strain)

In this case, while there is a change in the shape of the body, there is no change in its volume. It takes place by the movement of contiguous layers of the body, one over the other.



DEFORMATION OF A PARALLELOPIPED

Q^1 that is the planes of the two faces ABCD and PQRS can be said to have turned through an angle θ. This angle θ is called the angle of shear or shearing strain. Tangential stress is equal to the force F divided by area ‘a’ of the face APQB.

The rigidity modulus is defined as the ratio of the tangential stress to the shearing strain.

Hence **tangential stress** = $\frac{F}{a}$ $\theta = PP^1 / PS = \frac{x}{L}$

$$\textbf{Rigidity modulus } \boldsymbol{\eta} = \frac{\text{tangential stress}}{\text{shearing strain.}} = \frac{F/a}{\theta} = \frac{F/a}{x/L} = \frac{FL}{ax}\ \mathbf{N/m^2}$$

Longitudinal Strain co efficient:

If x/L is the Longitudinal strain produced due to Longintudinal stress T

The from Hookes law

$x/L \propto T$

$x/L = \alpha\ T$

Where α is called as Longitudinal strain co efficient.

Logitudinal strain co efficient (α) is defined as the Longitudinal strain produced per unit stress.

Lateral Strain co efficient:

When an elastic material subjected to tensile stress it produces longitudinal strain as well as lateral strain in the direction perpendicular to longitudinal strain in the material.

Lateral strain is defined as the ratio of change in the diameter to original diameter.

If 'd' is the change in the diameter and 'D' is the original diameter then

Lateral strain = d/D

If d/D is the lateral strain produced due to Longintudinal stress T

The from Hookes law

$d/D \propto T$

$d/D = \beta\ T$

Where β is called as Lateral strain co efficient.

Lateral strain co efficient (β) is defined as the Lateral strain produced per unit stress.

Poisson's Ratio (σ)

It is commonly observed fact that when we stretch string or wire it becomes longer but thinner. That is an increase in length is always accompanied by a decrease in its cross section. In other words a linear or a tangential strain produced in a wire is accompanied by a transverse or lateral strain of opposite kind in a direction right angle to the direction of applied force. This change which occurs in a direction perpendicular to the direction along which the deforming force is acting is called **lateral change**.

***Within elastic limits of a body, the ratio of lateral strain (β) to the longitudinal strain (α) is a constant and is called Poisson's ratio* (σ).**

If a deforming force acting on a wire of length 'L' produces a change in length 'x' accompanied by a change in diameter of 'd' in it which has a original diameter of 'D'

$\therefore$ **Poisson's ratio, σ =** $\dfrac{\textbf{Lateral strain}}{\textbf{Longitudinal strain}} = \dfrac{Ld}{xD}$

The lateral strain coefficient $\beta = \dfrac{d}{D}$ and Longitudinal strain coeficient $\alpha = \dfrac{x}{L}$,

$\therefore$ **Poisson's ratio, σ =** $\dfrac{Ld}{xD} = \dfrac{\beta}{\alpha}$

Poisson's ratio is a dimensionless quantity.

Limiting value of Poisson's ratio:

W.K.T.

$Y = 2\eta(1 + \sigma)$, and $Y = 3K(1-2\sigma)$

$\therefore 2\eta(1 + \sigma) = 3K(1-2\sigma)$

where k & η are essentially positive quantities.

(i) If σ be positive quantity, then the right hand side and left hand side expression must be positive. i.e. $(1-2\sigma) > 0$

Or $2\sigma < 1$

$\therefore \sigma < 0.5$

(ii) If σ is A negative quantity, then the right hand side and left hand side expression must be positive. i.e. $(1 + \sigma) > 0$

Or $\sigma > -1$

Thus the limiting value of Poisson's ratio is $-1 > \sigma < 0.5$

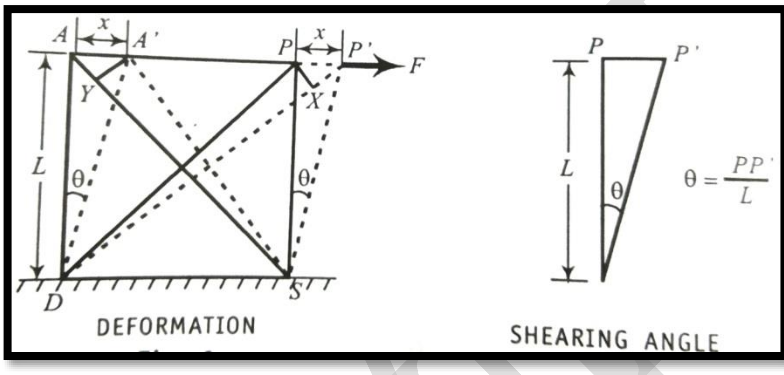
If σ = 0.5, then Bulk modulus is infinite. It means that, the substance is perfectly incompressible. Actually there is no substance which perfectly incompressible.

If σ = -1, the rigidity modulus is infinite. It means that, if a body is extended linearly, then it should also extend laterally. No substance exhibits such phenomenon.

In actual practice, the value of σ varies from 0.2 to 0.4 **(Practical limit)**

Equivalence of shear to compression and extension Strain

Consider a front face a cube ABPS, whose lower surface AS is fixed to a rigid support. Let a tangential force 'F' is applied at the upper surface along BP of the cube in a direction as shown in figure. The applied tangential force causes the relative displacements at different parts of the cube, so that, A moves to A' and P moves to P' through a small angle. Due to this the diagonal AS will be shortened to A'S and diagonal DP will be increased to a length DP'.let θ be the angle of shear which is very small in magnitude.



Let length of each side of the cube = L.

and PP'= *x*.

As θ is very small, from ΔSPP', we can write, $\theta = \tan\theta = x/L$.

As diagonal DP increases to DP' and diagonal AS is compressed to A'S,

We have Extension Strain along DP and compression strain along AS

Therefore extension Strain along DP

Elongation strain = P'X/DP

If L is the length of each side of the cube, then we have DP= $\sqrt{2}$ L (By Pythagoras theorem).

From Δ PP'X , P'X=PP' cos (PP'X)

but ∠ PP'X= ∠AP'D = ∠ APD = 45°

$$\therefore\ P'X = PP'\cos 45^\circ = \frac{PP'}{\sqrt{2}}$$

Let PP' = x

we get, $P'X = PP'\cos 45^\circ = \frac{x}{\sqrt{2}}$

$\therefore$ Elongation strain along DP = P'X/DP= $\frac{x/\sqrt{2}}{L\sqrt{2}} = \frac{x}{2L} = \frac{\theta}{2}$ (1)

Since x/L = θ

Similarly the compression strain along AS

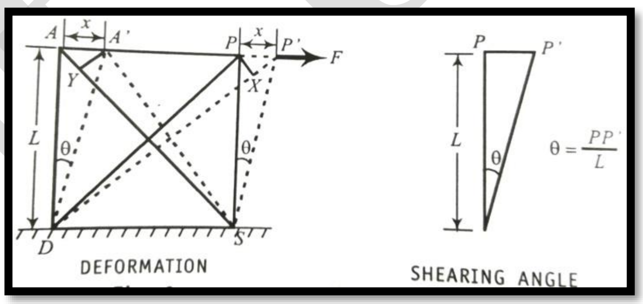
i.e Compresion strain = $\frac{A\prime Y}{AS} = \frac{\theta}{2}$ (2)

From (1) and (2) it is clear that a shear strain θ is equivalent to an extension strain and compression strain at right angles to each other and each of value θ/2.

Elongation strain + compression strain = $\frac{\theta}{2} + \frac{\theta}{2} = \theta$, the shearing strain.

Relation between η, α and β

Consider a cube with each of its sides of length L under the action of tangential stress T. let tangential force F be applied to its upper face. It causes the plane of the faces perpendicular to the applied force F turn through an angle θ. as a result diagonal AC undergoes contraction and diagonal DB[1]undergoes elongation of equal amount.



Now, shearing strain occurring along AP can be treated as equivalent to a longitudinal strain, along DP' and an equal lateral strain along the diagonal A'S i.e., perpendicular to DP. Let α and β be the longitudinal strain co efficient and lateral strain co efficients

since T is the applied stress, therefore extension produced for the length DP due to tensile stress

then Longitudinal Strain = T. DB.α

Lateral strain = T.DB.β.

$\therefore$ Total extension = Longitudinal strain along DP + Lateral strain Perpendiclar to Dp

$\therefore$ P'X = DP.T.(α+β),

P'X=PP' cos (PP'X)

but ∠ PP'X= ∠AP'D = ∠ APD = 45°

$\therefore$ P'X = PP'cos45° = $\frac{PP'}{\sqrt{2}}$

Let PP'= x

P'X = $\frac{x}{\sqrt{2}}$

And DP = $\sqrt{2}\,L$

$\therefore$ P'X=($\sqrt{2}\,L$).T(α+β),

Rearranging the terms

$$\therefore (\sqrt{2}L)T(\alpha+\beta)=\frac{x}{\sqrt{2}},$$

$$\text{Or, } \frac{1}{2}\frac{1}{(\alpha+\beta)}=\frac{TL}{x}=\frac{T}{(x/L)}=\frac{T}{\theta}=\eta$$

$$\therefore \eta=\frac{1}{2(\alpha+\beta)},$$

$$=\frac{1}{2\alpha(1+\beta/\alpha)},$$

$$\text{Or, } \eta=\frac{1/\alpha}{2(1+\sigma)}, \quad (\because \sigma=\beta/\alpha),$$

$$\eta=\frac{Y}{2(1+\sigma)}$$

$$or\ Y=2\eta(1+\sigma)$$

Relation between Y and α

Consider a cube of unit side subjected to unit tension along one side. Let α be the elongation per unit length per unit tension along the direction of the force. Therefore,

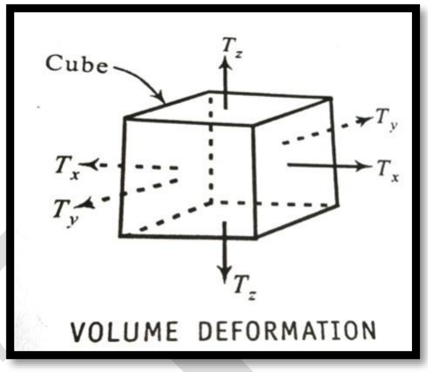
Stress = $\frac{\text{Force}}{\text{Area}}$ = 1

Similarly, linear strain = $\frac{\text{change in length}}{\text{original length}} = \frac{\alpha}{1}$ = α

$$\therefore \quad \mathbf{Y} = \frac{\text{Normal stress}}{\text{Longitudinal strain}} = \frac{1}{\alpha}$$

Relation between K, n and Y

Consider a cube of unit volume, as shown in the diagram. Let T_x T_y *and* T_z be the stress acting on the faces along X,Y and Z.



Each stress produces an extension in its own direction and a lateral contraction in the other two perpendicular directions. Let α be the elongation per unit length per unit stress along the direction of the forces and β be the contraction per unit length per unit stress in a direction perpendicular to the respective forces. Then stress like T_x produces an increase in length of αT_x in X-direction: but since other two stresses T_y and T_z are perpendicular to X-direction they produce a contraction of βT_y and βT_z respectively in the cube along X-direction .Hence, a length which was unity along X-direction becomes ,

$$1+ \alpha T_x - \beta T_y - \beta T_z.$$

Similarly along Y and Z directions the respective length become,

$$1+ \alpha T_y - \beta T_z - \beta T_x.$$

$$1+ \alpha T_z - \beta T_x - \beta T_y.$$

Hence the new volume of the cube is

$$= (1+ \alpha T_x - \beta T_y - \beta T_z)(1+ \alpha T_y - \beta T_z - \beta T_x)(1+ \alpha T_z - \beta T_x - \beta T_y)$$

Since α and β are very small, the terms which contain either powers of α and β, or their products can be neglected.

$\therefore$ New volume of the cube $= 1 + \alpha(T_x+ T_y+ T_z)- 2\beta(T_x+ T_y+ T_z)$,

$$= 1+ (\alpha-2\beta)(T_x+ T_y+ T_z)$$

If $T_x= T_y= T_z= T\sigma$

Then the new volume =1+ (α-2β) 3T

Since the cube under consideration is of unit volume, increase in volume = [1+3T (α-2β)]-1

= 3T (α-2β)

If instead of outward stress T, a pressure P is applied, the decrease in volume = 3P (α-2β).

$\therefore$ **Volume strain =** $\dfrac{\text{change in volume}}{\text{original volume}} = \dfrac{3\text{P}\,(\alpha - 2\beta)}{1}$

$\therefore$ **K =** $\dfrac{\text{Pressure}}{\text{Volume strain}} = \dfrac{\text{P}}{3\text{P}\,(\alpha - 2\beta)} = \dfrac{1}{3\,(\alpha - 2\beta)}$

$$\text{K} = \frac{1}{3\alpha(1-2\beta/\alpha)} = \frac{(1/\alpha)}{3(1-2\beta/\alpha)} = \frac{Y}{3(1-2\sigma)} \qquad (\because \sigma = \frac{\beta}{\alpha} \text{ and Y} = \frac{1}{\alpha})$$

Relation between Y, K and η

We know that $\eta = \dfrac{Y}{2(1+\sigma)}$ and K $= \dfrac{Y}{3(1-2\sigma)}$

Rearranging which we get,

$$\frac{Y}{\eta} = 2 + 2\sigma \text{ and } \frac{Y}{3K} = 1\text{-}2\sigma$$

Adding the above we get, $\dfrac{Y}{\eta} + \dfrac{Y}{3K} = 3,$

$$\text{Or } \frac{Y(3K+\eta)}{3\eta K} = 3$$

$$\therefore Y = \left[\frac{9\eta K}{3K+\eta}\right]$$

Relation between K , η and σ

We have the relations Y = 2η (1 + σ), and Y = 3K (1-2σ)

Equating the above equations we get,

$$2\eta + 2\eta\sigma = 3K - 6K\sigma,$$

$\therefore 2\eta\sigma + 6K\sigma = 3K - 2\eta,$

Or , $\sigma(2\eta + 6K) = 3K - 2\eta,$

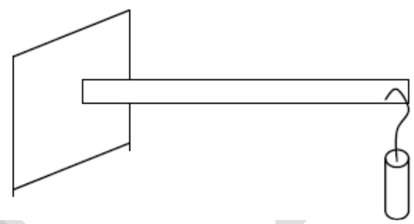
Or, $\sigma = \dfrac{3K - 2\eta}{2\eta + 6K}$

BENDING OF BEAMS

A homogenous body of uniform cross section whose length is large compared to its other dimensions is called a beam.

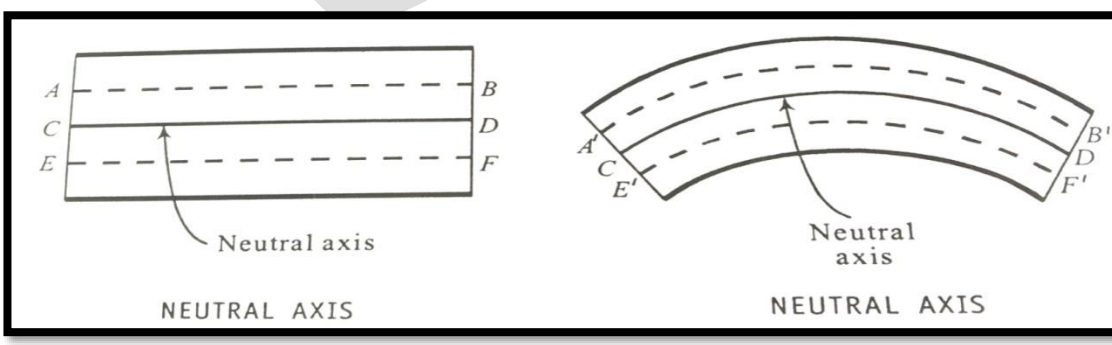
Neutral surface and neutral axis

Consider a uniform beam MN whose one end is fixed at M. The beam can be thought of as made up of a number of parallel layers and each layer in turn can be thought of as made up of a number of infinitesimally thin straight parallel longitudinal filaments or fibers arranged one closely next to the other in the plane of the layer.



If a load is attached to the free end of the beam, the beam bends. The successive layers along with constituent filaments are strained. A filament like AB of an upper layer will be elongated to A' B' and the one like EF of a lower layer will be contracted to E'F'. But there will always be a particular layer whose filaments do not change their length as shown for CD. Such a layer is called neutral surface and the line along which a filament of it is situated is called neutral axis.

Neutral Surface: It is that layer of a uniform beam which does not undergo any change in its dimensions, when the beam is subjected to bending within its elastic limit.

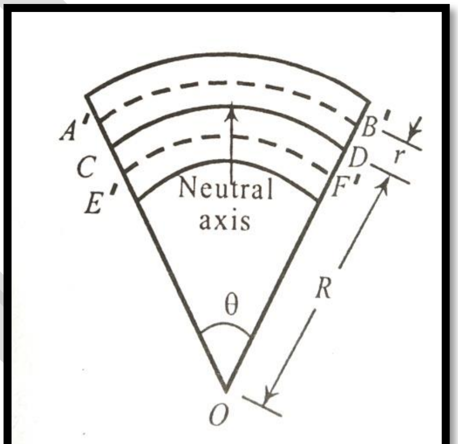


Neutral axis: It is a longitudinal line along which neutral surface is intercepted by any longitudinal plane considered in the plane of bending.

Bending moment of a beam:

Consider a uniform beam whose one end is fixed at M. If now a load is attached to the beam, the beam bends. The successive layers are now strained. A layer like AB which is above the neutral surface will be elongated to A' B' and the one like EF below neutral surface will be contracted to E'F'. CD is neutral surface which does not change its length.

The shape of each layers of the beam can be imagined to form part of concentric circles of varying radii. Let R be the radius of the circle to which the neutral surface forms a part.



PART OF BENT BEAM

$\therefore CD=R\theta$

where 'θ' is the common angle subtended by the layers at common center O of the circles. The layer AB has been elongated to A^1B^1.

$\therefore$ Change in length = A' B' -AB

But $AB=CD=R\theta$

If the successive layers are separated by a distance r then,

$$A'B' = (R+r)\theta$$

$\therefore$ Change in length $=(R+r)\theta-R\theta = r\theta$

But original length = $AB=R\theta$

$$\therefore \text{Linear strain} = \frac{r\theta}{R\theta} = \frac{r}{R}$$

Youngs Modulus Y= Longitudinal stress/linear strain

Longitudinal stress = Yx Linear strain

$$= Y\text{x}\,\frac{r}{R}$$

But stress $= \frac{F}{a}$

Where F is the force acting on the beam and a is the area of the layer AB.

$$\frac{F}{a} = \frac{Yr}{R} = \frac{Yar}{R}$$

Moment of this force about the neutral axis=F x its distance from neutral axis.

$$= F \times r = Yar^2 / R$$

Moment of force acting on the entire beam $= \Sigma \frac{Yar^2}{R}$

$$= \frac{Y}{R} \sum ar^2$$

The moment of inertia of a body about a given axis is given by Σmr^2, where Σm is the mass of the body. Similarly Σar^2 is called the geometric moment of Inertia Ig.

$Ig = \Sigma ar^2 = Ak^2$, where A is the area of cross section of the beam and k is the radius of gyration about the neutral axis.

$$\text{Moment of force} = \frac{Y}{R} Ig$$

1) Bending moment of rectangular beam $= \frac{Y}{R}\frac{bd^3}{12}$

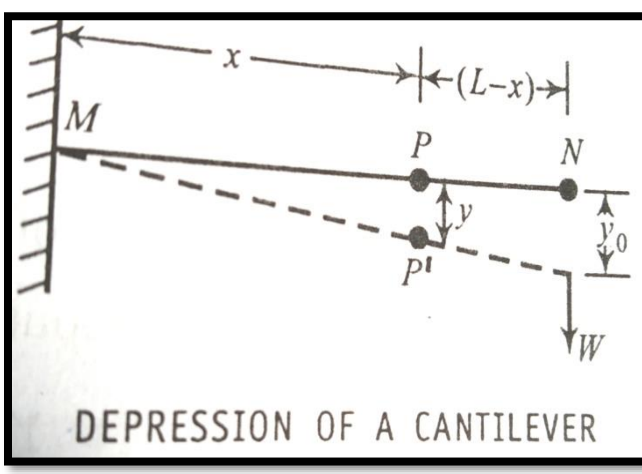
2) Bending moment of circular beam $= \frac{Y}{R}\frac{\pi\rho^4}{4}$

Single Cantilever

If one end of beam is fixed to a rigid support and its other end is loaded, then the arrangement is called single cantilever or cantilever.

Consider a uniform beam of length L fixed at M. Let a load W act on the beam at N.

Consider a point on the free beam at a distance x from the fixed end which will be at a distance (L-x) from N. Let P' be its



position after the beam is bent.

$\therefore$ Bending moment = Force x Perpendicular distance.

= W (L-x)

But bending moment of a beam is given by $\frac{Y}{R} Ig$

$$\frac{Y}{R} Ig = \text{W(L-x)} \qquad (1)$$

$$\frac{1}{R} = \frac{\text{W(L - x)}}{YIg} \qquad (2)$$

But if 'y' is the depression of the point P then it can be shown that

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \qquad (3)$$

where 'R' is the radius of circle to which the bent beam becomes a part.

Comparing equations (2)and (3)

$$\frac{d^2 y}{dx^2} = \frac{\text{W(L - x)}}{\text{YIg}}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{w(L-x)}{YI_g}$$

$$d\left(\frac{dy}{dx}\right) = \frac{w(Ldx - xdx)}{YI_g}$$

Integrating both sides

$$\frac{dy}{dx} = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right] + C_1 \qquad (4)$$

C_1 is constant of integration

But dy/dx is the slope of the tangent drawn to the bent beam at a distance x from the fixed end. When x=0, it refers to the tangent drawn at M, where it is horizontal. Hence (dy/dx)=0 at x=0. Introducing this condition in equation (4) we get $0=C_1$

Equation (4) becomes

$$\frac{dy}{dx} = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]$$

$$dy = \frac{W}{YIg}\left[Lx - \frac{x^2}{2}\right]dx$$

Integrating both sides we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + C_2 \qquad (5)$$

where C_2 is constant of integration, y is the depression produced at known distance from the fixed end. Therefore when x=0, it refers to the depression at M, where there is obviously no depression. Hence y=0 at x=0. Introducing this condition in equation (5) we get

$$y = \frac{W}{YIg}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right]$$

At the loaded end, $y=y_0$ and x=L

Therefore $$y_0 = \frac{W}{YIg}\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$$

Depression produced at loaded end is

$$y_0 = \frac{WL^3}{3YIg}$$

Therefore the young's modulus of the material of the cantilever is

$$Y = \frac{WL^3}{3y_0 Ig} \qquad (6)$$

Case (a):

If the beam is having rectangular cross-section, with breadth b and thickness d then,

$$\text{Ig} = \frac{bd^3}{12} \qquad (7)$$

Substituting equation (7) in equation (6) we get

$$Y = \frac{WL^3}{3y_0} x \frac{12}{bd^3}$$

$$Y = \frac{4WL^3}{Y_0 bd^3}$$

Case(b):

If the beam is having a circular cross section of radius r then,

$$Ig = \frac{\Pi r^4}{4} \qquad (8)$$

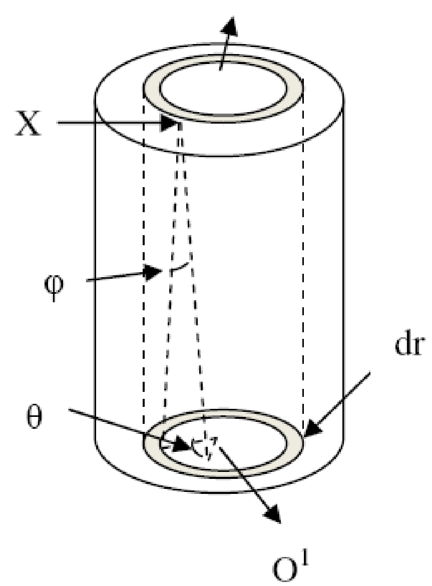
Substituting equation (8) in equation (7) we get

$$Y = \frac{4WL^3}{3\Pi y_0 r^4}$$

Torsion of a cylinder

A long body which is twisted around its length as an axis is said to be under torsion. The twisting is brought into effect by fixing one end of the body to a rigid support and applying a suitable couple at the other end. The elasticity of a solid, long uniform cylindrical body under torsion can be studied, by imagining it to be consisting of concentric layers of the material of which it is made up of. The applied twisting couple is calculated in terms of the rigidity modulus of the body.

Expression for the Torsion of a cylindrical rod



Consider a long cylindrical rod of length 'L' and radius 'R' rigidly fixed at its upper end. Let OO' be its axis. Imagine the cylindrical rod is made up of thin concentric hollow cylindrical layers each of thickness *'dr'*. The rod twisted at its lower end, and then the concentric layers slide one over the other. This movement will be zero at the fixed end and gradually increased along the downward direction. Let us consider one concentric circular layer of radius *'r'* and thickness *'dr'* . Any point 'X' on its uppermost part would remain fixed and a point like ‘B’ at its bottom moves to ‘B'’.

Now, $B\hat{X}B' = \phi$ gives the angle of shear.

Since ϕ is also small, the movement length $BB' = L\phi$.

Also, if $B\hat{O}'B' = \theta$, the length B B' = rθ.

$$\therefore L\phi = r\theta$$

Now, the cross sectional area of the layer under consideration is 2πr dr . If ‘F’ is the shearing force, then the shearing stress T is given by

$$T = \frac{Force}{Area} = \frac{F}{2\pi r dr}$$

∴Shearing force F = T(2πrd$r$)

∴ Rigidity modulus n = Shearing stress/shearing strain.

$$n = \frac{T}{\phi}$$

$$\therefore T = n\phi = \frac{nr\theta}{L}$$

$$\therefore F = \frac{nr\theta}{L}\left(2\pi r dr\right) = \frac{2\pi n\theta}{L} r^2 dr$$

The moment of the force about $OO' = \left(\frac{2\pi n\theta}{L} r^2 dr\right) r = \frac{2\pi n\theta}{L} r^3 dr$

This is only for the one layer of the cylinder.

Therefore, twisting couple acting on the entire cylinder $= \int_0^R \frac{2\pi n\theta}{L} r^3 dr$

$$= \frac{2\pi n\theta}{L}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\pi n R^4 \theta}{2L}$$

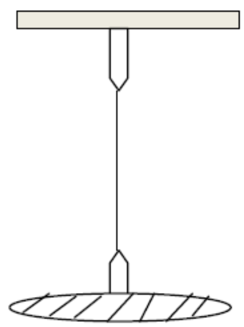
Couple per unit twist is given by C=Total twisting couple / angle of twist.

$$C = \frac{\pi n R^4 \theta / 2L}{\theta}$$

$$C = \left(\frac{\pi n R^4}{2L} \right)$$

Torsion Pendulum

Torsion pendulum consists of a heavy metal disc is suspended by means of a wire. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strain. The restoring couple of the wire tries to bring the wire back to the original position. Therefore disc executes torsional oscillations about the mean position.



Let θ be the angle of twist made by the wire and C be the couple per unit twist.

Then the restoring couple per unit twist = Cθ.

Therefore the angular acceleration produced by the restoring couple in the wire. a = $\frac{d^2\theta}{dt^2}$

Let I be the moment of inertia of the wire about the axis. Therefore, we have, I $\left(\frac{d^2\theta}{dt^2}\right)$ = -Cθ

The above relation shows that the angular acceleration is proportional to angular displacement and is always directed towards the mean position. The negative sign indicates that the couple tends to decrease the twist on the wire. Therefore, the motion of the disc is always simple harmonic motion (SHM).

Therefore, the time period of oscillator is given by relation,

$$T = 2\pi \sqrt{\frac{displacement}{Accleration}} = 2\pi \sqrt{\frac{\theta}{\left(\frac{C}{I} X\ \theta\right)}} = 2\pi \sqrt{\frac{I}{C}}$$

Application of torsion pendulum

1) Determination of moment of inertia of an irregular body

a) Time period (T_1) of pendulum is determined by fixing a regular body at the free end of torsion pendulum.

b) Similarly the time period (T_2) is regular body is determined.

If I_1 and I_2 are the moment of inertia of regular and irregular body respectively, then

$$T_1 = 2\pi\sqrt{\frac{I_1}{C}}$$

$$T_2 = 2\pi\sqrt{\frac{I_2}{C}}$$

$$\therefore \frac{T_1^2}{T_2^2} = \frac{I_2}{I_1}$$

The moment of inertia of the regular body about any axis can be determined by knowing its mass and dimensions. This moment of inertia of the irregular body is calculated.

$$\therefore I_2 = I_1 \frac{T_1^2}{T_2^2}$$

2) Determination of Torsion rigidity

The Time period of torsion pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

$$\therefore\ T^2 = 4\pi^2\frac{I}{C}$$

$$C = 4\pi^2\frac{I}{T^2}$$

Where, I is the moment of inertia of the regular body and C is the couple per unit twist of the wire.

But $\quad C = \dfrac{\pi n R^4}{2L}$

$$\therefore \quad 4\pi^2 \frac{I}{T^2} = \frac{\pi n R^4}{2L}$$

$$\text{or} \quad n = \frac{8\pi L}{R^4}\left(\frac{I}{T^2}\right)$$