MODULE III

MAXWELL'S EQUATIONS

Maxwell's equation provide a complete description of electromagnetic phenomena and includes all modern information and communication technologies. The maxwell equations are the basic tool for designing all electrical and electronics components from mobile phones to satellites.

FUNDAMENTALS OF VECTOR CALCULUS:

Vector: Any vector has both magnitude and direction. It is represented by drawing an arrow in a suitable coordinate System. (Fig 1.)

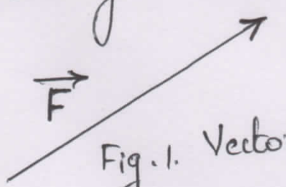


Fig. 1. Vector representation.

(i) Magnitude Of a Vector: The magnitude of the vector is taken care of by making the length of the arrow (= R the distance between A and B) numerically equal to or proportional to the magnitude of the vector $\vec{F}$. (Fig 2.)

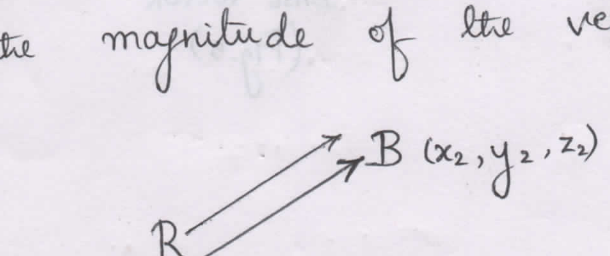
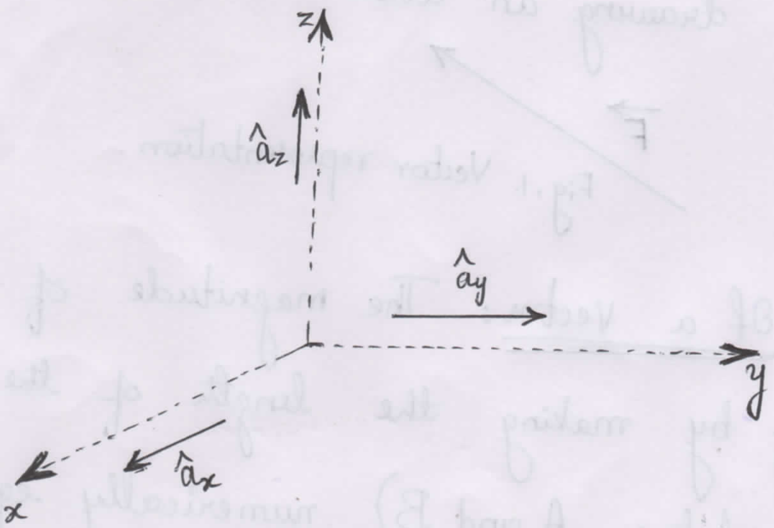


Fig 2. Magnitude of the vector.

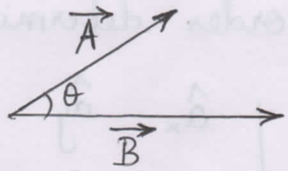
(ii) Direction Of the vector: The orientation Of $\vec{F}$ with respect to the coordinates is taken at the same inclination as described in the given situation. However, representing the direction of $\vec{F}$ is actually achieved by assigning its direction to what we call a unit vector.

(iii) Base Vectors: Base vectors are same as unit vectors but oriented strictly along the coordinates in the given coordinate system and pointing away from the origin. In rectangular coordinate system we can represent the base vectors as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$, along the x, y and z coordinates. (Fig 5).



BASE VECTOR
(Fig. 5)

Scalar Product or Dot Product: The scalar product or dot product of two vectors is defined as the product of their magnitudes and of the cosine of the smaller angle between them.

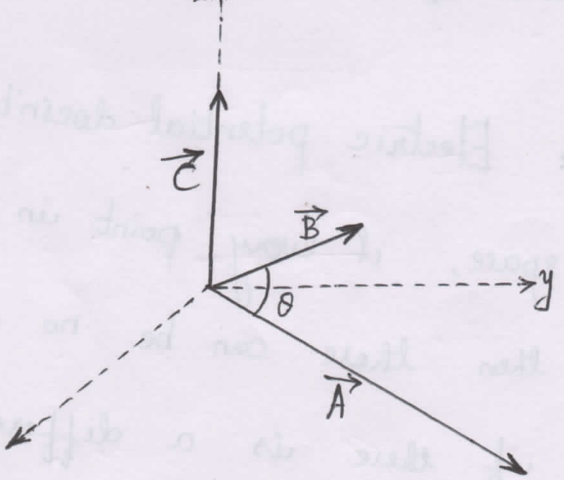


(Fig.6) Dot product

If $\vec{A}$ and $\vec{B}$ are two vectors inclined at an angle θ (Fig.6) their dot product is given as,

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta .$$

Cross Product or Vector Product: Given two vectors $\vec{A}$ and $\vec{B}$, their cross product is a single vector $\vec{C}$ whose magnitude is equal to the product of the magnitude of $\vec{A}$ and the magnitude of $\vec{B}$ multiplied by the sine of the smaller angle between them. The direction of $\vec{C}$ is perpendicular to the plane which has both $\vec{A}$ and $\vec{B}$ such that, $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right handed system (Fig7).



Cross Product.

(Fig.7)

$$\therefore \quad \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

$$\& \quad \vec{A} \times \vec{B} = \vec{C}$$

In terms of the components of $\vec{A}$ and $\vec{B}$, the vector $\vec{C}$ can be expressed as a third order determinant expressed as,

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operator ∇ : ∇ is a mathematical operator. It is called del (sometimes called nabla) and is meant to carry out a speific vector calulus operation. If it is a Cartesian coordinate problem, then the operation is as per the equation given below:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are the base vectors.

Gradient, and ∇ : Electric potential doesn't have a direction. In Certain region of space, if every point in it is at the same electric potential, then there can be no electric field. On the other hand, if there is a difference of potential between any two points in the region then, an electric field does exist between them. The actual direction of the field will be in the direction in which maximum decrease of potential is established. The rate of change of potential decides the strength of the field $\vec{E}$. The relation is given as,

$$\vec{E} = -\frac{\partial V}{\partial r}\,\hat{a}_r .$$

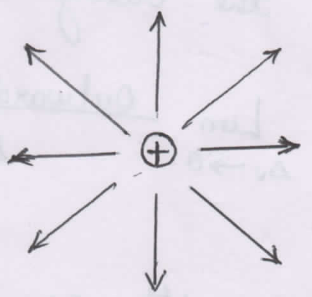
$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right),$$

$$= -\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)V$$

Thus, $\vec{E} = -\nabla V$ (grad V).

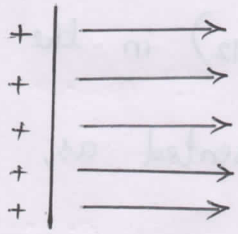
Convention for Directions of Field in Electrostatics:

In electrostatics we assume that the field diverges radially from a positive charge (Fig 8). For a positively charged plane, the field points away from the plane (Fig 9). normally.



Field lines due to a positive charge

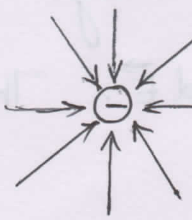
Fig. 8.



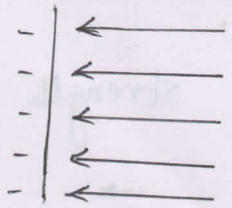
Field due to a positively charged plane.

Fig. 9.

In the case of negative charges, it is just the opposite. (Figs. 10 and 11).



Field lines due to a negative charge
Fig. 10.



Field lines due to a negatively charged plane
Fig. 11.

The direction of the field line at any given point is the direction along which a positive charge would experience the force when placed in the field at that point.

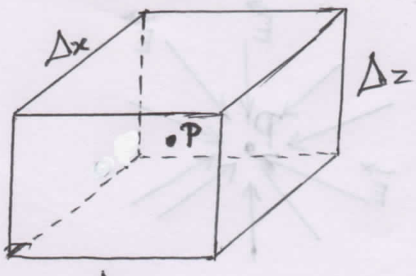
Divergence, $\nabla \cdot \vec{A}$: The divergence of a vector field $\vec{A}$ at a given point P means, it is the outward flux per unit volume as the volume shrinks to zero about P.

Considering an elementary volume Δv around a point P (Fig. 12) in the given space, the divergence at P can be represented as,

$$\text{div } \vec{A} = \lim_{\Delta v \to 0} \frac{\text{Outward flux of } \vec{A}}{\Delta v}$$

Mathematically, it could be rewritten as,

$$\text{div } \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v} \qquad \text{——— ①}$$



Elementary Volume Around P
Fig. 12

Now, by considering a rectangular parellelopiped around the given point P, (Fig.12) as the elementary volume Δv, and working the total outward flux from all its six faces, it is possible to show that,

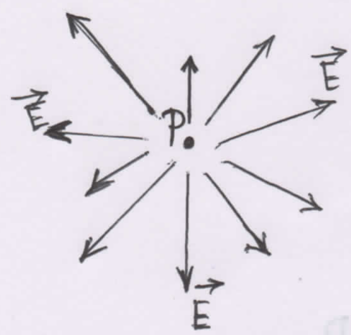
$$\lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot \vec{ds}}{\Delta v} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Bigg|_{at\, P}$$

Since as per equation (1) the left side is divergence, we can write

$$\text{divergence of } \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right).$$

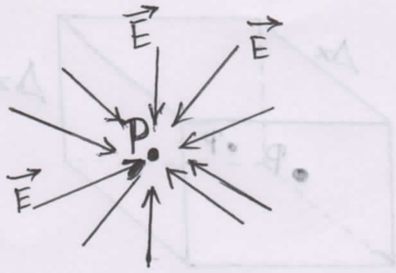
In other words, divergence of $\vec{A} = \nabla \cdot \vec{A}$

Physical Significance Of Divergence: The divergence of the vector field $\vec{A}$ is a measure of how much the field diverges or emanates from that point. If there are positive charges densely packed at a point, then a large no. of field lines diverge from that point. In other words, there is more divergence. The field $\vec{E}$ indicates a source of positive charges at P negative divergence indicating the presence of negative charges at P.
A vector field whose divergence is zero, is called solenoidal field.



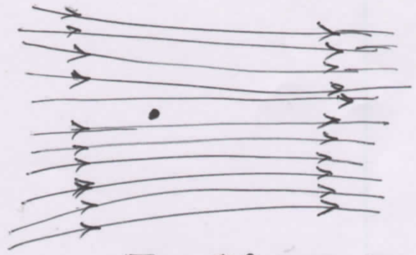
Positive Divergence

Fig. 13A



Negative Divergence

Fig. 13 B



Zero Divergence

Fig 13. C.

Curl $\nabla \times \vec{A}$: The curl of a vector field $\vec{A}$ at a given point P, means, it is the maximum circulation of $\vec{A}$ per unit area as the area shrinks to zero about P. Curl $\vec{A}$ is represented as a vector whose direction is normal to the area around P when the area is oriented to make the circulation maximum.

It can be represented as,

$$\text{curl } \vec{A} = \lim_{\Delta s \to 0} \frac{\text{Max. circulation around P}}{\Delta S}$$

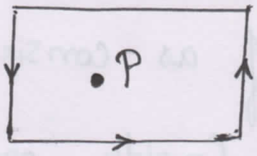
Mathematically we can write,

$$\text{Curl } \vec{A} = \left(\lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot dl}{\Delta S} \right) \hat{a}_n$$

where, the elementary area ΔS is bounded by the curve $L = \oint dl$, and $\hat{a}_n$ is the unit vector normal to ΔS.

By considering a rectangular elementary area across the point P as ΔS (Fig. 14) and working the closed line integral about the 4 sides of the boundary line,

$$\left(\lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot dl}{\Delta s} \right) \hat{a}_n = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

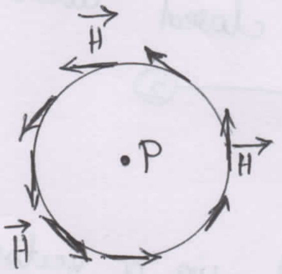


Δy

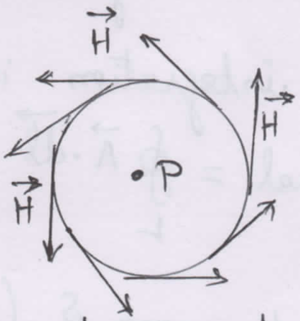
$$\text{curl of } \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\text{curl of } \vec{A} = \nabla \times \vec{A}$$

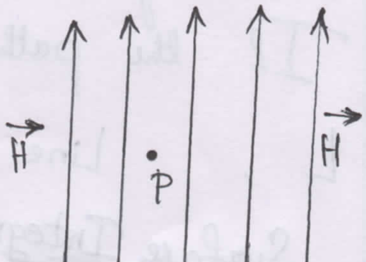
Physical Significance Of curl: The curl of a vector field $\vec{A}$ at a point P is a measure of how much the field curls (circulates) around P. Let the field be a magnetic field $\vec{H}$ around the point P. A vector field whose curl is zero is called irrotational.



Curl of $\vec{H}$
Fig. 15A



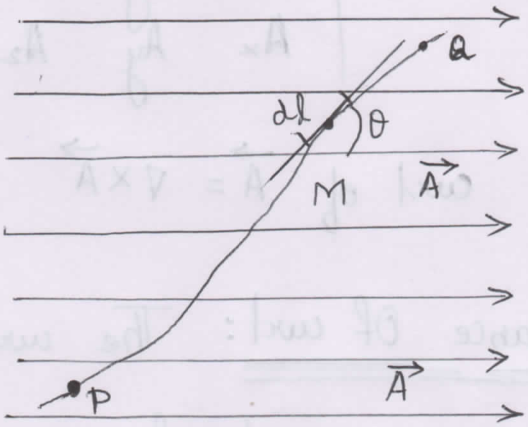
Larger curl
Fig. 15B



A field of zero curl
Fig. 15.C

Linear Integral (or Line Integral):

Consider a linear path of length L from P to Q in a vector field $\vec{A}$ (Fig 16). The line could be thought of as consisting of small elementary lengths dl. Consider one such element at M. At M, draw a tangent. Let the tangent make an angle θ with $\vec{A}$. Then we have $\vec{A} \cdot \vec{dl} = A\, dl \cos\theta$.



Line Integration Fig.16

The dot product $\vec{A} \cdot \vec{dl}$ between P to Q becomes the line integral. $\therefore$ Line integral $= \int_P^Q \vec{A} \cdot \vec{dl}$ ——— (1)

If the path of integration is a closed curve of length L, Line integral $= \oint_L \vec{A} \cdot \vec{dl}$ ——— (2)

Surface Integral:

Consider a surface of area S (Fig.18) in a vector field $\vec{A}$. The surface could be thought of as made up of a number of elementary surfaces each of area dS. Let $\hat{a}_N$ be the unit normal to a dS at M. In a vector field, the elementary surface dS acts as a vector $\vec{dS}$ given as.

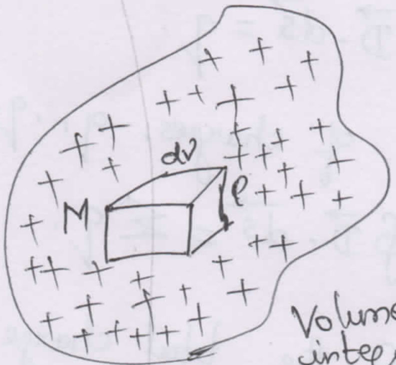
$$\vec{dS} = dS \cdot \hat{a}_N$$

$\vec{A} \cdot \vec{dS}$, it represents the flux of the vector field $\vec{A}$ through $\vec{dS}$. The flux of the field $\vec{A}$ through the surface S can be obtained by integrating $\vec{A} \cdot \vec{dS}$ over the entire surface S.

$$\psi = \int_S \vec{A} \cdot \vec{dS} \quad \text{——— (3)}$$

$$\boxed{\psi = \oint_S (\vec{A} \cdot \hat{a}_N)\, dS}$$

Volume Integral: Consider a volume V in which the charges are distributed uniformly, it is referred as volume charge distribution. Consider an elementary volume dv at M. Let the charge density at M be ρ_v. Then the volume integral of ρ_v over the volume v is $\oint_v \rho_v \, dv$.



Volume integration.

LAWS OF ELECTROSTATICS, MAGNETISM & FARADAY'S LAW

Gauss' law in electrostatics, Gauss' law for magnetic field, Ampere's law and Faraday's law form the foundations of Maxwell's equations. Bio-Savart's law which is the fundamental law in magnetostatics.

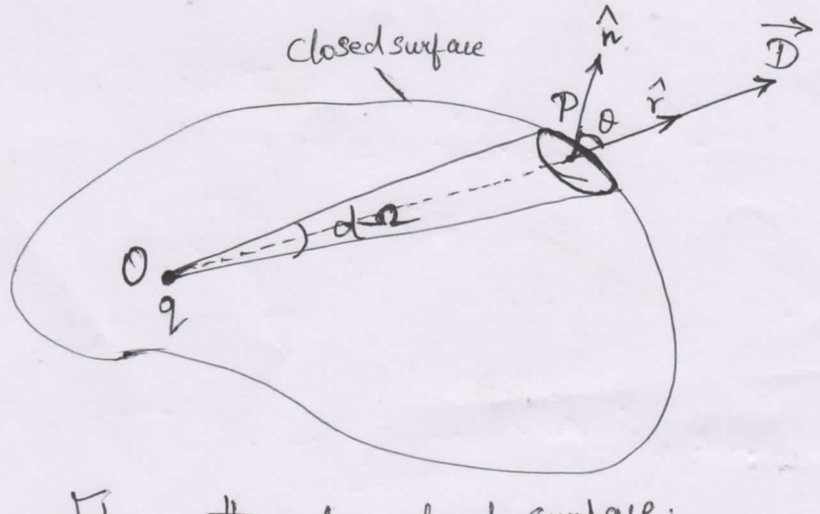
Gauss' Law in electrostatics: Consider electric charges in a certain region. A closed surface of any shape can be imagined surrounding those charges. Such a surface is called "Gaussian surface". The closed surface to be made up of a number of elementary surfaces each of area ds. If $\vec{D}$ is the flux density at ds. The total electric flux over the entire surface.

$$\psi = \oint_S \vec{D} \cdot \vec{ds} = q.$$

If there are a number of charges, $q_1, q_2 \dots$

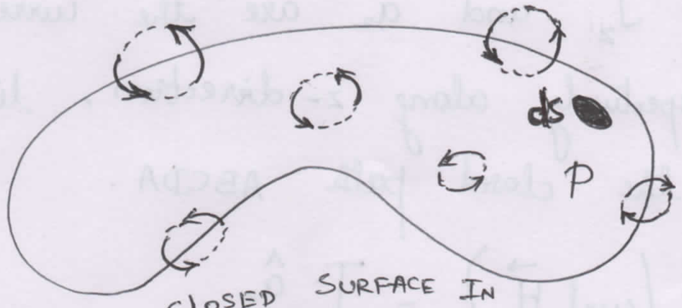
$$\psi = \oint \vec{D} \cdot \vec{ds} = \Sigma q.$$

or, $\psi = Q$ the total charge enclosed.



Flux through closed surface.

Gauss' Law for Magnetic Fields:

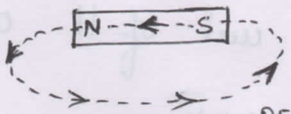


CLOSED SURFACE IN
A MAGNETIC FIELD

The flux over the entire closed surface,

total outward flux = total inward flux

+ve -ve



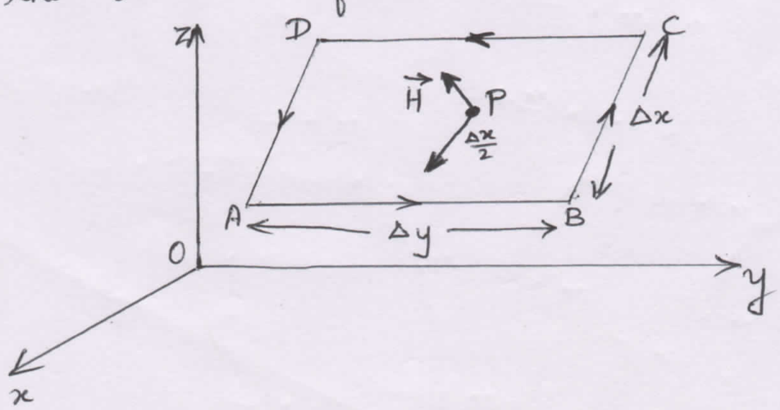
CLOSED PATH OF
A FLUX LINE

∴ The total flux summed over the entire Gaussian surface = 0.

$\nabla \cdot \vec{B} = 0$ where, $\vec{B}$ is the magnetic flux density.

This is one of the four Maxwell's equations.

Ampere's Law: current density $\vec{J}$ is defined as the current per unit area of cross-section of an imaginary plane held normal to the direction of current in a current carrying conductor.



Consider a point P in a magnetic field $\vec{H}$. curl $\vec{H}$ of the rectangular loop ABCD around P in a plane parallel to x-y plane. If J_z and $\hat{a}_z$ are the current density and unit vector respectively along z-direction, line integral of $\vec{H} \cdot \vec{dl}$ over the closed path ABCDA.

$$\left(\text{curl } \vec{H}\right)_1 = J_z \hat{a}_z$$

$$\left(\text{curl } \vec{H}\right)_2 = J_x \hat{a}_x$$

$$\left(\text{curl } \vec{H}\right)_3 = J_y \hat{a}_y$$

are the components of curl of $\vec{H}$ around the point P.

curl $\vec{H}$ is represented as $\nabla \times \vec{H}$

$$\therefore \nabla \times \vec{H} = J_x \hat{a}_x + J_y \hat{a}_y + I_z \hat{a}_z$$

$$\text{Or, } \nabla \times \vec{H} = \vec{J}$$

where, $\vec{J}$ is the current density vector with components J_x, J_y & J_z.

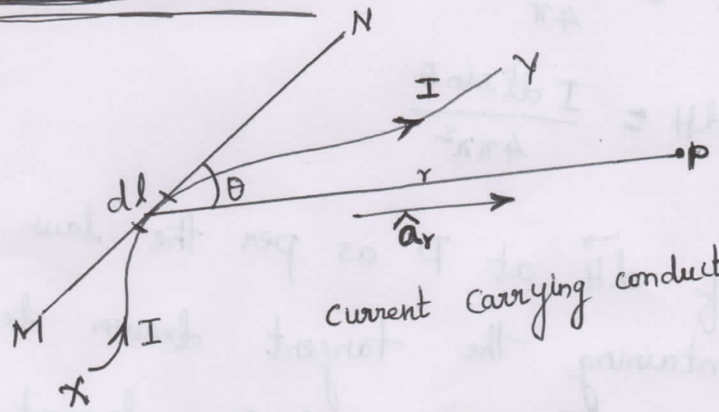
BIOT-SAVART'S LAW:

Magnetic field Intensity $\vec{H}$ and Magnetic flux Density $\vec{B}$:

The magnetic field at a point is dealt with in terms of two vectors $\vec{H}$ and $\vec{B}$. The vector $\vec{H}$ refers to the magnetic field intensity or magnetic field. $\vec{B}$ also refers to the magnetic field, and is referred to as magnetic flux density or magnetic induction. They are related by

$\vec{B} = \mu \vec{H}$, where μ is the permeability of the medium in vacuum, $\vec{B} = \mu_0 \vec{H}$, where μ_0 is the permeability of vaccum $= 4\pi \times 10^{-7}$ H/m.

Biot-Savart's Law:



Let XY be a conductor carrying a current I. Consider a differential element of length dl. Let P be a point chosen arbitrarily. Let the distance Op be r, and the angle between MN and Op be θ.

Statement Of Biot-Savart's law: The magnitude of the magnetic field intensity $\vec{dH}$ at a point due to the current in the differential element is directly proportional to the product of the current I, the magnitude of the length of the differential element dl, and the sine of the angle between the tangent drawn to the element, and the line joining the point and the element, (ie., $\sin\theta$), and it is inversely proportional to the square of the distance between them.

$$\therefore dH \propto \frac{I\,dl \sin\theta}{r^2},$$

$$dH = K \frac{I\,dl \sin\theta}{r^2},$$

$$K = \frac{1}{4\pi}$$

$$dH = \frac{I\,dl \sin\theta}{4\pi r^2}$$

The direction of $\vec{dH}$ at P as per the law is perpendicular to the plane containing the tangent drawn to the element and the line joining the point and the element.

In vector notation, $\vec{dH} = \dfrac{I\,\vec{dl} \times \hat{a}_r}{4\pi r^2}$,

where, the vector $\vec{dl}$ is directed along the direction of current, and $\hat{a}_r$ is the unit vector.

Faraday's Law Of Electromagnetic Induction :

Faraday's experiments revealed that whenever there is a change of magnetic flux linkage in a circuit, an emf is induced in it which results in the flow of current. The flow of current itself sets up a magnetic field. Lenz found that, the direction of magnetic field due to the current flow is such that it opposes the change causing it. i.e., it decreases the magnetic field if the magnetic flux is increasing and increases the magnetic field if the magnetic flux is decreasing. Lenz stated that, the induced e.m.f. is in such a direction as to oppose the change causing it. Faraday's law can be stated as "the magnitude of the induced e.m.f in a circuit is equal to the rate of change of the magnetic flux through it, and its direction opposes the flux change".

$\therefore$ The induced emf e is expressed as,

$$e = -\frac{d\phi}{dt}.$$

If we consider a coil with N turns, then the emf induced across the coil is, $e = -N\dfrac{d\phi}{dt}$.

Integral and Point (or Differential) Forms Of Faraday's Law:

Consider a loop of a conducting material which is stationary.

As per Faraday's law, $e = -\dfrac{d\phi}{dt}$, where $e \rightarrow$ induced emf

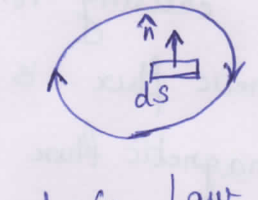
$\phi \rightarrow$ magnetic flux.

$$\frac{d\phi}{dt} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

induced emf in the loop is given by

$$e = \oint \vec{E} \cdot dL$$

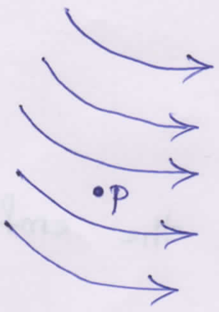
$$\oint \vec{E} \cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$



This is Faraday's Law in integral form.

GAUSS DIVERGENCE THEOREM

Divergence of $\vec{D}$: consider a point P in a region where there are charges. Let ρ_v be the charge density at P. Due to the charges there will be electric field around P. Then,



$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial z}{\partial z} \right) = \rho_v ,$$

$$\nabla \cdot \vec{D} = \rho_v$$

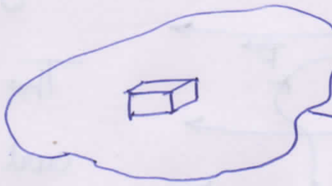
The above equation is known as Maxwell's first equation

$\nabla \cdot \vec{D}$ is the divergence of $\vec{D}$.

Statement Of Gauss' Divergence Theorem: The integral of the normal component of the flux density over any closed surface in an electric field is equal to the volume integral of the divergence of the flux throughout the space enclosed by the surface.

$$\oint_S \vec{D} \cdot d\vec{s} = \int_v \nabla \cdot \vec{D}\, dv$$

Proof: Consider a gaussian surface in a region with certain charge density.



Let ΔQ be the charge within the element. If ρ is the charge density, and since ρ can vary continuously in the volume, we have,

$$\rho_v = \lim_{\Delta v \to 0} \left[\frac{\Delta Q}{\Delta v}\right] = \frac{dQ}{dv}$$

$$dQ = \rho_v \, dv.$$

If Q is the total charge enclosed by the gaussian surface, then,

$$Q = \int dQ = \int_v \rho_v \, dv.$$

But from Maxwell's first equation

$$Q = \int_v \nabla \cdot \vec{D}\, dv.$$

From Gauss' law,

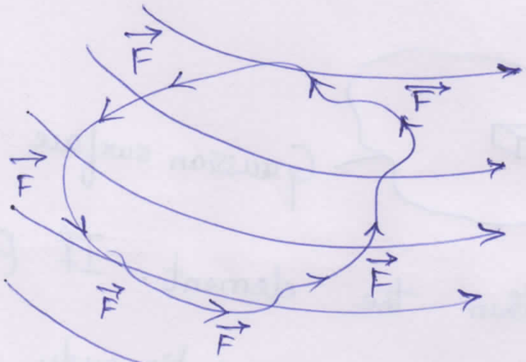
$$\oint_S \vec{D} \cdot d\vec{s} = Q.$$

$$\oint_S \vec{D}\cdot\vec{ds} = \int_V \nabla\cdot\vec{D}\, dv$$

This is, Gauss' divergence theorem.

STOKE'S THEOREM: This provides a way of relating line integral to a surface integral in cases where curl of a vector exists in a vector field.

$$\int(\nabla\times\vec{F})\cdot\vec{ds} = \oint\vec{F}\cdot\vec{dl}$$

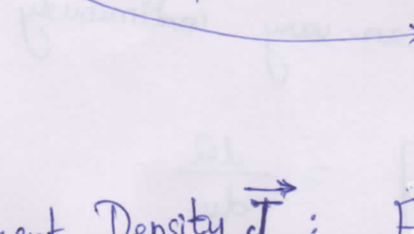


The curl of $\vec{F}$ and sum of curl within any chosen surface then, the sum will be equal to just the circulation of $\vec{F}$ around the boundary of the chosen surface.

Current Density $\vec{J}$: From the discussion of Ampere's law, the current density is the current per unit area of cross-section. If ρ_v is the charge density & $\vec{v}$ is the velocity of charge flow, then

$$\vec{J} = \rho_v\vec{v}$$

Equation Of Continuity: For direct current, if we consider a closed surface at some part of it, the charge flow into the surface is equal to the charge outflow, which means there is no net charge within the surface. (Kirchoff's law)



$$\nabla\cdot\vec{J} = 0.$$

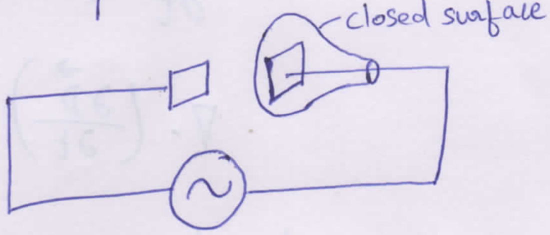
Consider the ampere's law under time varying conditions for the field. $\nabla \times \vec{H} = \vec{J}$. Taking the divergence

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}$$

But, as per rules of vector analysis, the divergence of a curl for any field is always zero.

$$\nabla \cdot \vec{J} = 0.$$

This condition holds good under static conditions. Consider an ac circuit with a capacitor. Consider a closed surface which encloses



only one plate of the capacitor. Whenever there is any current flow into the closed surface, it is not accompanied by a simultaneous current outflow through any part of the surface and vice versa. $\nabla \cdot \vec{J} = 0$ fails in ac circuit.

Equation Of Continuity: Consider a closed surface enclosing certain amount of charge. According to equation of continuity, if there is any charge outflow through this surface, it must be accompanied by a simultaneous reduction of charge within the surface. $\boxed{\nabla \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}}$, where, $\vec{J}$ is the current density & ρ_v is the charge density.

This is equation of continuity.

Displacement Current:

(Modification Of Ampere's Law to suit the Time Varying condition):

To make the ampere's law to work under time varying field conditions, Maxwell suggested the following treatment.

We know from Gauss' Law, $\nabla \cdot \vec{D} = \rho_v$.

Differentiate w.r.t. time, $\frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \frac{\partial \rho_v}{\partial t}$

$$\nabla \cdot \left(\frac{\partial \vec{D}}{\partial t}\right) = \frac{\partial \rho_v}{\partial t}$$

From equation of continuity,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t},$$

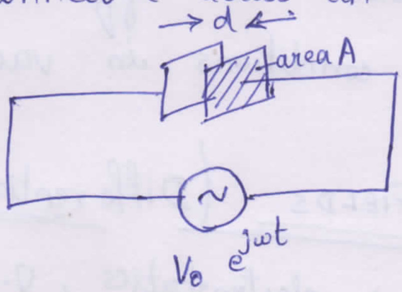
$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0.$$

For the time varying case, $\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$, must be considered instead of $\nabla \cdot \vec{J} = 0$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

This equation is known as Maxwell - Ampere's law.

Expression for Displacement Current: Consider a parallel plate capacitor connected across an a.c. source.



(i) Conduction current: Let the potential be, $V = V_0 e^{j\omega t}$

$$E = \frac{V}{d}$$

$$D = \frac{\varepsilon V}{d} = \frac{\varepsilon}{d} V_0 e^{j\omega t}$$

Displacement current density is given by $\left(\frac{\partial D}{\partial t}\right)$.

If I_D is the displacement current, then $\left(\frac{\partial D}{\partial t}\right) = \frac{I_D}{A}$.

$$\therefore I_D = \left(\frac{\partial D}{\partial t}\right) A = \frac{\partial}{\partial t}\left(\frac{\varepsilon}{d} V_0 e^{j\omega t}\right) A$$

$$I_D = \frac{j\omega \varepsilon A}{d} V_0 e^{j\omega t}$$

Displacement current is the correction factor in Maxwell's equation that appears in time-varying condition but doesn't describe any movement of charges though it has an associated magnetic field.

MAXWELL'S EQUATIONS

Four Maxwell's equations in differential form for time-varying fields and in conditions in vaccum as follows.

TIME-VARYING FIELDS (Differential form or Point form):

1. From Gauss' Law in electrostatics, $\nabla \cdot \vec{D} = \rho_v$

2. From Faraday's law, $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

3. From Gauss' law for magnetic fields, $\nabla \cdot \vec{B} = 0$

4. From Ampere's law, $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

STATIC FIELDS: (Differential form or Point form):

1. $\nabla \cdot \vec{D} = \rho_v$
2. $\nabla \times \vec{E} = 0$
3. $\nabla \cdot \vec{B} = 0$
4. $\nabla \times \vec{H} = \vec{J}$

Wave Equations in Differential Form in Free Space in terms Of Field:

Consider the two curl equations of Maxwell,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But, $D = \varepsilon E$, and $B = \mu H$.

$$\therefore \nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \text{---} \quad (1)$$

$$\nabla \times \vec{E} = -\mu \frac{\partial H}{\partial t} \quad \text{--- (2)}$$

Taking curl for both sides of Eq. (2).

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H}) \quad \text{--- (3)}$$

Since,
$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 E$$
$$= \nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E}$$

(Since, as per Maxwell's equation, $\nabla \cdot \vec{D} = \rho_v$ or $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$).

$\therefore$ From Eqs (3) and (4),

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

Using Eq (1) in the right side, we have.

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\varepsilon}\right)$$

The above equation represents the wave equation in $\vec{E}$ for a medium with constant μ and ε, i.e, a homogeneous and isotropic medium. If we consider free space, i.e, space where there are no charges, or currents, then $\rho_v = 0$, and $J = 0$

$$\boxed{\nabla^2 E - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

Plane Electromagnetic waves in vacuum:

Consider a plane electromagnetic wave along x-direction of wavelength λ. Let the electric field associated with the wave be oriented parallel to y-direction and be denoted as E_y. Let A be its amplitude.

$$\vec{E_y} = A \cos\left[\frac{2\pi}{\lambda}(x-ct)\right]\hat{a}_y$$

and $$\vec{B_z} = \frac{1}{c} A \cos\left[\frac{2\pi}{\lambda}(x-ct)\right]\hat{a}_z$$

Since $\vec{E_y}$ and $\vec{B_z}$ perpendicular to each other in the above electromagnetic wave in vacuum.

two equations, we have,

$$c = \frac{E_y}{B_z}. \quad \text{where, } E_y = |\vec{E_y}|, \ \& \ B_z = |\vec{B_z}|.$$

The classical wave equation for an oscillating physical quantity $\vec{F}$ is represented as,

$$\nabla^2 F - \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2} = 0$$

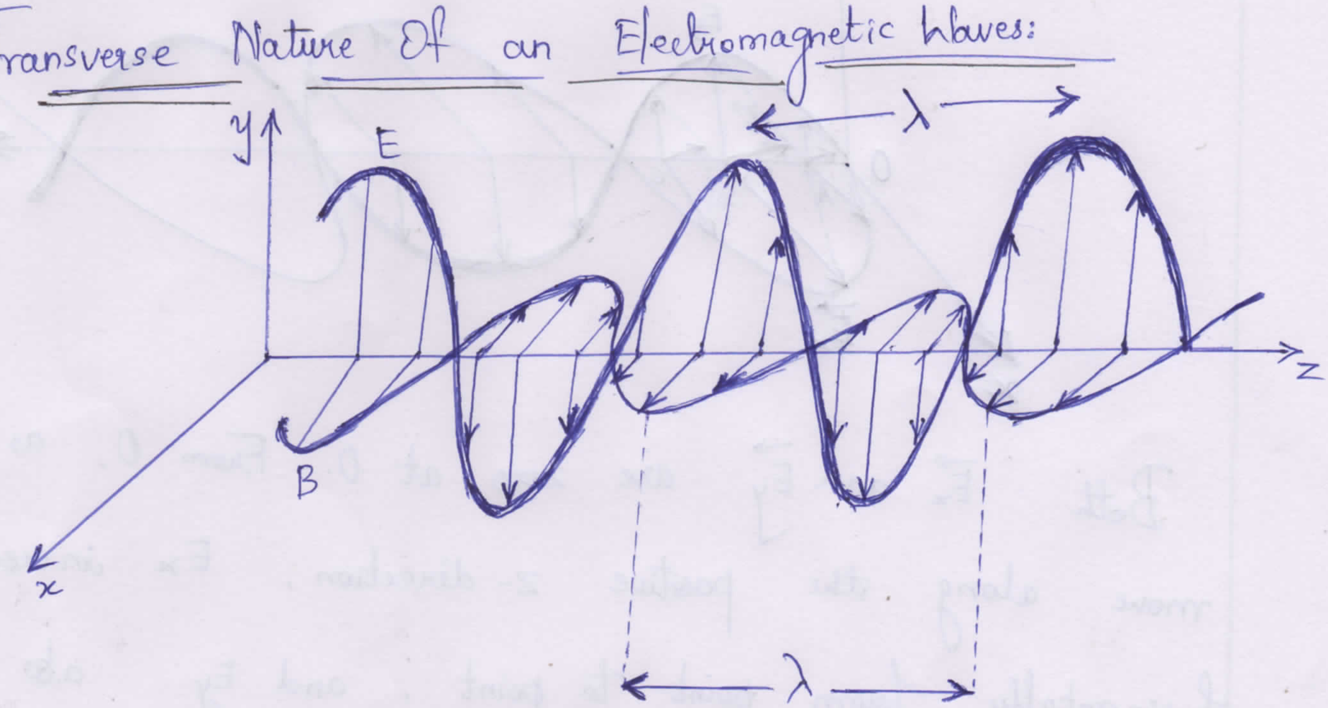
$$\nabla^2 E - \mu t \frac{\partial^2 E}{\partial t^2} = 0$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

POLARIZATION OF EM WAVES

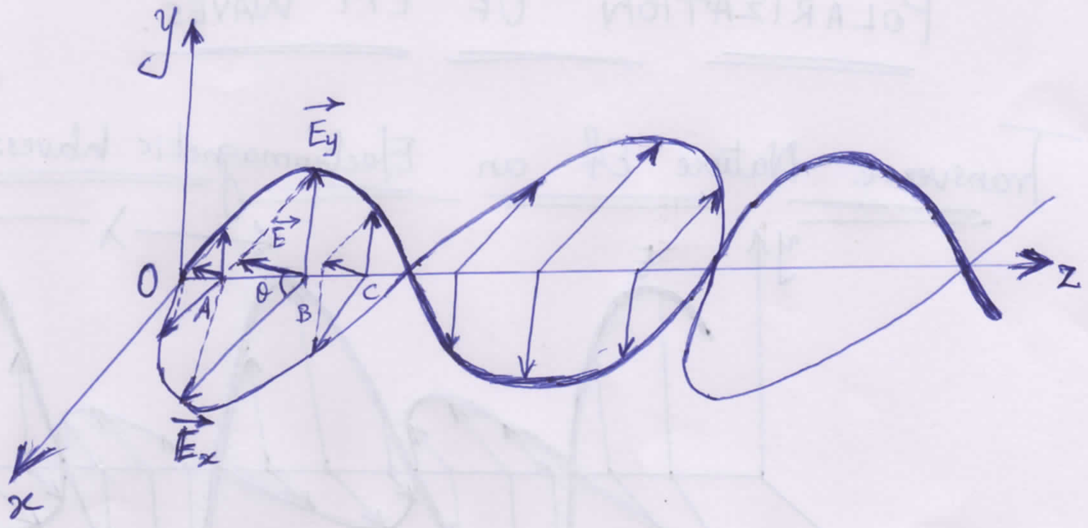
Transverse Nature Of an Electromagnetic Waves:



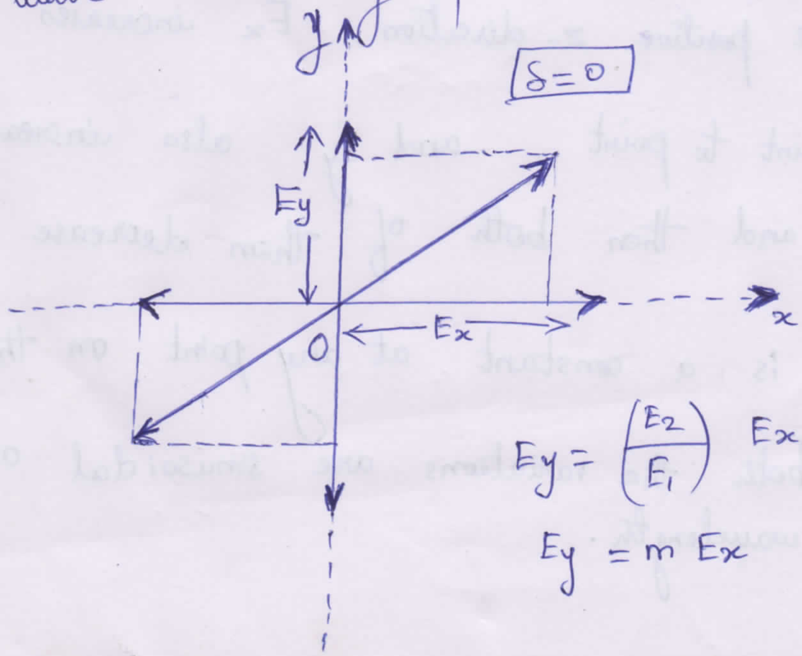
The velocity of movement of any electromagnetic wave, irrespective of wavelength or intensity, is same in a given medium.

Linear Polarization: Let us consider E_x and E_y components which are in identical phase, but of different amplitudes. Both $\vec{E_x}$ and $\vec{E_y}$ are zero at O. From O, as we move along the positive z-direction, E_x increases horizontally from point to point, and E_y also increases vertically upto B, and then both of them decrease simultaneously.

$\left|\frac{E_y}{E_x}\right|$ is a constant at any point on the z-axis since, both the variations are sinusoidal and of same wavelength.



Both $\vec{E_x}$ and $\vec{E_y}$ are zero at O. From O, as we move along the positive z-direction, E_x increases horizontally from point to point, and E_y also increases vertically upto B, and then both of them decrease simultaneously. $\left|\frac{E_y}{E_x}\right|$ is a constant at any point on the z-axis since, both the variations are sinusoidal and of same wavelength. The resultant $\vec{E}$ also varies along with $\vec{E_x}$ and $\vec{E_y}$ but always oriented at an inclination of θ to E_x. Thus, we say that the wave is linearly polarized.

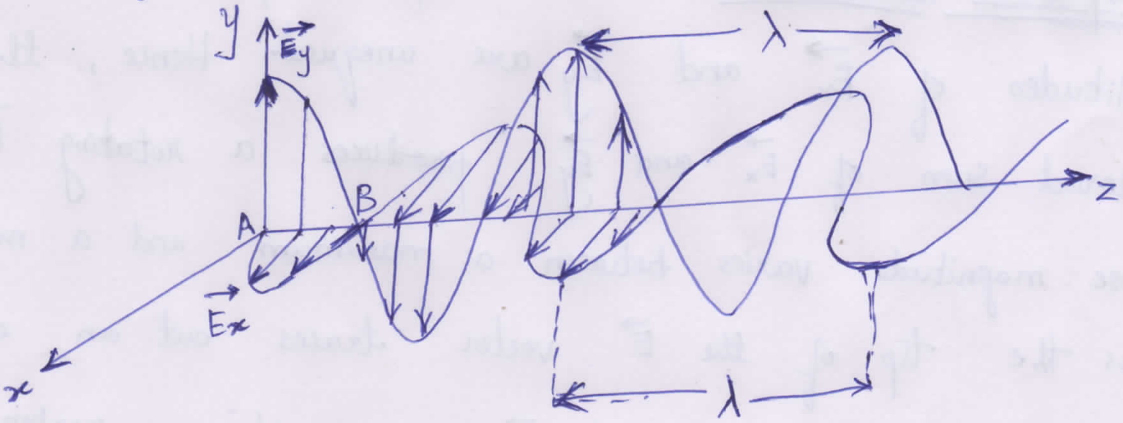


$$E_y = \left(\frac{E_2}{E_1}\right) E_x$$

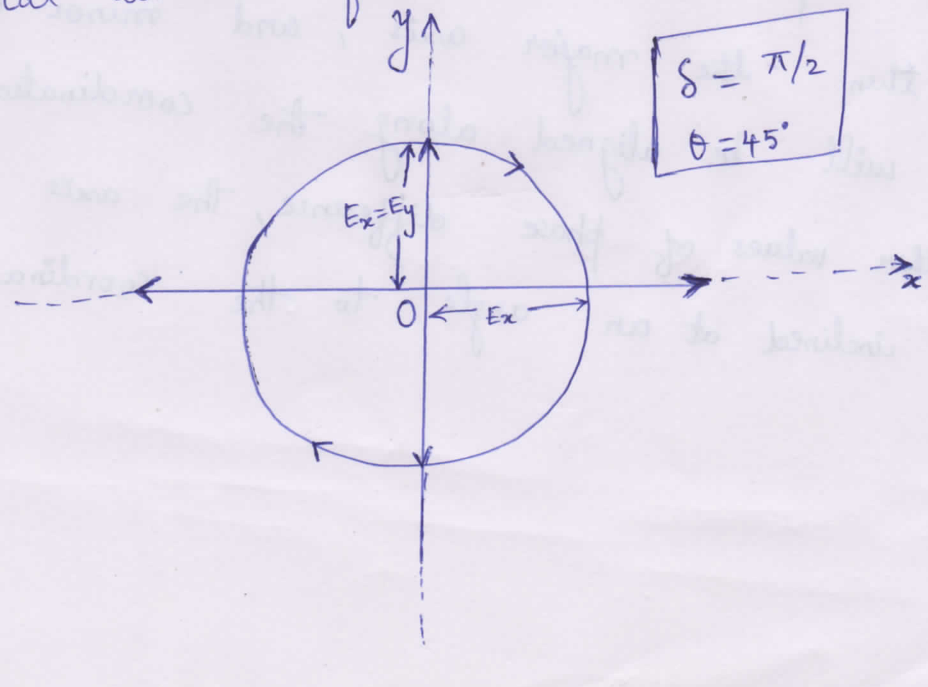
$$E_y = m\, E_x$$

Circular Polarization:

Let us consider the two components $\vec{E_x}$ and $\vec{E_y}$ to be of equal amplitudes i.e., $E_1 = E_2$. But, Let one of them, say $\vec{E_y}$, be ahead of $\vec{E_x}$ by a phase of 90°.



At point A, E_y is maximum but E_x is zero. Hence the resultant field $\vec{E}$ is vertically upwards. At the point B, it is vice-versa, and $\vec{E}$ becomes horizontal. Between A and B, $\vec{E_y}$ and $\vec{E_x}$ make the field $\vec{E}$ tilt progressively towards horizontal direction from A to B.

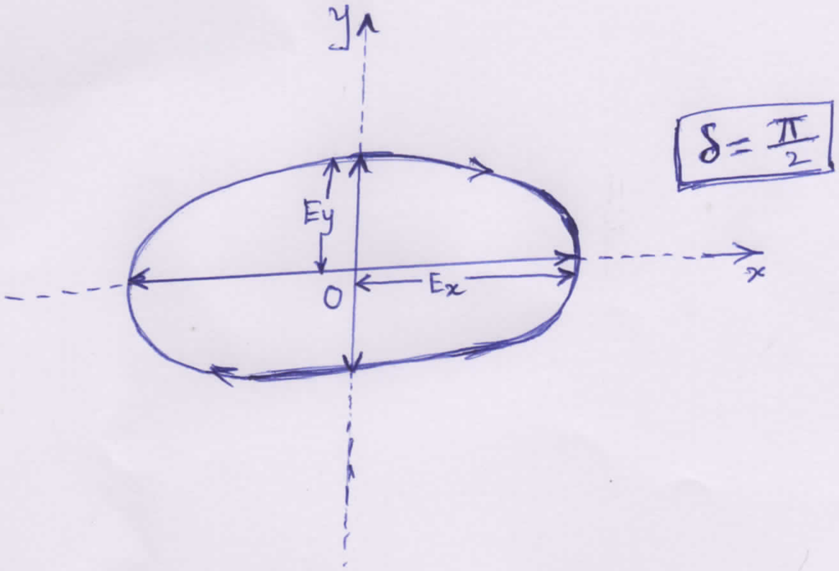


The condition for circular polarization for a wave travelling in z-direction is that, the components $\vec{E_x}$ and $\vec{E_y}$ must have a constant phase difference of 90° and their amplitudes are equal.

Elliptical Polarization: For elliptical polarization, the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are unequal. Hence, the vectorial sum of $\vec{E_x}$ and $\vec{E_y}$ produces a rotating $\vec{E}$-vector whose magnitude varies between a maximum and a minimum. Thus the tip of the $\vec{E}$ vector traces out an ellipse in the projected image of $\vec{E}$ on a plane perpendicular to the z-axis. Thus the wave is elliptically polarized.

Hence, the condition for elliptical polarization for a wave travelling in z-direction is that, the components $\vec{E_x}$ and $\vec{E_y}$ must have a constant non-zero phase difference, and their amplitudes are unequal. If the phase difference is 90°, then the major axis, and minor axis of the ellipse will be aligned along the coordinates axes.

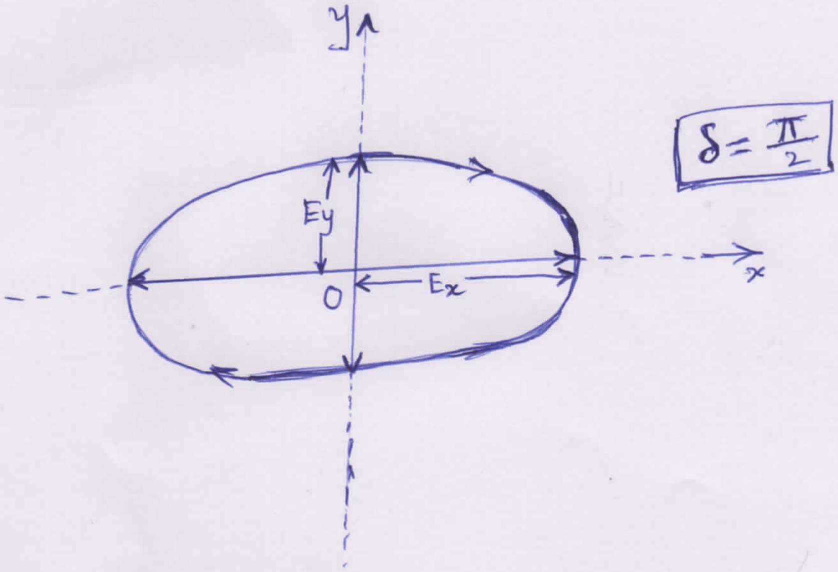
For other values of phase difference, the axis of the ellipse will be inclined at an angle to the coordinate axes.



For elliptical polarization, the two components have unequal amplitudes ie., $E_x \neq E_y$. The phase difference $\delta \neq 0$.

Here, Let us assume $\delta = \pi/2$. For these conditions, it can be shown that, $\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1.}$

This is the equation for an ellipse. Hence the wave is elliptically polarized.



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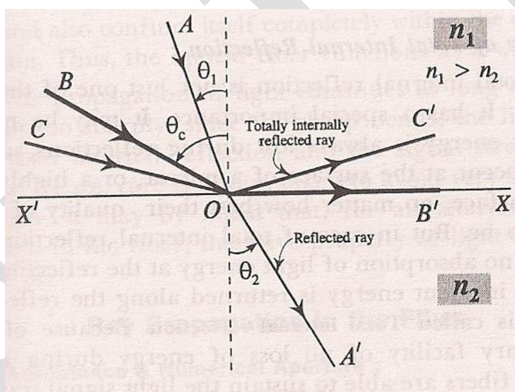
This is the equation for an ellipse. Hence the wave is elliptically polarized.

OPTICAL FIBERS

Total Internal Reflection:

When a ray of light travels from denser to rarer medium it bends away from the normal. As the angle of incidence increases in the denser medium, the angle of refraction also increases. For a particular angle of incidence called the "*critical angle*", the refracted ray grazes the surface separating the media or the angle of refraction is equal to 90°. If the angle of incidence is greater than the critical angle, the light ray is reflected back to the same medium. This is called *"Total Internal Reflection".*

XX^1 is the surface separating medium of refractive index n_1 and medium of refractive index n_2, $n_1 > n_2$. AO and OA^1 are incident and refracted rays. θ_1 and θ_2 are angle of incidence and angle of refraction, $\theta_2 > \theta_1$. For the ray BO, θ_c is the critical angle. OB^1 is the refracted ray which grazes the interface. The ray CO incident with an angle greater than θ_c is totally reflected back along OC^1.



TOTAL INTERNAL REFLECTION

From Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

For total internal reflection,

$$\theta_1 = \theta_c \text{ and } \theta_2 = 90°$$

$$n_1 \sin\theta_c = n_2 \quad \text{(because } \sin 90° = 1\text{)}$$

$$\theta_c = \sin^{-1}(n_2/n_1)$$

In total internal reflection there is no loss or absorption of light energy.

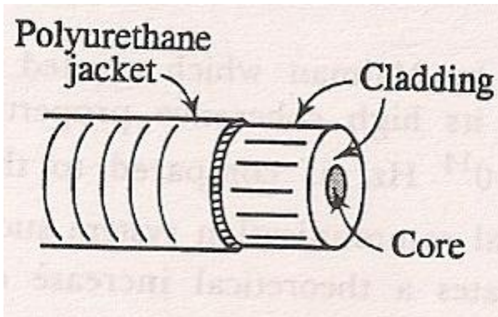
The entire energy is returned along the reflected light. Thus is called Total internal reflection.

Optical Fibers:

Optical fiber is made from transparent dielectrics. It works on the principle of Total internal reflection (TIR).

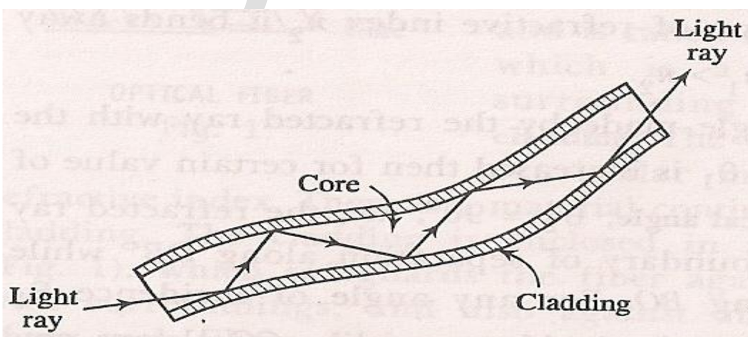
Construction and propagation mechanism:

Optical fiber is cylindrical in shape. The inner cylindrical part is called as core of refractive index n_1. The outer part is called as cladding of refractive index n_2, $n_1>n_2$. There is continuity between core and cladding. Cladding is enclosed inside a polyurethane jacket. Number of such fibers is grouped to form a cable.



OPTICAL FIBER

The light entering through one end of core strikes the interface of the core and cladding with angle greater than the critical angle and undergoes total internal reflection. After series of such total internal reflection, it emerges out of the core. Thus the optical fiber works as a waveguide. Care must be taken to avoid very sharp bends in the fiber because at sharp bends, the light ray fails to undergo total internal reflection.

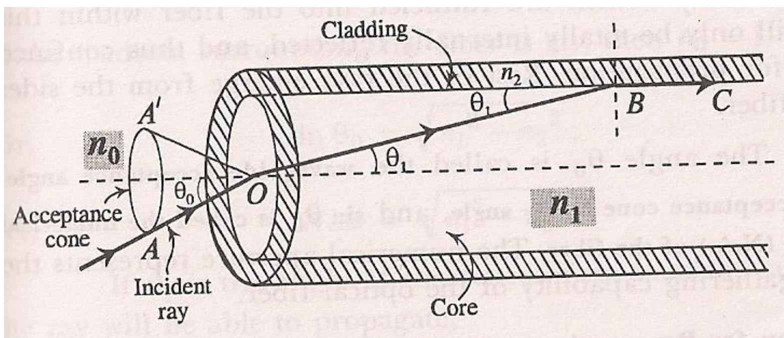


FIBER AS A WAVEGUIDE

Expression for Numerical Aperture of an optical fiber:

Consider a light ray AO incident at an angle 'θ_0' enters into the fiber. Let 'θ_1' be the angle of refraction for the ray OB. The refracted ray OB incident at a critical angle (90°- θ_1) at B grazes the interface between core and cladding along BC. If the angle of incidence is greater than critical angle, it undergoes total internal reflection.

Now if AO is rotated around the fiber axis by keeping θ_0 same then it describes a conical surface through which the beam converges at wide angles into the core and will only be totally internally reflected. Thus θ_0 is called the waveguide acceptance angle and $\sin\theta_0$ is called the numerical aperture.



ACCEPTANCE CONE

Let n_0, n_1 and n_2 be the refractive indices of the medium, core and cladding respectively.

From Snell's law,

$$n_o\sin\theta_0 = n_1\sin\theta_1 \rightarrow (1)$$

At B the angle of incidence is (90 - θ_1)

From Snell's law,

$$n_1\sin(90-\theta_1) = n_2\sin 90$$

$$n_1\cos\theta_1 = n_2$$

$$\cos\theta_1 = n_2 / n_1 \rightarrow (2)$$

From eqn (1)

$$Sin\theta_0 = \frac{n_1}{n_0}\sin\theta_1$$

$$= \frac{n_1}{n_o}\sqrt{1-cos^2\theta_1} \rightarrow (3)$$

Using eqn (2) in (3)

$$Sin\,\theta_0 = \frac{n_1}{n_0}\sqrt{1-\frac{n_2^2}{n_1^2}}$$

$$= \frac{\sqrt{n_1^2 - n_2^2}}{n_o} \rightarrow (4)$$

If the surrounding medium is air, $n_o = 1$

$$Sin\,\theta_0 = \sqrt{n_1^2 - n_2^2}$$

But $\sin\theta_0$ is called numerical aperture.

$$\text{N.A} = \sqrt{n_1^2 - n_2^2}$$

Therefore for any angle of incidence θ_i equal to or less than θ_0, the incident ray is able to propagate if,

$$\theta_i < \theta_0$$

$$\text{Sin}\,\theta_i < \text{Sin}\,\theta_0$$

$$sin\theta_i < \sqrt{n_1^2 - n_2^2}$$

$sin\theta_i < N.A$ is the condition for propagation

Acceptance angle is defined as the maximum angle that a light ray can have relative to the axis of the fiber and propagate through the fiber.

Numerical aperture indicates the ability of the optical fiber to accept light i.e the light gathering capability of the optical fiber. The sign of the acceptance angle also called numerical aperture.

<u>Fractional Index Change:</u>

“It is the ratio of the refractive index difference between the core and cladding to the refractive index of the core of an optical fiber”. $\Delta = \dfrac{n_1 - n_2}{n_1}$

Relation between N.A and Δ:

$$\text{Consider } \Delta = \frac{n_1 - n_2}{n_1}$$

$$n_1 - n_2 = \Delta n_1$$

We have

$$\text{N.A} = \sqrt{n_1^2 - n_2^2}$$

$$= \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

Considering $n_1 \approx n_2$

$$= \sqrt{(n_1 + n_2)\Delta n_1}$$

$$N.A = \sqrt{2n_1^2 \Delta}$$

$$N.A = n_1\sqrt{2\Delta}$$

Increase in the value of Δ increases N.A

It enhances the light gathering capacity of the fiber. Δ value cannot be increased very much because it leads to intermodal dispersion intern signal distortion.

Modes of propagation and V-number:

The number of paths supported for propagation in the fiber is known as modes of propagation and is determined by a parameter called V-number.

If the surrounding medium is air, then v- number is given by

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where 'd' is the core diameter, n_1 and n_2 are refractive indices of core and cladding respectively, 'λ' is the wavelength of light propagating in the fiber.

$$V = \frac{\pi d}{\lambda}(NA)$$

If the fiber is surrounded by a medium of refractive index n_0, then,

$$V = \frac{\pi d}{\lambda}\frac{\sqrt{n_1^2 - n_2^2}}{n_0}$$

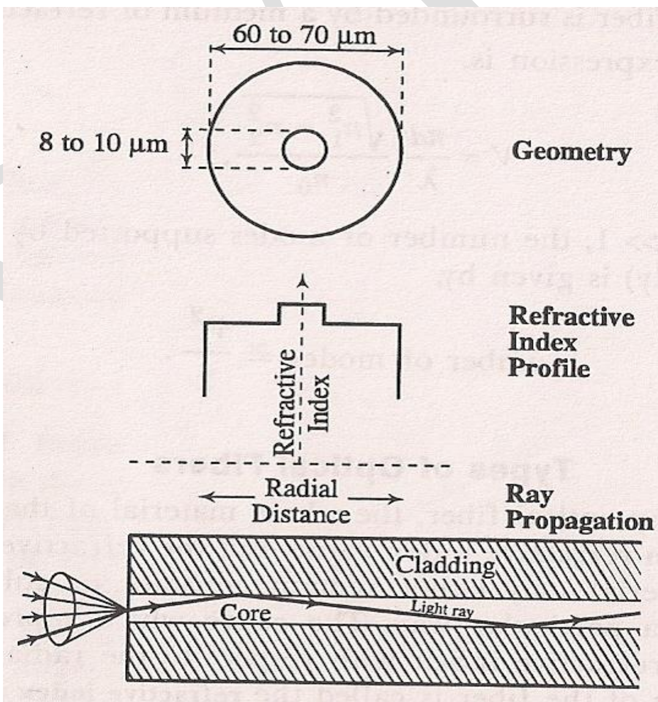
For V >1, the number of modes supported by the fiber is given by, number of modes ≈ $V^2/2$.

Types of optical fibers:

In an optical fiber the refractive index of cladding is uniform and the refractive index of core may be uniform or may vary in a particular way. The curve which represents the variation of refractive index with respect to the radial distance from the axis of the optical fiber is called refractive index profile.

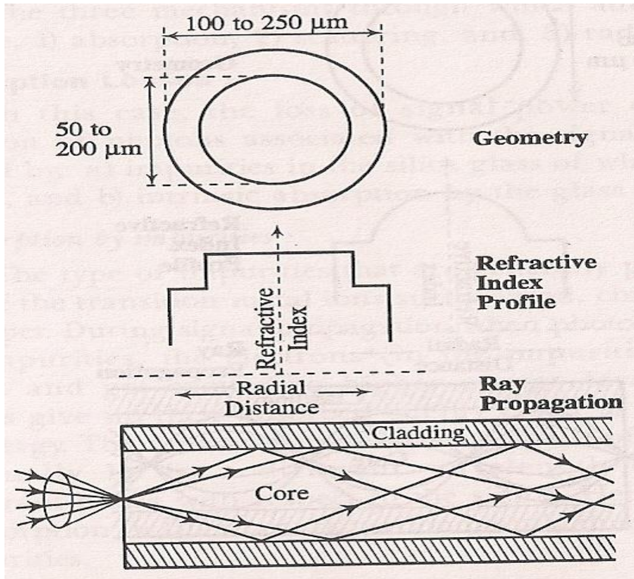
Following are the different types of fibers:

***1) Single mode fiber*:** Refractive index of core and cladding has uniform value. the refractive index of core is greater than that of cladding, hence R I profile takes the form of a step .the diameter of the core is about 8-10μm and that of cladding is 60-70μm.Because of its narrow core it can guide just a single mode. LASER is used as source. They are used in submarine.



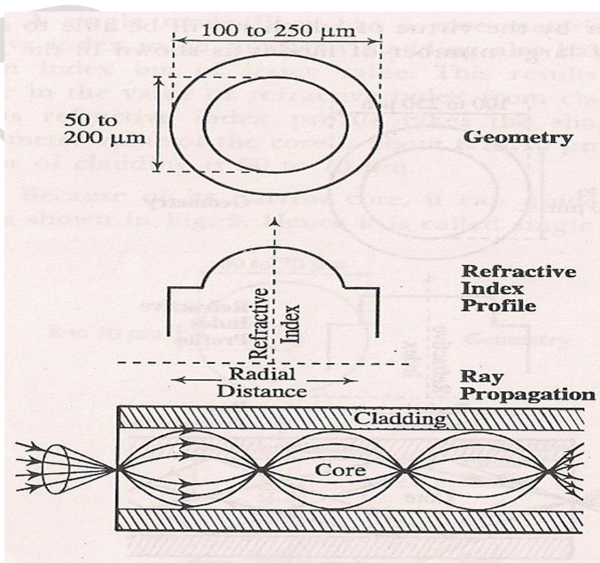
SINGLE MODE FIBER

2) <u>Step index multimode fiber:</u> It is similar to single mode fiber but core has large diameter. It can guide large number of modes hence it is called as multimode
fiber. In this type of fiber the path lengths for different modes are different. Therefore they reach the end with time delay. Laser or LED is used as a source of light. It has an application in data links.



STEP - INDEX MULTIMODE FIBER

3)<u>Graded index multimode fiber:</u> It is also called GRIN. The refractive index of core decreases from the axis towards the core cladding interface. The refractive index profile is shown in figure. In this type of fiber all the modes will have same optical path length and reach the other end at the same time. Laser or LED is used as a source of light. It is the expensive of all. It is used in telephone trunk between central offices.



GRADED - INDEX MULTIMODE FIBER

Attenuation:

Attenuation is the loss of optical power as light signal travels through a fiber. It is expressed in decibel/kilometer [db/km]. If P_{in} is the input power and P_{out} is the output power after passing through a fiber of length 'L', the mean attenuation constant or coefficient 'α' of the fiber, in units of db/km is given by

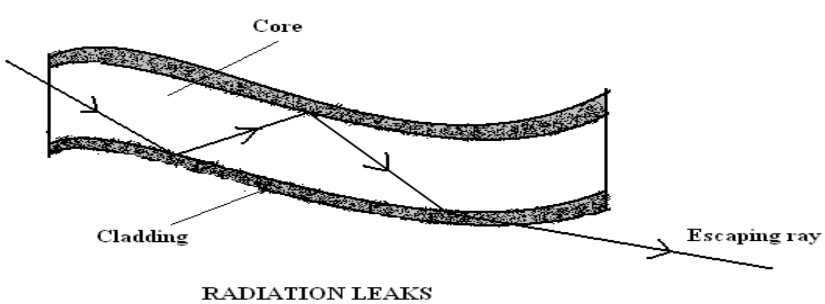
$$\alpha = -\frac{10}{L}\log_{10}\left(\frac{p_{out}}{p_{in}}\right) \qquad \text{dB/km}$$

Attenuation can be caused by three mechanisms.

1. **Absorption loss: -** Absorption of photons by impurities like metal ions such as iron, chromium, cobalt and copper in the silica glass of which the fiber is made of. During signal processing photons interact with electrons of impurity atoms. The atoms are excited and de-excite by emitting photons of different wavelengths. Hence it is a loss of energy. The other impurity such as hydroxyl ions (OH) causes significant absorption loss. The absorption of photons by fiber material itself is called intrinsic absorption.

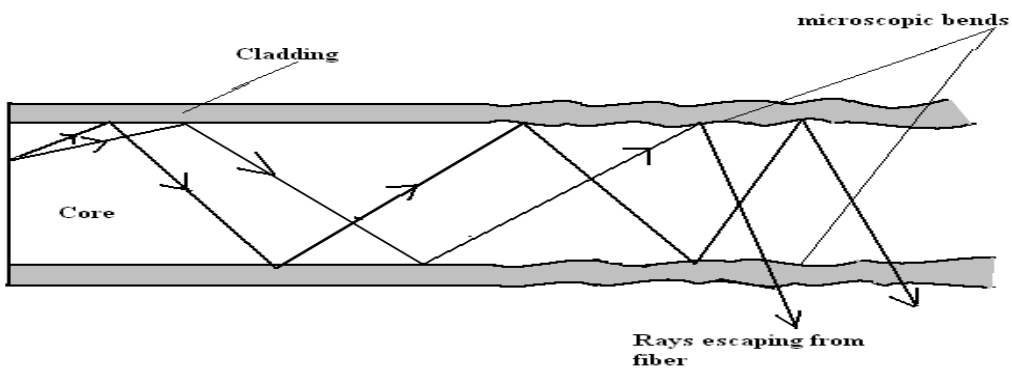
2. **Scattering loss:** When the wavelength of the photon is comparable to the size of the particle then the scattering takes place known as Rayleigh scattering. This is because of the non uniformity in manufacturing, the refractive index of the material changes leads to a scattering of light when light enters such non-uniform regions. It is inversely proportional to the fourth power of wavelength. Scattering of photons also takes place due to trapped gas bubbles which are not dissolved at the time of manufacturing.

3. **Radiation losses:** Radiation losses occur due to macroscopic bends and microscopic bends.
   Macroscopic bending: All optical fibers are having critical radius of curvature provided by the manufacturer. If the fiber is bent below that specification of radius of curvature, the light ray incident on the core cladding interface will not satisfy the condition of TIR. This causes loss of optical power.



RADIATION LEAKS

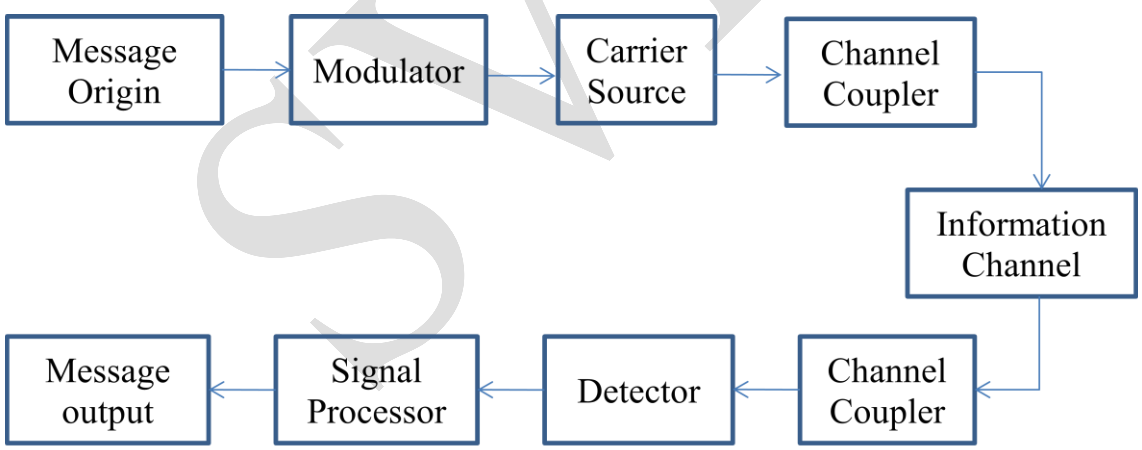
Microscopic bending: Optical power loss in optical fibers is due to non-uniformity of the optical fibers when they are laid. Non uniformity is due to manufacturing defects and also lateral pressure built up on the fiber. The defect due to non uniformity (micro endings) can be overcome by introducing optical fiber inside a good strengthen polyurethane jacket.



MICROSCOPIC BENDING LOSS

Applications:

Point to Point Communication using optical fibers :



Optical fiber communication system consists of transmitter, information channel and receiver. Transmitter converts an electrical signal into optical signal. Information channel carries the signal from transmitter to receiver. The receiver converts optical signal to electrical form. The block diagram of optical fiber communication system is shown in fig.

Message origin: It converts a non electrical message into an electrical signal.

Modulator: It converts the electrical message into proper format and it helps to improve the signal onto the wave which is generated by the carrier source.

There are two types of format. They are Analog and digital. Analog signal is continuous and it doesn't make any change in the original format. But digital signal will be either in ON or OFF state.

Carrier source: It generates the waves on which the data is transmitted. These carrier waves are produced by the electrical oscillator. Light emitting diodes (LED) and laser diodes (LD) are the different sources.

Channel Coupler: (Input) The function of the channel coupler is to provide the information to information channel. It can be an antenna which transfers all the data.

Information channel: It is path between transmitter and receiver. There are two types of information channel. They are guided and unguided. Atmosphere is the good example for unguided information channel. Co-axial cable, two-wire line and rectangular wave guide are example for guided channel.

Channel Coupler: (Output) The output coupler guides the emerged light from the fiber on to the light detector.

Detector: The detector separates the information from the carrier wave. Here a photo-detector converts optical signal to electronic signal.

Signal processor: Signal processor amplifies the signals and filters the undesired frequencies.

Message output: The output message will be in two forms. Either person can see the information or hear the information. The electrical signal can be converted into sound wave or visual image by using CRO.

Advantages of optical communication system:

1) It carries very large amount of information in either digital or analog form due to its large bandwidth.
2) The materials used for making optical fiber are dielectric nature. So, it doesn't produces or receives any electromagnetic and R-F interferences.
3) Fibers are much easier to transport because of their compactness and lightweight.
4) It is easily compatible with electronic system.
5) It can be operated in high temperature range.
6) It does not pick up any conducted noise.
7) Not affected by corrosion and moisture.
8) It does not get affected by nuclear radiations.
9) No sparks are generated because the signal is optical signal.

- Note: - Optical fibers are used in sensors like pressure sensor, voltage sensor and current sensors.

  Optical fibers are used in local networks like data link purpose.