Networks Assignment 2

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April 2, 2022

The first letter of my first name is M=13
The first 2 letters of my surname K=11

$$p_1 = n_1(mod\ 6) = 13(mod\ 6) = 5$$
$$p_2 = n_2(mod\ 6) = 11(mod\ 6) = 1$$

Question 1

The Watts-Strogatz clustering coefficient wants to find the clustering node proportional $C_i = \frac{\text{number of transitive relations of node } i}{\text{total number of possible transitive relations of node } i} = \frac{2t_i}{k_i(k_i-1)}$
where t_i denotes the number of triangles attached to node i of degree k_i.

In this case we have two types nodes which we are going to name them type i and j. The top nodes we have type i and for bottom nodes we have type j.

Starting with the top node we can observe that every node have one.

$$\text{Type } i$$
$$C_i = \frac{2(t_i)}{k_i(k_i-1)} = \frac{2(1)}{2(2-1)} = 1$$

There are $n-2$ nodes of type i, $\sum C_i = (n-2) \times 1 = n-2$

$$\text{Type } j$$
$$C_j = \frac{2(t_j)}{k_j(k_j-1)} = \frac{2(n-2)}{(n-1)(n-1-1)} = \frac{2(n-2)}{(n-1)(n-2)} = \frac{2}{n-1}$$

Now we need to sum up C_i and C_j

$$\bar{C} = \frac{1}{n}\sum_i C_i$$
$$\bar{C} = \frac{1}{n}\left(\frac{2}{n-1} + n - 2\right) = \frac{2}{n(n-1)} + 1 - \frac{2}{n}$$

and we need to find the limit

$$\lim_{n\to\infty}\left(\frac{2}{n(n-1)} + 1 - \frac{2}{n}\right) = 1$$

In this clustering we have $t = |C_3|$ as the total number of triangles and $|P_2|$ is the number of paths of length 2 in the network. Thus

$$C = \frac{3t}{|P_2|} = \frac{3|C_3|}{|P_2|}$$

Type j

$$\binom{n-1}{2} \times 2 = \frac{(n-1)(n-2)}{2} \times 2 = (n-1)(n-2)$$

Type i

$$\binom{2}{2} \times (n-2) = n-2$$

Now for $|P_2|$ we need to sum up Type i and j. $|P_2| = (n-2) + (n-1)(n-2) = (n-2)(1+n-1) = n(n-2)$ and $|C_3| = n-2$. Hence, the Newman clustering coefficient $C = \frac{3|C_3|}{|P_2|} = \frac{3(n-2)}{(n-2)n} = \frac{3}{n}$. Taking the limit will give us the following

$$\lim_{n\to\infty}\left(\frac{3}{n}\right) = 0$$

Question 2

G is a connected k-regular network with n nodes and that its adjacency matrix A has only four distinct eigenvalues namely $\lambda_1 = 10, \lambda_2 = 4, \lambda_3 = 1$ and $\lambda_4 = -2$. The spectrum of G network is $\sigma(A) = \{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$.

To find the number of nodes we have $p_1 + p_2 + p_3 + p_4 = n$. In this case we have $p_1 = 1, p_2 = 2, p_3 = 4, p_4 = 11$,therefore, the number of nodes is $n = 1 + 2 + 4 + 11 = 18$.

The Gershgorin disks tells us that in a connected G network $|\lambda_1 \leq k|$ and since $A\bar{e} = k\bar{e}$ we know that $\lambda_1 = k$. So, since $tr(A^2)$ double counts we will find the edges of G using $\frac{nk}{2} = \frac{18\times 10}{2} = 90$ edges.

Now the number of triangles t is given by $t = \frac{1}{6}tr(A^3) = \frac{1}{6}(k^3 + p_2\lambda_2^3 + p_3\lambda_3^3 + p_4\lambda_4^3) = \frac{1}{6}((10)^3 + 2(4)^3 + 4(1)^3 + 11(-2)^3) = 174$ triangles.

To calculate the total number of distinct 4-cycles $\frac{1}{8}\left(k^4 + p_2\lambda_2^4 + p_3\lambda_3^4 + p_4\lambda_4^4 - nk(2k-1)\right) = \frac{1}{8}\left((10)^4 + 2(4)^4 + 4(1)^4 + 11(-2)^4 - 18 \times 10(2(10)-1)\right) = 909$ squares.

The network G have 18 nodes,90 edges, 174 triangles and 909 squares.

Question 3

We start by constructing the distance the distance matrix of the network in the graph 5.

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 & 3 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 3 & 3 & 3 & 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Now the vector of distance-sum of each node is then $\bar{s} = D\bar{e} = (\bar{e}^T D)^T$.

$$\bar{s} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 & 3 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 3 & 3 & 3 & 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 & 3 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\bar{s} = [10\ 13\ 14\ 15\ 15\ 14\ 13\ 17\ 15]^T$

$$\bar{s}\begin{bmatrix} 0.8 & 0.615 & 0.571 & 0.533 & 0.533 & 0.571 & 0.615 & 0.471 & 0.471 \end{bmatrix}$$

Now the closeness of the node i in this network is calculated by the

$$CC(i) = \frac{n-1}{s(i)}$$

.

$$\begin{aligned}
CC(1) &= \frac{8}{10} = 0.8\\
CC(2) &= \frac{8}{13} = 0.615\\
CC(3) &= \frac{8}{14} = 0.571\\
CC(4) &= \frac{8}{15} = 0.533\\
CC(5) &= \frac{8}{15} = 0.533\\
CC(6) &= \frac{8}{14} = 0.571\\
CC(7) &= \frac{8}{13} = 0.615\\
CC(8) &= \frac{8}{17} = 0.471\\
CC() &= \frac{8}{17} = 0.471
\end{aligned}$$

The full vector of closeness centralities

$$CC = \begin{bmatrix}0.8 & 0.615 & 0.571 & 0.533 & 0.533 & 0.571 & 0.615 & 0.471 & 0.471\end{bmatrix}^T$$

The betweenness centrality of node 1 of the network with graph 1

$$BC(i) = \sum_j \sum_k \frac{\rho(j,i,k)}{\rho(j,k)}, i \neq j \neq k$$

where $\rho(j,k)$ is number of shortest paths connecting the node j to the node k, and $\rho(j,i,k)$ is the number of the shortest paths that pass through node i in the network.

(j,k)	$\rho(j,1,k)$	$\rho(j,k)$	$\dfrac{\rho(j,1,k)}{\rho(j,k)}$
(2,6)	1	1	1
(2,7)	1	1	1
(2,8)	1	1	1
(3,4)	1	2	$\frac{1}{2}$
(3,5)	1	2	$\frac{1}{2}$
(3,6)	1	1	1
(3,7)	1	1	1
(3,8)	1	1	1
(4,5)	1	2	$\frac{1}{2}$
(4,6)	1	1	1
(4,7)	1	1	1
(4,8)	1	1	1
(5,6)	1	1	1
(5,7)	1	1	1
(5,8)	1	1	1
(6,7)	1	1	1
(6,8)	1	1	1
			$BC(1) = 15.5$