Financial Maths Assignment 2

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Question 1

(a)To calculate $d(t^2W(t))$ we are going to use the Ito's Lemma which is defined by the following expression $d(t^2W(t)) = \dfrac{\partial f}{\partial t}dt + \dfrac{\partial f}{\partial W(t)}dW(t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial W(t)^2}dW(t)^2$ and expand it as:

$$\begin{aligned} d(t^2W(t)) &= 2tW(t)dt + t^2dW(t) + \frac{1}{2}\cdot 0dW(t)^2 \\ &= 2tW(t)dt + t^2dW(t) \end{aligned}$$

(b)Now if we integrate $d(t^2W(t))$ and have the following:

$$\begin{aligned} \int_0^T d(t^2W(t)) &= 2\int_0^T tW(t)dt + \int_0^T t^2dW(t) \\ \left[t^2W(t)\right]_0^T &= 2\int_0^T tW(t)dt + \int_0^T t^2dW(t) \\ T^2W(T) &= 2\int_0^T tW(t)dt + \int_0^T t^2dW(t) \end{aligned}$$

We need to make $\displaystyle\int_0^T tW(t)dt$ the subject of the equation to find $I(T)$,

$$\begin{aligned} 2\int_0^T tW(t)dt &= T^2W(T) - \int_0^T t^2dW(t) \\ 2\int_0^T tW(t)dt &= T^2W(T) - \int_0^T t^2dW(t) \\ \int_0^T tW(t)dt &= \frac{1}{2}T^2W(T) - \frac{1}{2}\int_0^T t^2dW(t) \\ \therefore I(T) := \int_0^T tW(t)dt &= \frac{1}{2}T^2W(T) - \frac{1}{2}\int_0^T t^2dW(t) \end{aligned}$$

(c) Using (b) to simplify the equation taking common factor and rearranging some terms

$$
\begin{aligned}
I(T) := \int_0^T tW(t)dt = \frac{1}{2}T^2W(T) - \frac{1}{2}\int_0^T t^2 dW(t) \\
I(T) &= \frac{1}{2}T^2W(T) - \frac{1}{2}\int_0^T t^2 dW(t) \\
&= \frac{1}{2}\left[T^2W(T) - \int_0^T t^2 dW(t)\right] \\
&= \frac{1}{2}\left[\int_0^T T^2 dW(t) - \int_0^T t^2 dW(t)\right] \\
I(T) &= \frac{1}{2}\int_0^T \left(T^2 - t^2\right) dW(t)
\end{aligned}
$$

(d) First we will show the expected value to be zero $E(I(T)) = 0$

$$
\begin{aligned}
E(I(T)) &= E\left[\frac{1}{2}\int_0^T \left(T^2 - t^2\right) dW(t)\right] \\
&= \frac{1}{2}\int_0^T \left(T^2 - t^2\right) E(dW(t)) \\
&= \frac{1}{2}\int_0^T \left(T^2 - t^2\right) E(W(t+dt) - W(t)) \\
&= \frac{1}{2}\int_0^T \left(T^2 - t^2\right) \cdot 0 \\
E(I(T)) &= 0 \quad \blacksquare
\end{aligned}
$$

Now we need to find the variance

$$
\begin{aligned}
var(I(T)) &= E(I(T)^2) - E(I(T))^2 \\
&= E(I(T)^2) - 0 \\
&= E\left[\frac{1}{2}\int_0^T \left(T^2 - t^2\right) dW(t)\right]^2 \\
&= E\left[\frac{1}{4}\int_0^T \left(T^2 - t^2\right)^2 dt\right] \\
&= \frac{1}{4}\int_0^T \left(T^2 - t^2\right)^2 dt \\
&= \frac{1}{4}\int_0^T \left(T^4 - 2T^2t^2 + t^4\right) dt \\
&= \frac{1}{4}\left[\left[T^4 t\right]_0^T - \left[\frac{2}{3}T^2t^3\right]_0^T + \left[\frac{t^5}{5}\right]_0^T\right] \\
&= \frac{1}{4}\left[T^5 - 2\frac{T^5}{3} + \frac{T^5}{5}\right] \\
&= \frac{1}{4}\left[\frac{8T^5}{15}\right] \\
var(I(T)) &= \frac{2}{15}T^5 \quad \blacksquare
\end{aligned}
$$

Question 2

(a) We are given the equation $S(t) = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}$ now we are going to use the Ito's Lemma

$$
\begin{aligned}
dS(t) &= \frac{\partial S(t)}{\partial t}dt + \frac{\partial S(t)}{\partial W(t)}dW(t) + \frac{1}{2}\frac{\partial^2 S(t)}{\partial W(t)^2}dW(t)^2 \\
&= -\frac{1}{2}\sigma^2 S(t) + \sigma S(t)dW(t) + \frac{1}{2}\sigma^2 S(t)dW(t)^2 \\
&= -\frac{1}{2}\sigma^2 S(t) + \sigma S(t)dW(t) + \frac{1}{2}\sigma^2 S(t)dt \\
\therefore dS(t) &= \sigma S(t)dW(t)\blacksquare
\end{aligned}
$$

$$S(t) = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}$$

at t=0

$$S(0) = S_0 e^{-\frac{1}{2}\sigma^2 \cdot 0 + \sigma W(0)} = S_0 e^0$$

$$S(0) = S_0$$

Question 4

(a) We are given that $d(e^{bt}r)$ derive the integrated form of the Ornstein Uhlenbeck as follows:

$$
\begin{aligned}
d(e^{bt}r) &= \frac{\partial(e^{bt}r)}{\partial t}dt + \frac{\partial(e^{bt}r)}{\partial r}dr + \frac{1}{2}\frac{\partial^2(e^{bt}r)}{\partial r^2}dr^2 \\
&= bre^{bt}dt + e^{bt}dr + 0
\end{aligned}
$$

We are given that $dr = b(a-r)dt + \sigma dW(t)$ and we are going to use it as follows:

$$
\begin{aligned}
d(e^{bt}r) &= bre^{bt}dt + e^{bt}(b(a-r)dt + \sigma dW(t)) \\
&= e^{bt}(brdt + badt - brdt + \sigma dW) \\
d(e^{bt}r) &= e^{bt}badt + e^{bt}\sigma dW(t)
\end{aligned}
$$

Now we need to integrate both sides with limits from 0 to t

$$
\begin{aligned}
\int_0^t d(e^{bs}r) &= \int_0^t e^{bs}bads + \int_0^t e^{bs}\sigma dW(s) \\
\left[e^{bs}r\right]_0^t &= ba\left[\frac{e^{bs}}{b}\right]_0^t + \sigma\int_0^t e^{bs}dW(s) \\
re^{bt} - re^0 &= ae^{bt} - ae^0 + \sigma\int_0^t e^{bs}dW(s) \\
re^{bt}e^{-bt} &= re^{-bt} + ae^{bt}e^{-bt} - ae^{-bt} + e^{-bt}\sigma\int_0^t e^{bs}dW(s) \\
r &= re^{-bt} + a - ae^{-bt} + e^{-bt}\sigma\int_0^t e^{bs}dW(s) \\
r &= re^{-bt} + a - ae^{-bt} + \sigma\int_0^t e^{b(s-t)}dW(s)
\end{aligned}
$$