Proposal
Comparative Study of the Heisenberg Spin Chain using IonQ and the Six-Vertex Model using QuEra

Musfar Muhamed Kozhikkal

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Abstract

Quantum spin chains and vertex models play a crucial role in understanding quantum many-body systems and integrability in physics. This proposal outlines a comprehensive study aimed at simulating two cornerstone models of quantum many-body physics: the Heisenberg spin chain and the Six-Vertex model. The study leverages the capabilities of two leading quantum simulation platforms, IonQ's trapped-ion system for the Heisenberg spin chain and QuEra's neutral atom processor for the Six-Vertex model. We explore their deep mathematical connections, including the Yang-Baxter equation, transfer matrices, and Bethe Ansatz solutions. By comparing simulation results with established analytical results that have been subject to mathematical investigation for decades, this work will assess the performance, accuracy, and practical limitations of current quantum hardware. This study will provide new insights into the experimental realization of integrable models, error resilience in quantum simulators, and the future direction of quantum simulation in condensed matter physics.

Contents

1 Introduction

Quantum many-body physics has long been a subject of intense study due to its rich phenomenology and profound implications for understanding fundamental interactions in nature. Two models that have contributed significantly to this field are the Heisenberg spin chain and the Six-Vertex model. The Heisenberg spin chain, which models interacting spins on a one-dimensional lattice, has been central to exploring quantum magnetism and entanglement properties. The Six-Vertex model-often associated with ice-type models, has provided critical insights into phase transitions and critical phenomena within statistical mechanics.

Recent advances in quantum computing have opened the possibility of simulating these models using quantum hardware. Platforms like IonQ and QuEra now allow us to experimentally implement models that were previously only accessible through theoretical and numerical methods. The goal of this research is to implement the Heisenberg spin chain in IonQ's trapped-ion quantum processor and the Six-Vertex model in QuEra's neutral atom array, followed by a detailed comparative analysis of the simulation results. This comparative study not only tests the robustness of long-established mathematical predictions but also benchmarks the current state of quantum simulation technology.

The project is motivated by decades of mathematical research, such as those detailed in [1] and [2],which have provided exact solutions and integrability conditions for these models. However, despite these theoretical advances, experimental verifications remain challenging because of the complexity of many-body quantum systems. In this proposal, we bridge the gap between theory and experiment by leveraging state-of-the-art quantum simulators. The expected outcomes include enhanced understanding of quantum correlations, entanglement properties, and the inherent limitations imposed by current quantum hardware.

2 Background and Literature Review

2.1 Historical Context and Significance

The study of spin chains and vertex models has a rich history in both statistical mechanics and quantum physics. The Heisenberg model was initially proposed to understand ferromagnetism and antiferromagnetism in materials. Over the years, it has evolved into a paradigmatic model for exploring quantum phase transitions and entanglement. The model's solvability via the Bethe Ansatz technique has provided exact solutions that serve as benchmarks for numerical and experimental studies.

The Six-Vertex model emerged from investigations into the statistical mechanics of ice. It captures the essential physics of systems with local constraints, where each vertex obeys the ice rule (two arrows in, two arrows out). Early work by Lieb and others established the integrability of the model and laid the groundwork for understanding phase transitions in two-dimensional systems. Both models, despite their apparent simplicity, encapsulate complex phenomena that challenge our understanding of many-body interactions.

2.2 Recent Developments and Computational Advances

In recent years, quantum computing platforms have made significant strides, enabling experimentalists to probe many-body systems with unprecedented precision. IonQ's trapped-ion systems, known for their high-fidelity quantum gates, offer an excellent platform for simulating spin models such as the Heisenberg chain. In contrast, QuEra's neutral atom processors, with their capability of handling large arrays of atoms, provide a promising architecture for simulating lattice-based models like the Six-Vertex model.

The literature reveals several attempts to simulate these models numerically using classical computers, but only a few have explored the experimental implementation on quantum hardware. Prior work such as [1] has established exact results for the Heisenberg spin chain, while studies like [2] have delved into the vertex models. The current project seeks to extend these findings by directly comparing quantum hardware simulations with established analytical predictions.

2.3 Critical Gaps and Research Motivation

Despite theoretical progress, the experimental simulation of many-body systems remains challenging due to issues like decoherence, gate errors, and scalability. The primary motivation of this project is to understand how these limitations manifest in practice and how different quantum platforms handle these challenges. By comparing the performance of the IonQ's and QuEra's systems, our objective is to identify the strengths and limitations

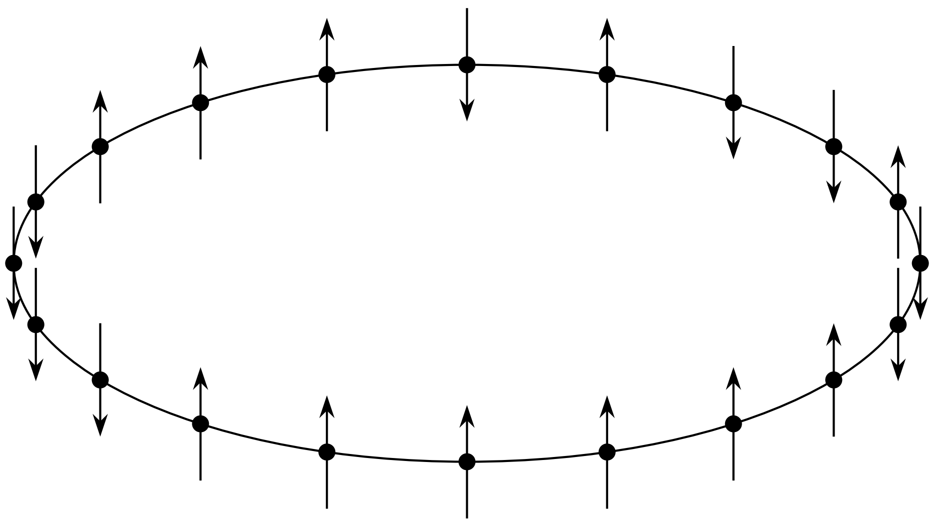

Figure 1: Heisenberg Spin Chain

of each approach and provide guidance for future developments in quantum simulation technology.

3 Theoretical Framework

3.1 Heisenberg Spin Chain

Quantum spin chain are specific example of exactly solvable ("quantum integrable") systems in 1+1 spacetime dimension. Imagine a ring of atoms with periodic boundary conditions, each of which possesses a quantum "degrees of freedom" called a spin which can point in two directions (up or down), like in Figure 1. Here quantum means one allow all complex linear superpositions of the different possible spin configurations of the ring which forms the physical state space.

The dynamics of the system, which means the evolution of a particular state in time, is governed by the Schrödinger equation. This equation involves the Hamiltonian, an operator over the state space that encodes the microscopic interaction between the quantum spins. The most studied example is the Heisenberg spin-chain. Werner Heisenberg developed the Heisenberg model, which is a statistical mechanical model used in the study of critical points and phase transitions of magnetic systems in which the spins of magnetic systems are treated quantum mechanically.

Quantum mechanically the dominant coupling between two dipoles may cause nearest-neighbors to have lowest energy when they are aligned. Under this assumption, we can say the magnetic interaction occurs only between neighboring dipoles and on a 1-dimensional periodic lattice, the Hamiltonian can be written as follows

$$\hat{H} = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} - h \sum_{j=1}^{N} \sigma_j \tag{1}$$

Here J is the coupling constant and dipoles are represented by classical vectors (or "spins") σ_j with the periodic boundary condition $\sigma_{N+1} = \sigma_1$.

The Heisenberg model treats the spins quantum-mechanically by replacing the spin by a quantum operator acting upon the tensor product $(\mathbb{C}^2)^{\otimes N}$, of dimension 2^N. Now recall the Pauli spin-1/2 matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and for $1 \leq j \leq N$ and $a \in \{x, y, z\}$ denote $\sigma_j^a = I^{\otimes j-1} \otimes \sigma^a \otimes I^{\otimes N-j}$, where I is the 2×2 identity matrix.

Given a choice of real-valued coupling constants J_x, J_y, and J_z, the Hamiltonian is given by

$$\hat{H} = -\frac{1}{2} \sum_{j=1}^{N} (J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z + h\sigma_j^z) \tag{3}$$

where h on the RHS indicates the external magnetic field with periodic boundary conditions. $\sigma^{x,y,z}$ are the 2 by 2 complex Pauli-matrices and the lower indices indicate on which atom in the ring the matrices act.

The Heisenberg chain only involves nearest neighbor interaction. Each spin can "rotate" by the Pauli matrices in different direction. Depending on the coupling constants J_x, J_y and J_z in front of each term certain spin-arrangements are particularly favorable in the sense that they possess a minimal energy. Inorder to compute these energies and associated stationary states we have to solve the eigenvalue problem of the above Hamiltonian.

The model can be named based on the values of J_x, J_y and J_z. If $J_x \neq J_y \neq J_z$, the model is called the Heisenberg XYZ model and for the case of $J = J_x = J_y \neq J_z = \Delta$ it is the Heisenberg XXZ model. And when $J_x = J_y = J_z = J$ it is called Heisenberg XXX model.

The spin 1/2 Heisenberg model in one dimension may be solved exactly using the Bethe ansatz [3]. In the algebraic formulation, these are related to particular Quantum affine algebras and Elliptic Quantum Group in the XXZ and XYZ cases respectively [4]. Other approaches do so without Bethe ansatz [5].

In the case of Heisenberg XXX model, the physics strongly depends on the sign of the coupling constant J and the dimension of the space. The ground state is always ferromagnetic for positive J and for negative J the ground state is antiferromagnetic in two and three dimensions. The nature of correlations in the antiferromagnetic Heisenberg model depends on the spin of the magnetic dipoles in one dimension. Short-range order is present if the spin is integer and quasi-long range order exhibits for a half-integer spin system.

Some applications for this are as follows:

- An important object is the entanglement entropy. In order to describe it, we subdivide the unique ground state into a block (several sequential spins) and the environment (the rest of the ground state). The entropy of the block can be considered as entanglement entropy. At zero temperature in the critical region (thermodynamic limit), it scales logarithmically with the size of the block. As increases the logarithmic dependence changes into a linear function. For large temperatures, linear dependence follows from the second law of thermodynamics.

- The Heisenberg model provides an important and tractable theoretical example for applying Density Matrix Renormalization.

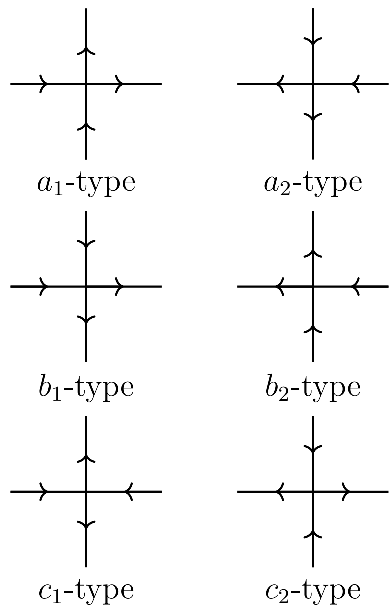

Figure 2: Types of vertices in 6-vertex model.

- The six-vertex model can be solved using the Algebraic Bethe Ansatz for the Heisenberg Spin Chain.
- The half-filled Hubbard model in the limit of strong repulsive interactions can be mapped to a Heisenberg model with $J < 0$ representing the strength of the superexchange interaction.

3.2 Six-Vertex Model

The vertex model is a representation of particles or atoms by vertices in a type of statistical mechanical model where the vertex is associated with the Boltzmann weights [6, 7]. This contrasts with a nearest-neighbour model, such as the Ising model, in which the energy, and thus the Boltzmann weight of a statistical microstate is attributed to the bonds connecting two neighbouring particles. The energy associated with a vertex in the lattice of particles is thus dependent on the state of the bonds that connect it to adjacent vertices. It turns out that every solution of the Yang–Baxter equation with spectral parameters in a tensor product of vector spaces $V \otimes V$ yields an exactly solvable vertex model.

Six-vertex model is a square lattice model where each vertex is constrained by the so-called ice rules: exactly two arrows point into and two arrows point out of each vertex (that is, number of incoming and outgoing arrows are equal for each vertices.) [8]. The possible lattice vertices are given Figure 2.

Here we can see that the net arrow at any vertex is zero. From this we can draw a 6-vertex model corresponding to a general 4×4 square lattice as in Figure 3

The six-vertex model is closely related not only to the XXZ and XXX Heisenberg chains, but in essence this is the same thing.

The statistical weight associated with each vertex configuration is determined by parameters that capture the underlying physical interactions. The partition function of the model can be computed exactly using the transfer matrix method, which has led to

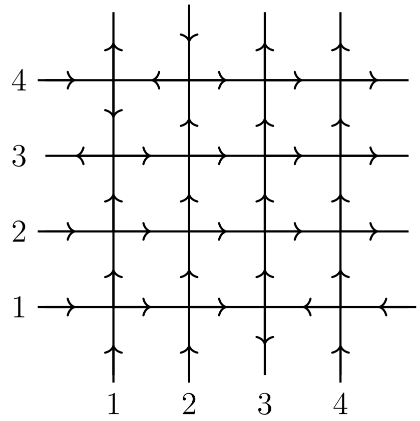

Figure 3: 4×4 lattices of 6-vertex model.

a complete characterization of its phase diagram. The integrability of the model makes it an ideal candidate for benchmarking quantum simulations against analytical results.

3.3 Interrelation Between the Models

Mathematically, there is a deep connection between the Heisenberg spin chain and the Six-Vertex model. In many formulations, the transfer matrix of the Six-Vertex model commutes with the Hamiltonian of the Heisenberg chain, implying a shared integrable structure. This correspondence has been studied extensively in the literature and provides the theoretical foundation for our comparative analysis. The ability to simulate both models on quantum hardware will allow us to directly observe this correspondence and ro understand how hardware-specific errors affect the manifestation of integrable structures.

4 Research Objectives and Hypotheses

The overarching objective of this project is to bridge theoretical predictions with experimental realizations by comparing two quantum many-body models on state-of-the-art quantum simulators. The specific objectives are:

- **Implementation:** Develop quantum circuits that simulate the Heisenberg spin chain on IonQ's trapped-ion platform and the Six-Vertex model on QuEra's neutral atom system.
- **Validation:** Compare the simulation results with exact analytical solutions, focusing on observables such as ground state energies, correlation functions, and entanglement measures.
- **Error Analysis:** Evaluate the impact of hardware-specific noise and imperfections on the simulation outcomes, and develop error mitigation strategies.
- **Comparative Study:** Analyze the similarities and differences in the performance of the two quantum simulators, with a focus on their ability to capture the integrable properties of the underlying models.
- **Broader Impact:** Assess the implications of the findings for the future of quantum simulation in condensed matter physics and quantum information science.

The working hypothesis is that, while both platforms will successfully capture the qualitative features of the respective models, differences in hardware architecture and error profiles will lead to quantitative variations. By systematically analyzing these differences, we expect to gain insights into optimizing quantum simulations for complex many-body systems.

5 Methodology

5.1 IonQ Implementation of the Heisenberg Spin Chain

5.1.1 Mapping the Model to Qubits

The first step involves mapping the spin operators $S_i^{x,y,z}$ to the qubit operators. This is typically achieved through the Jordan-Wigner transformation, which allows the spin chain to be represented in terms of Pauli matrices. The resulting Hamiltonian is then decomposed into a series of unitary operations that can be implemented using standard quantum gates.

5.1.2 Quantum Circuit Design

We will design a quantum circuit that approximates the time-evolution operator

$$U = e^{-iHt} \approx \prod_k e^{-iJ_k(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z)dt}, \tag{4}$$

using Trotter-Suzuki decomposition. The circuit will be optimized to minimize gate depth and error accumulation. Special attention will be paid to two-qubit gate fidelities, as these are critical for accurately capturing the dynamics of the spin chain.

5.1.3 Experimental Considerations

IonQ's trapped-ion system is known for its long coherence times and high-fidelity operations. However, challenges remain in scaling the number of qubits and managing crosstalk between ions. We will perform simulations on increasingly larger chains to test the scalability of our approach. Error mitigation techniques, such as dynamical decoupling and pulse shaping, will be employed to reduce the impact of decoherence.

5.1.4 Data Acquisition and Analysis

Observables such as the ground state energy and spin-spin correlation functions will be measured using standard quantum tomography techniques. The experimental data will be compared with theoretical predictions derived from the Bethe Ansatz. Statistical analysis will be applied to assess the reproducibility and reliability of the measurements.

5.2 QuEra Implementation of the Six-Vertex Model

5.2.1 Mapping the Vertex Model to a Spin System

The Six-Vertex model can be reformulated as an effective spin system by associating arrow configurations with spin states. This mapping allows us to express the partition function of the vertex model in terms of a transfer matrix that resembles the Hamiltonian of a spin chain. This equivalence is crucial for leveraging the quantum simulation techniques developed for the Heisenberg model.

5.2.2 Quantum Simulation on a Neutral Atom Platform

QuEra's neutral atom arrays offer a scalable platform for simulating lattice models. By programming the positions and interactions of individual atoms, we can emulate the vertex configurations and the corresponding transfer matrix. The experimental protocol involves initializing the system in a well-defined state, evolving it under the designed Hamiltonian, and performing measurements to extract physical observables.

5.2.3 Circuit and Control Strategies

Implementing the Six-Vertex model requires precise control over the inter-atomic interactions. We will develop pulse sequences and control protocols that mimic the effective Hamiltonian of the vertex model. Techniques such as Rydberg blockade will be exploited to enforce the ice rules at each vertex, ensuring that the simulation accurately reflects the constraints of the Six-Vertex model.

5.2.4 Measurement and Data Processing

The primary observables for the Six-Vertex model include vertex configurations, correlation lengths, and phase transition markers. High-resolution imaging will be used to capture the spatial distribution of atoms and infer the arrow configurations. Data processing algorithms will then reconstruct the effective spin correlations and compare them to the predictions of the transfer matrix formalism.

5.3 Simulation Details and Data Analysis

Both implementations will involve extensive numerical simulations prior to hardware execution. Classical simulation tools will be used to benchmark the quantum circuits and predict the expected outcomes. Once the experiments are conducted, we will perform a detailed statistical analysis to quantify the agreement between experimental data and theoretical models. Techniques such as least-squares fitting, error propagation, and Monte Carlo sampling will be employed to ensure robustness in the analysis.

6 Comparative Analysis

The comparative analysis will focus on several key observables:

- **Ground State Energy:** Both models will be used to calculate the ground state energy, and the results will be compared with exact analytical solutions.
- **Correlation Functions:** Spin-spin correlation functions in the Heisenberg chain and vertex-vertex correlations in the Six-Vertex model will be studied. Differences in the decay of correlations will provide insights into the influence of hardware-specific noise.

- **Entanglement Entropy:** The entanglement entropy serves as a measure of quantum correlations in the system. We will compare the entanglement profiles generated on both platforms to assess their ability to capture long-range quantum correlations.

- **Error Profiles:** By analyzing the error rates and decoherence effects in both implementations, we aim to understand how hardware limitations affect the simulation outcomes.

This multi-faceted comparison is designed to not only validate the quantum simulation techniques but also highlight the practical differences between trapped-ion and neutral atom platforms. The results are expected to provide a clear picture of the current state of quantum simulation and point to areas where improvements are necessary.

7 Significance and Impact

The significance of this project lies in its dual contribution to both fundamental physics and quantum technology:

7.1 Advancing Fundamental Understanding

By experimentally simulating the Heisenberg spin chain and Six-Vertex model, this research will provide direct evidence of the integrable structures predicted by decades of mathematical research. Confirming these predictions on quantum hardware will validate long-standing theoretical frameworks and potentially reveal new phenomena arising from hardware-induced perturbations.

7.2 Benchmarking Quantum Simulators

The comparative analysis of IonQ and QuEra platforms will serve as a benchmark for the performance of different quantum computing architectures. Understanding the strengths and limitations of these platforms is crucial for guiding future developments in quantum hardware, particularly in the context of scaling up quantum simulations for more complex many-body systems.

7.3 Implications for Quantum Information Science

Many-body physics and quantum information science share common challenges, such as dealing with entanglement and decoherence. Insights gained from this study will not only advance our understanding of condensed matter systems but also inform the design of more robust quantum algorithms and error correction techniques. The outcomes of this project are expected to have far-reaching implications for both theoretical research and practical applications in quantum computing.

8 Timeline and Resource Allocation

8.1 Project Timeline

The project is envisioned to span over a period of 24 months, divided into the following phases:

1. **Literature Review and Preliminary Simulations (Months 1):** Comprehensive review of existing literature and benchmarking of simulation codes on classical computers.

2. **Design and Development (Months 2-3):** Development of quantum circuits and control protocols for both the Heisenberg spin chain and Six-Vertex model.

3. **Experimental Implementation (Months 4-5):** Execution of the quantum simulations on IonQ and QuEra platforms, with iterative refinement based on preliminary results.

4. **Data Analysis and Comparative Study (Months 6-8):** Detailed analysis of experimental data, statistical comparison with theoretical models, and evaluation of hardware-specific effects.

5. **Writing and Publication (Months 9):** Preparation of the final report, publications in peer-reviewed journals, and presentations at relevant conferences.

8.2 Resource Allocation

The project will require access to quantum computing hardware (IonQ and QuEra systems), high-performance classical computing resources for simulations, and specialized software such as Cirq, PennyLane, and QuEra Bloqade.

9 Risk Assessment and Mitigation Strategies

9.1 Potential Challenges

Several challenges may arise during the project:

- **Hardware Limitations:** Both quantum platforms have inherent limitations, including decoherence, gate infidelities, and qubit connectivity issues.

- **Scalability:** Simulating larger systems may be challenging due to the exponential growth of the Hilbert space and the limitations in current hardware.

- **Error Mitigation:** The success of the simulations depends on effective error mitigation techniques, which may require further methodological development.

9.2 Mitigation Strategies

To address these challenges, we plan to:

- Develop adaptive error correction and mitigation strategies based on real-time feedback from the quantum hardware.

- Start with smaller system sizes to validate our methods before scaling up.
- Collaborate with hardware experts at IonQ and QuEra to optimize experimental protocols and leverage the latest improvements in quantum control.

10 Conclusion

This proposal presents a detailed plan to perform a comparative study of the Heisenberg spin chain and the Six-Vertex model using leading quantum simulation platforms. By implementing these models on trapped-ion system of IonQ and neutral atom processor of QuEra, we aim to validate decades of theoretical predictions, benchmark the performance of current quantum hardware, provide insights that will guide future research and bridge the gap between theory and experiment in quantum many-body physics. The outcomes of this project have the potential to impact not only condensed matter physics but also the broader field of quantum information science, paving the way for more accurate and scalable quantum simulations.

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