Heisenberg Spin Chain and Six-Vertex Model

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by

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Chapter 1

Introduction

An integrable system can be defined as a special class of system whose algebra dominates it's dynamics. And the main feature of this systems are the large number of symmetries which makes it solvable. Integrable systems are very useful in understanding a physical model in a exactly solvable and simplified setting[1]. A little more physical definition given in [2] is "A mechanical system is called integrable if we can reduce its solution to a sequence of quadratures."

In physics, integrable models are seem in fields like classical mechanics fluid dynamics, 2d quantum field theories, condensed matter physics, gauge theories, string theory and so on. In Mathematics the scope of integrable systems range from complex analysis to differential geometry.

Heisenberg XXX and XXZ spin chain models are integrable systems that are been widely studied through Quantum Inverse Scattering Method (QISM). The Quantum Inverse Scattering Method is a method to find the exact solution in quantum field theory and statistical physics for two-dimensional models. The QISM appears due to the quantization of the inverse scattering method which is classical. The classical inverse scattering method are also a well developed method useful in solving partial differential equations. [3]

The quantum nonlinear Schrodinger equation ($i\partial_t\psi = -\partial_x^2\psi + 2c\psi^\dagger\psi\psi$) and the sine-Gordon equation ($\partial_t^2 u - \partial_x^2 u + m^2\sin(\beta u)/\beta = 0$) are some models which are solved by QISM. QISM is now a branch of mathematical physics which is well developed and is useful in explaining the Bethe Ansatz in an algebraic nature. The QISM exactly calculate some physical characteristics of particles like momentum, dressed energy and the S-matrix.

Quantum or classical many particle system are very difficult to solve except for a few exceptions. We faces often invincible difficulties in the computations of relevant physical quantities of these systems. Consider correlation functions as an example which encode the probability to find two spins in the ring separated by a distance x to be aligned (either at a certain time or in a quantum statistical ensemble). In general these quantities can only be computed numerically or perturbatively where the system is almost "free", where it is close to a non-interacting ring of atoms.

Numerical computation running into difficulties with growing number of atoms and perturbation theory works only for small values of coupling constant. Some essential physical effects are missed out in these both approaches. The significance of "exactly solvable" or "quantum integrable" system result from the fact they provide non-trivial interacting system where exact solutions can be obtained for all values of coupling constant. These models can provide valuable insights into transport phenomena, magnetic and electric properties, can be discussed as toy models for quantum computers and many constructs in computations related to string theory. Nowadays they are more than that of a purely theoretical construct for example the Heisenberg spin-chain can be experimentally realized in condensed matter systems and the even correlation functions can be measured in the Lab.

Besides this physical motivation quantum spin-chain study is mathematically very interesting and have lot of applications. Many methods used to solve physical models lead to several algebraic structures which turned into research areas in pure mathematics most famously quantum groups or algebras. Both Physicists and mathematicians are involved in this research area tries to bridge the gap between abstract mathematical theories and concrete physical applications over the past years.

Chapter 2

Spin Chains

Quantum spin chain are specific example of exactly solvable ("quantum integrable") systems in 1+1 spacetime dimension. Picture a ring of atoms with periodic boundary conditions each of which possesses a quantum "degrees of freedom" called a spin which can point in two directions (up or down). Here quantum means one allow all complex linear superpositions of the different possible spin configurations of the ring which forms the physical state space.

Thy dynamics of the system that means the evolution of a particular state in time is governed by the Schrödinger equation. This equation involves the Hamiltonian, an operator over the state space encoding the microscopic interaction between the quantum spins. The most studied example is the Heisenberg spin-chain.

Werner Heisenberg developed Heisenberg model, which is a statistical mechanical model used in the study of critical points and phase transitions of magnetic systems in which the spins of magnetic systems are treated quantum mechanically.

Quantum mechanically the dominant coupling between two dipoles may cause nearest-neighbors to have lowest energy when they are aligned. Under this assumption, we can say the magnetic interaction occurs only between neighboring dipoles and on a 1-dimensional periodic lattice, the Hamiltonian can be written as follows

$$\hat{H} = -J\sum_{j=1}^{N}\sigma_j\sigma_{j+1} - h\sum_{j=1}^{N}\sigma_j \tag{2.1}$$

Here J is the coupling constant and dipoles are represented by classical vectors (or "spins") σ_j with the periodic boundary condition $\sigma_{N+1} = \sigma_1$.

The Heisenberg model treats the spins quantum-mechanically by replacing the spin by a quantum operator acting upon the tensor product $(\mathbb{C}^2)^{\otimes N}$, of dimension 2^N. Now recall the the Pauli spin-1/2 matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

and for $1 \leq j \leq N$ and $a \in \{x, y, z\}$ denote $\sigma_j^a = I^{\otimes j-1} \otimes \sigma^a \otimes I^{\otimes N-j}$, where I is the 2×2 identity matrix.

Given a choice of real-valued coupling constants J_x, J_y, and J_z, the Hamiltonian is given by

$$\hat{H} = -\frac{1}{2}\sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z + h\sigma_j^z) \tag{2.3}$$

where h on the RHS indicates the external magnetic field with periodic boundary conditions. $\sigma^{x,y,z}$ are the 2 by 2 complex Pauli-matrices and the lower indices indicate on which atom in the ring the matrices act.

The Heisenberg chain only involves nearest neighbor interaction. Each spin can "rotate" by the Pauli matrices in different direction. Depending on the coupling constants J_x, J_y and J_z in front of each term certain spin-arrangements are particularly favorable in the sense that they possess a minimal energy. Inorder to compute these energies and associated stationary states we have to solve the eigenvalue problem of the above Hamiltonian.

The model can be named based on the values of J_x, J_y and J_z. If $J_x \neq J_y \neq J_z$, the model is called the Heisenberg XYZ model and for the case of $J = J_x = J_y \neq J_z = \Delta$ it is the Heisenberg XXZ model. And when $J_x = J_y = J_z = J$ it is called Heisenberg XXX model.

The spin 1/2 Heisenberg model in one dimension may be solved exactly using the Bethe ansatz.[4] In the algebraic formulation, these are related to particular Quantum affine algebras and Elliptic Quantum Group in the XXZ and XYZ cases respectively.[5] Other approaches do so without Bethe ansatz.[6]

In the case of Heisenberg XXX model the physics strongly depends on the sign of the coupling constant J and the dimension of the space. The ground state is always ferromagnetic For positive J and for negative J the ground state is antiferromagnetic in two and three dimensions. The nature of correlations in the antiferromagnetic Heisenberg model depends on the spin of the magnetic dipoles in one dimension. Short-range order is present if the spin is integer and quasi-long range order exhibits for half-integer spin system.

Some applications for this are:

- One important object is entanglement entropy. In order to describe it we subdivide the unique ground state into a block (several sequential spins) and the environment (the rest of the ground state). The entropy of the block can be considered as entanglement entropy. At zero temperature in the critical region (thermodynamic limit) it scales logarithmically with the size of the block. As increases the logarithmic dependence changes into a linear function. For large temperatures linear dependence follows from the second law of thermodynamics.
- The Heisenberg model provides an important and tractable theoretical example for applying Density Matrix Renormalisation.
- The six-vertex model can be solved using the Algebraic Bethe Ansatz for the Heisenberg Spin Chain.

- The half-filled Hubbard model in the limit of strong repulsive interactions can be mapped onto a Heisenberg model with $J < 0$ representing the strength of the superexchange interaction.

2.1 Yang-Baxter Equation

Yang-Baxter Equation for $V_1 \otimes V_2 \otimes V_3$ is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.4}$$

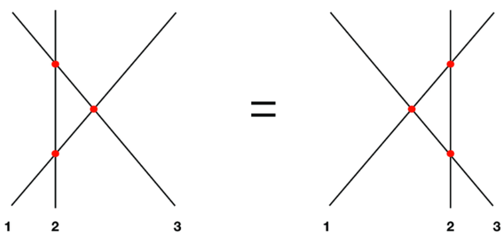


FIGURE 2.1: Picture representation Yang-Baxter Equation for $V_1 \otimes V_2 \otimes V_3$

$R-$Matrix are the solutions of Yang-Baxter equation. The R-Matrix of XXX chain has a form

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} \tag{2.5}$$

where variable λ is called the spectral parameter and permutation matrix $\mathcal{P}$ is

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

Hence

$$R(\lambda) = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix} \tag{2.7}$$

We will rewrite this matrix by using $n(\sigma^x) = 1, n(\sigma^y) = \frac{\lambda}{\lambda+\eta}, n(\sigma^z) = \frac{\eta}{\lambda+\eta}$. And we make notation for elements in R_{12}, R_{13} and R_{23} as

$$\begin{aligned} \lambda_1 &= \frac{\lambda_{12}}{\lambda_{12}+\eta}, \qquad \lambda_2 = \frac{\lambda_{13}}{\lambda_{13}+\eta}, \qquad \lambda_3 = \frac{\lambda_{23}}{\lambda_{23}+\eta}, \\ \eta_1 &= \frac{\eta}{\lambda_{12}+\eta}, \qquad \eta_2 = \frac{\eta}{\lambda_{13}+\eta}, \qquad \eta_3 = \frac{\eta}{\lambda_{23}+\eta}, \end{aligned} \tag{2.8}$$

So, we have $V \otimes V$ R- Matrix as

$$R_{12}(\lambda_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & \eta_1 & 0 \\ 0 & \eta_1 & \lambda_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{13}(\lambda_{13}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \eta_2 & 0 \\ 0 & \eta_2 & \lambda_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{23}(\lambda_{23}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_3 & \eta_3 & 0 \\ 0 & \eta_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

Now we have to make this in the space $V_1 \otimes V_2 \otimes V_3$, we tensor product with I in the trival space.

$$R_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & \eta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & \eta_1 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_1 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \eta_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & \eta_2 & 0 \\ 0 & \eta_2 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta_2 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_3 & \eta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & \eta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.10}$$

Puting this in the Yang-Baxter Equation (2.4) gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2\lambda_3 & \lambda_2\eta_3 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \lambda_1\eta_3+\eta_1\eta_2\lambda_3 & \lambda_1\lambda_3+\eta_1\eta_2\eta_3 & 0 & \eta_1\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2 & 0 & \lambda_1\eta_2\eta_3+\eta_1\lambda_3 & \lambda_1\eta_2\lambda_3+\eta_1\eta_3 & 0 \\ 0 & \eta_1\eta_3+\lambda_1\eta_2\lambda_3 & \eta_1\lambda_3+\lambda_1\eta_2\eta_3 & 0 & \lambda_1\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_1\lambda_2 & 0 & \eta_1\eta_2\eta_3+\lambda_1\lambda_3 & \eta_1\eta_2\lambda_3+\lambda_1\eta_3 & 0 \\ 0 & 0 & 0 & \eta_2 & 0 & \lambda_2\eta_3 & \lambda_2\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2\lambda_3 & \eta_1\eta_2\lambda_3+\lambda_1\eta_3 & 0 & \lambda_1\eta_2\lambda_3+\eta_1\eta_3 & 0 & 0 & 0 \\ 0 & \lambda_2\eta_3 & \eta_1\eta_2\eta_3+\lambda_1\lambda_3 & 0 & \lambda_1\eta_2\eta_3+\eta_1\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2 & 0 & \eta_1\lambda_2 & \eta_2 & 0 \\ 0 & \eta_2 & \eta_1\lambda_2 & 0 & \lambda_1\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_1\lambda_3+\lambda_1\eta_2\eta_3 & 0 & \lambda_1\lambda_3+\eta_1\eta_2\eta_3 & \lambda_2\eta_3 & 0 \\ 0 & 0 & 0 & \eta_1\eta_3+\lambda_1\eta_2\lambda_3 & 0 & \lambda_1\eta_3+\eta_1\eta_2\lambda_3 & \lambda_2\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparing the two sides of the Yang-Baxter Equation (2.4), we get the relation for the spectral parameter of the R-Matrix for XXX chain as

$$\lambda_{13} = \lambda_{12} + \lambda_{23} \tag{2.11}$$

Similary we can do for XXZ chain by putting,

$$\begin{aligned} \lambda_1 &= \frac{\sinh(\lambda_{12})}{\sinh(\lambda_{12}+\eta)}, \qquad \lambda_2 = \frac{\sinh(\lambda_{13})}{\sinh(\lambda_{13}+\eta)}, \qquad \lambda_3 = \frac{\sinh(\lambda_{23})}{\sinh(\lambda_{23}+\eta)}, \\ \lambda_1 &= \frac{\sinh(\eta)}{\sinh(\lambda_{12}+\eta)}, \qquad \lambda_1 = \frac{\sinh(\eta)}{\sinh(\lambda_{13}+\eta)}, \qquad \lambda_1 = \frac{\sinh(\eta)}{\sinh(\lambda_{23}+\eta)} \end{aligned} \tag{2.12}$$

The rest of the calcution is same.

2.2 RTT- Relation

The $RTT-$relation is given as

$$R_{ab}(\lambda-\mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda-\mu) \tag{2.13}$$

To show this we use

$$T(\lambda) = L_N(\lambda)\dots L_1(\lambda) = R_{0N}\Big(\lambda-\frac{\eta}{2}\Big)\dots R_{01}\Big(\lambda-\frac{\eta}{2}\Big) \tag{2.14}$$

where the $L-$operator is given by

$$L_n(\lambda) = \begin{pmatrix} \lambda+\frac{\eta}{2}\sigma_n^z & \eta\sigma_n^- \\ \eta\sigma_n^+ & \lambda-\frac{\eta}{2}\sigma_n^z \end{pmatrix} = R_{on}\Big(\lambda-\frac{\eta}{2}\Big) \tag{2.15}$$

The space V_0 is the auxiliary linear space in $V_0\otimes V_1\otimes\cdots\otimes V_N$ and $V_k\sim\mathbb{C}$

By writing V_3 as V_0 in Eqn(2.4), we get Yang-Baxter Equation as

$$R_{12}(\lambda-\mu)R_{10}(\lambda-\eta)R_{20}(\mu-\eta) = R_{20}(\mu-\eta)R_{10}(\lambda-\eta)R_{12}(\lambda-\mu)$$

and using $R(\lambda-\eta)=L(\lambda)$, for N=2 this equation takes the form

$$R_{12}(\lambda-\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda-\mu) \tag{2.16}$$

Here we can see that L_1 acts on $V_1\otimes V_0$ and L_2 acts on $V_2\otimes V_0$.

Consider another $L-$operator as L', which acts on V' like L'_1 acts on $V_1\otimes V'$ and L_2 acts on $V_2\otimes V'$. Since the space where the operators L_1 and L'_2 acts on are different, we get

$$[L(\lambda), L'(\mu)] = 0, \qquad \forall\lambda,\mu \tag{2.17}$$

Now, we will consider the product of L and L' for N=2 in the RTT-relation (2.13)

$$
\begin{aligned}
R_{12}(\lambda-\mu)L_1(\lambda)L_1'(\lambda)L_2(\mu)L_2'(\mu) &= R_{12}(\lambda-\mu)L_1(\lambda)L_2(\mu)L_1'(\lambda)L_2'(\mu) \\
&= L_2(\mu)L_1(\lambda)R_{12}(\lambda-\mu)L_1'(\lambda)L_2'(\mu) \\
&= L_2(\mu)L_1(\lambda)L_2'(\mu)L_1'(\lambda)R_{12}(\lambda-\mu) \\
&= L_2(\mu)L_2'(\mu)L_1(\lambda)L_1'(\lambda)R_{12}(\lambda-\mu)
\end{aligned} \tag{2.18}
$$

This shows that if two operators satisfies RTT-relation, then their products will also satisfy the RTT-relation.

2.3 Transfer Matrix

Tranfer matrix $\mathcal{T}$ is the trace of monodromy matrix.

$$
\mathcal{T}(\lambda) = \mathrm{Tr}T(\lambda) = \mathrm{Tr}\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = A(\lambda) + D(\lambda) \tag{2.19}
$$

Taking the RTT-relation (2.13)

$$
\begin{aligned}
T_1(\lambda)T_2(\mu) &= R_{12}^{-1}(\lambda-\mu)T_2(\mu)T_1(\lambda)R_{12}(\lambda-\mu) \\
\implies \mathrm{Tr}(T_1(\lambda)T_2(\mu)) &= \mathrm{Tr}(R_{12}^{-1}(\lambda-\mu)T_2(\mu)T_1(\lambda)R_{12}(\lambda-\mu)) \\
&= \mathrm{Tr}(T_2(\mu)T_1(\lambda)R_{12}(\lambda-\mu)R_{12}^{-1}(\lambda-\mu)) \\
&= \mathrm{Tr}(T_2(\mu)T_1(\lambda)) \\
\implies \mathrm{Tr}(T_1(\lambda))\mathrm{Tr}(T_2(\mu)) &= \mathrm{Tr}(T_2(\mu))\mathrm{Tr}(T_1(\lambda)) \\
\implies \mathcal{T}(\lambda)\mathcal{T}(\mu) &= \mathcal{T}(\mu)\mathcal{T}(\lambda) \\
\implies [\mathcal{T}(\lambda), \mathcal{T}(\mu)] &= 0
\end{aligned} \tag{2.20}
$$

Using $N = 2$ in (2.14) and substituting (2.15), we get

$$\mathcal{T}(\lambda) = \mathrm{Tr}(L_2(\lambda)L_1(\lambda)) = \left(\lambda + \frac{\eta}{2}\sigma_2^z\right)\left(\lambda + \frac{\eta}{2}\sigma_1^z\right) - \left(\lambda - \frac{\eta}{2}\sigma_2^z\right)\left(\lambda - \frac{\eta}{2}\sigma_1^z\right) + \eta^2(\sigma_2^+\sigma_1^- + \sigma_2^-\sigma_1^+) \tag{2.21}$$

We get the XXX spin chain by putting $\lambda = 0$, that is

$$\frac{4}{\eta^2}\mathcal{T}(0) = 2(\sigma_2^x\sigma_1^x + \sigma_2^y\sigma_1^y + \sigma_2^z\sigma_1^z) \tag{2.22}$$

To get the Hamiltonian, we will define the transfer matrix in terms of the permutation matrix defined in Eqn.(2.6) with the properties

$$\mathcal{P}^2 = 1, \qquad \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{12} = \mathcal{P}_{13} \tag{2.23}$$

also we have

$$\mathcal{T}(\lambda) = \mathrm{Tr}_0 T(\lambda) = \mathrm{Tr}_0(R_{0N}(\lambda - \frac{\eta}{2}) \ldots R_{01}(\lambda - \frac{\eta}{2})) \tag{2.24}$$

Here trace over the space 0 is the auxiliary space.

If we take $\lambda = 0$, then Eqn.(2.5) will become

$$R(\lambda)\Big|_{\lambda=0} = R(0) = \eta\mathcal{P} \tag{2.25}$$

that is $R_{12}(0) = \eta\mathcal{P}_{12}$, then by putting $\lambda = \eta/2$ we get monodromy matrix as

$$T\left(\frac{\eta}{2}\right) = R_{0N}(0) \ldots R_{02}(0)R_{01}(0) = \eta^N\mathcal{P}_{0N} \ldots \mathcal{P}_{02}\mathcal{P}_{01} \tag{2.26}$$

$$T^{-1}\left(\frac{\eta}{2}\right) = \frac{1}{\eta^N}\mathcal{P}_{01}\mathcal{P}_{02} \ldots \mathcal{P}_{0N} \tag{2.27}$$

as we get $T(\eta/2)T^{-1}(\eta/2) = 1$, and the transfer matrix will be

$$\mathcal{T}\left(\frac{\eta}{2}\right) = \eta^N\mathrm{Tr}_0(\mathcal{P}_{0N} \ldots \mathcal{P}_{02}\mathcal{P}_{01}) \tag{2.28}$$

by using the properties of the transfer matrix we get,

$$\begin{aligned}
\mathcal{P}_{0,N}\mathcal{P}_{0,N-1} &= \mathcal{P}_{0,N}\mathcal{P}_{0,N-1}\mathcal{P}_{0,N}\mathcal{P}_{0,N} = \mathcal{P}_{N,N-1}\mathcal{P}_{0,N} \\
\mathcal{P}_{0,N}\mathcal{P}_{0,N-2} &= \mathcal{P}_{N,N-2}\mathcal{P}_{0,N}, \qquad \ldots \qquad , \mathcal{P}_{0,N}\mathcal{P}_{0,1} = \mathcal{P}_{N,1}\mathcal{P}_{0,N} \qquad (2.29)
\end{aligned}$$

this will simplify the above equation into

$$\begin{aligned}
\mathcal{T}\left(\frac{\eta}{2}\right) &= \eta^N \mathrm{Tr}_0(\mathcal{P}_{N,N-1}\mathcal{P}_{N,N-2}\ldots\mathcal{P}_{N,2}\mathcal{P}_{N,1} \\
&= \eta^N \mathcal{P}_{N,N-1}\mathcal{P}_{N,N-2}\ldots\mathcal{P}_{N,2}\mathcal{P}_{N,1}\mathrm{Tr}_0(\mathcal{P}_{0,N}) \\
&= \eta^N \mathcal{P}_{N,N-1}\mathcal{P}_{N,N-2}\ldots\mathcal{P}_{N,2}\mathcal{P}_{N,1} \qquad (2.30)
\end{aligned}$$

and

$$\mathcal{T}^{-1}\left(\frac{\eta}{2}\right) = \frac{1}{\eta^N}\mathcal{P}_{N,1}\mathcal{P}_{N,2}\ldots\mathcal{P}_{N,N-2}\mathcal{P}_{N,N-1} \qquad (2.31)$$

We get for any artitary $k \in \{1, 2, \ldots, N\}$, then

$$\mathcal{T}^{-1}\left(\frac{\eta}{2}\right) = \frac{1}{\eta^N}\mathcal{P}_{k+1,k+2}\mathcal{P}_{k+1,k+3}\ldots\mathcal{P}_{k+1,N}\mathcal{P}_{k+1,1}\mathcal{P}_{k+1,2}\cdots\mathcal{P}_{k+1,k-1}\mathcal{P}_{k+1,k} \qquad (2.32)$$

The derivative of the transfer matrix with respect to λ at $\lambda = \eta/2$ can be written as

$$\begin{aligned}
\frac{d}{d\lambda}\mathcal{T}(\lambda)\Big|_{\lambda=\eta/2} &= \frac{d}{d\lambda}\mathrm{Tr}_0\big(R_{0N}\big(\lambda-\eta/2\big)\ldots R_{01}\big(\lambda-\eta/2\big)\big)\Big|_{\lambda=\eta/2} \\
&= \eta^{N-1}\sum_{k=1}^{N}\mathrm{Tr}_0(\mathcal{P}_{0,N}\ldots\mathcal{P}_{0,k+1}R'_{0,k}(0)\mathcal{P}_{0,k-1}\ldots\mathcal{P}_{0,1}) \\
&= \eta^{N-1}\sum_{k=1}^{N}R'_{k+1,k}(0)\mathcal{P}_{k+1,k-1}\ldots\mathcal{P}_{k+1,2}\mathcal{P}_{k+1,1}\mathcal{P}_{k+1,N}\mathcal{P}_{k+1,N-1}\ldots\mathcal{P}_{k+1,k+2} \qquad (2.33)
\end{aligned}$$

We know

$$\frac{d}{d\lambda}\log\mathcal{T}(\lambda)\Big|_{\lambda=\eta/2} = \frac{d}{d\lambda}\mathcal{T}(\lambda)\Big|_{\lambda=\eta/2}\mathcal{T}^{-1}(\lambda)\Big|_{\lambda=\eta/2} \qquad (2.34)$$

by substituting Eqn.(2.33),(2.32) and using $R'_{k+1,k}(0) = 1$, we get

$$\begin{aligned} \frac{d}{d\lambda} \log \mathcal{T}(\lambda)\Big|_{\lambda=\eta/2} &= \frac{\eta^{N-1}}{\eta^N} \sum_{k=1}^{N} R'_{k+1,k}(0)\mathcal{P}_{k+1,k} \\ &= \frac{\eta^{N-1}}{\eta^N} \sum_{k=1}^{N} \mathcal{P}_{k+1,k} \end{aligned} \tag{2.35}$$

We have hamiltonian of XXX $\frac{1}{2}-$spin Heisenberg chain as

$$H = \sum_{k=1}^{N} H_{k,k+1} \tag{2.36}$$

Then

$$\begin{aligned} H_{k,k+1} &= \sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y + \sigma_k^z \otimes \sigma_{k+1}^z \\ &= 2\mathcal{P}_{k,k+1} - \mathbb{I} \end{aligned} \tag{2.37}$$

So by using Eqn.(2.35)

$$\begin{aligned} H &= \sum_{k=1}^{N} 2\mathcal{P}_{k,k+1} - \sum_{k=1}^{N} \mathbb{I} \\ &= 2\eta \frac{d}{d\lambda} \log \mathcal{T}(\lambda)\Big|_{\lambda=\eta/2} - N \end{aligned} \tag{2.38}$$

Chapter 3

Algebraic Bethe Ansatz

Let us discuss the mathematical aspects involved in solving the eigenvalue problem of the Heisenberg spin-chain. It is better to introduce them by giving a rough overview over the historical development.

Bethe's 1931 work on the isotropic case ($g_x = g_y = g_z$) (the XXX model) had a major impact and was the starting point for many of the subsequent developments in this area. He made an "ansatz" for the stationary states of the XXX spin-chain to be a superposition of plane waves whose momenta/wave vectors have to satisfy an intricate set of non-linear equations, called Bethe's equations. This approach is nowadays referred to as "coordinate Bethe ansatz" and has been applied to to numerous other quantum integrable systems. The combinatorics and the algebraic aspects behind Bethe's ansatz are of great mathematical importance.

This is an original method for constructing the eigenfunctions of the quantum Hamiltonian of a Heisenberg spin chain and gave rise to a new approach to the study of a wide class of quantum systems. Despite the fact that the models solved by the Bethe ansatz are $(1 + 1)$-dimensional, they find a rather wide application in different areas of quantum physics, for example, in condensed matter physics, models of superconductivity and non-linear optics. Moreover, at the beginning of the 21st century, it was unexpectedly found that this method is very effective in solving a number of problems in theories of higher dimensions, in particular, in supersymmetric gauge field theories and string theory.

Another important work related to integrable system is by Onsager's 1944 solution of the planar Ising model. His paper contains the star-triangle relation (a precursor of the famous Yang-Baxter equation) and an infinite dimensional algebra, today called Onsager's algebra which is a special quotient of the sl_2 loop algebra. The planar Ising model is a model in classical statistical mechanics and there was no expectation that this model would have anything to do with quantum spin chain. The relation is purely mathematical (the statistical transfer matrix shares a set of common eigenvectors with the quantum spin-chain Hamiltonian) and was made apparent through Baxter's seminal works in the 1970's, who took many of Onsager's techniques to the next level by generalizing them as well as adding numerous new ideas to the subject area.

Another milestone in the mathematical development was the introduction of the "quantum inverse scattering method" (QISM) by the Faddeev-school and this laid the foundation for many algebraic structures. The method introduces a spectrum generating algebra. Central to this approach is the aforementioned Yang-Baxter equation and Baxter's idea of commuting transfer matrices. In this case, many important properties of physical systems can be established already at the level of algebra, without using its concrete representation. This approach was called algebraic Bethe ansatz. If very briefly, then the algebraic Bethe is a method of working with a special operator algebra describing a rather wide class of quantum systems.

An overview of Heisenberg spin-chains is given in the table follows

Model	Couplings	Algebra
XXX	$g_x = g_y = g_z$	sl_2 Yangian
XXZ	$g_x = g_y$	affine quantum algebra of sl_2
XYZ	all independent	elliptic algebra of sl_2

TABLE 3.1: Spin Chains and corresponding algebras

3.1 Commutation Relations for the Monodromy Matrix

The RTT-relation can written as

$$R(\lambda-\mu)(T(\lambda)\otimes\mathbb{I})(\mathbb{I}\otimes T(\mu)) = (\mathbb{I}\otimes T(\mu))(T(\lambda)\otimes\mathbb{I})R(\lambda-\mu) \tag{3.1}$$

where

$$T(\lambda)\otimes\mathbb{I} = \begin{pmatrix} A(\lambda) & 0 & B(\lambda) & 0 \\ 0 & A(\lambda) & 0 & B(\lambda) \\ C(\lambda) & 0 & D(\lambda) & 0 \\ 0 & C(\lambda) & 0 & D(\lambda) \end{pmatrix}, \mathbb{I}\otimes T(\mu) = \begin{pmatrix} A(\lambda) & B(\lambda) & 0 & 0 \\ C(\lambda) & D(\lambda) & 0 & 0 \\ 0 & 0 & A(\lambda) & B(\lambda) \\ 0 & 0 & C(\lambda) & D(\lambda) \end{pmatrix} \tag{3.2}$$

Then

$$\begin{gathered}
R(\lambda-\mu)(T(\lambda)\otimes\mathbb{I})(\mathbb{I}\otimes T(\mu)) = \\
\begin{pmatrix}
(\lambda-\mu+\eta)A(\lambda)A(\mu) & (\lambda-\mu+\eta)A(\lambda)B(\mu) & (\lambda-\mu+\eta)B(\lambda)A(\mu) & (\lambda-\mu+\eta)B(\lambda)B(\mu) \\
\substack{(\lambda-\mu)A(\lambda)C(\mu) \\ +\eta C(\lambda)A(\mu)} & \substack{(\lambda-\mu)A(\lambda)D(\mu) \\ +\eta C(\lambda)B(\mu)} & \substack{(\lambda-\mu)B(\lambda)C(\mu) \\ +\eta C(\lambda)B(\mu)} & \substack{(\lambda-\mu)B(\lambda)D(\mu) \\ +\eta D(\lambda)B(\mu)} \\
\substack{\eta A(\lambda)C(\mu) \\ +(\lambda-\mu)C(\lambda)A(\mu)} & \substack{\eta A(\lambda)D(\mu) \\ +(\lambda-\mu)C(\lambda)B(\mu)} & \substack{\eta B(\lambda)C(\mu) \\ +(\lambda-\mu)C(\lambda)B(\mu)} & \substack{\eta B(\lambda)D(\mu) \\ +(\lambda-\mu)D(\lambda)B(\mu)} \\
(\lambda-\mu+\eta)C(\lambda)C(\mu) & (\lambda-\mu+\eta)C(\lambda)D(\mu) & (\lambda-\mu+\eta)D(\lambda)C(\mu) & (\lambda-\mu+\eta)D(\lambda)D(\mu)
\end{pmatrix} \\
(\mathbb{I}\otimes T(\mu))(T(\lambda)\otimes\mathbb{I})R(\lambda-\mu) = \\
\begin{pmatrix}
(\lambda-\mu+\eta)A(\mu)A(\lambda) & \substack{\eta A(\mu)B(\lambda) \\ +(\lambda-\mu)B(\mu)A(\lambda)} & \substack{(\lambda-\mu)A(\mu)B(\lambda) \\ +\eta B(\mu)A(\lambda)} & (\lambda-\mu+\eta)B(\lambda)B(\mu) \\
(\lambda-\mu+\eta)C(\mu)A(\lambda) & \substack{\eta C(\mu)B(\lambda) \\ +(\lambda-\mu)D(\mu)A(\lambda)} & \substack{(\lambda-\mu)C(\mu)B(\lambda) \\ +\eta D(\mu)A(\lambda)} & (\lambda-\mu+\eta)D(\lambda)B(\mu) \\
(\lambda-\mu+\eta)A(\mu)C(\lambda) & \substack{\eta A(\mu)D(\lambda) \\ +(\lambda-\mu)B(\mu)C(\lambda)} & \substack{(\lambda-\mu)A(\mu)D(\lambda) \\ +\eta B(\mu)C(\lambda)} & (\lambda-\mu+\eta)B(\lambda)D(\mu) \\
(\lambda-\mu+\eta)C(\mu)C(\lambda) & \substack{\eta C(\mu)D(\lambda) \\ +(\lambda-\mu)D(\mu)C(\lambda)} & \substack{(\lambda-\mu)C(\mu)D(\lambda) \\ +\eta D(\mu)C(\lambda)} & (\lambda-\mu+\eta)D(\mu)D(\lambda)
\end{pmatrix}
\end{gathered} \tag{3.3}$$

From these equation we get the relations

$$
\begin{aligned}
\left[A(\lambda), A(\mu)\right] &= 0 \\
\left[B(\lambda), B(\mu)\right] &= 0 \\
\left[C(\lambda), C(\mu)\right] &= 0 \\
\left[D(\lambda), D(\mu)\right] &= 0 \\
A(\lambda)B(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}B(\mu)A(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}A(\mu)B(\lambda) \\
B(\lambda)A(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}A(\mu)B(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}B(\mu)A(\lambda) \\
A(\lambda)C(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}C(\mu)A(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}A(\mu)C(\lambda) \\
C(\lambda)A(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}A(\mu)C(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}C(\mu)A(\lambda) \\
B(\mu)D(\lambda) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}D(\lambda)B(\mu) + \frac{\eta}{(\lambda-\mu+\eta)}B(\lambda)D(\mu) \\
D(\mu)B(\lambda) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}B(\lambda)D(\mu) + \frac{\eta}{(\lambda-\mu+\eta)}D(\lambda)B(\mu) \\
C(\lambda)D(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}D(\mu)C(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}C(\mu)D(\lambda) \\
D(\lambda)C(\mu) &= \frac{(\lambda-\mu)}{(\lambda-\mu+\eta)}C(\mu)D(\lambda) + \frac{\eta}{(\lambda-\mu+\eta)}D(\mu)C(\lambda) \\
\left[A(\lambda), D(\mu)\right] &= \frac{\eta}{(\lambda-\mu)}(C(\mu)B(\lambda) - C(\lambda)B(\mu)) \\
\left[D(\lambda), A(\mu)\right] &= \frac{\eta}{(\lambda-\mu)}(B(\mu)C(\lambda) - B(\lambda)C(\mu)) \\
\left[B(\lambda), C(\mu)\right] &= \frac{\eta}{(\lambda-\mu)}(D(\mu)A(\lambda) - A(\lambda)D(\mu)) \\
\left[C(\lambda), B(\mu)\right] &= \frac{\eta}{(\lambda-\mu)}(A(\mu)D(\lambda) - A(\lambda)D(\mu))
\end{aligned}
\tag{3.4}
$$

3.2 Construction

We shall diagonalize the transfer matrix.

The strategy is to identify certain creation operators with which to construct states, as one does for simple harmonic oscillators.

These states operators, depends on parameters. The states are eigenstates of the transfer matrix if the parameters are solutions of a set of equations, namely the Bethe Ansatz equation.

Monodromy matrix $T_0(\lambda)$ is a 2×2 matrix in the auxiliary space (Hilbert space), whose elements are operators on the quantum space $V^{\otimes N}$ and can be written as

$$T_0(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{3.5}$$

This monodromy matrix statisfy the RTT-relation in Eqn.(2.13). Transfer matrix the trace of monodromy matrix as given in Eqn.(2.19). We use the transfer matrix to construct the eigenstates as

$$\mathcal{T}(\lambda) = A(\lambda) + D(\lambda) = \mathrm{Tr}T(\lambda) \tag{3.6}$$

In the framwork of the algebraic Bethe Ansatz, we impose some small requirements on this space as

$$\begin{aligned} A(\lambda)\,|0\rangle &= a(\lambda)\,|0\rangle\,, \\ D(\lambda)\,|0\rangle &= d(\lambda)\,|0\rangle\,, \\ C(\lambda)\,|0\rangle &= 0, \\ B(\lambda) &\neq C^{\dagger}(\lambda) \end{aligned} \tag{3.7}$$

That is, $C(\lambda)$ is the annilation operator for this space. $a(\lambda)$ and $d(\lambda)$ are some functions of λ whose explicit form depends on the specific model. $|0\rangle$ is the vacuum eigenvector for the operator $A(\lambda)$ and $D(\lambda)$.

The action of the operator $B(\lambda)$ on the vacuum is free, hence it is considered as the creation operator.

These are the requirements we impose on the hilbert space $\mathcal{H}$. These elements of monodromy matrix acts in the $\mathcal{H}$ with the vector $|0\rangle$.

For XXX-spin chain, we take the vacuum state with all spin-up

$$|0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \begin{pmatrix}1\\0\end{pmatrix}_2 \otimes\cdots\otimes \begin{pmatrix}1\\0\end{pmatrix}_N \tag{3.8}$$

Now lets see how the action of product of operators on the vacuum

$$\begin{aligned} B(\mu)B(\lambda)\left|0\right\rangle &= B(\lambda)B(\mu)\left|0\right\rangle, \\ A(\mu)D(\lambda)\left|0\right\rangle &= a(\mu)d(\lambda)\left|0\right\rangle, \end{aligned} \tag{3.9}$$

To get the $A(\mu)B(\lambda)$, we use the Eqn.(3.4)

$$\begin{aligned} A(\mu)B(\lambda)\left|0\right\rangle &= \frac{(\lambda-\mu+\eta)}{(\lambda-\mu)}B(\lambda)A(\mu)\left|0\right\rangle - \frac{\eta}{(\lambda-\mu)}B(\mu)A(\lambda)\left|0\right\rangle \\ &= a(\mu)\frac{(\lambda-\mu+\eta)}{(\lambda-\mu)}B(\lambda)\left|0\right\rangle - a(\lambda)\frac{\eta}{(\lambda-\mu)}B(\mu)\left|0\right\rangle \end{aligned} \tag{3.10}$$

Similarly for $A(\mu)B(\lambda_1)B(\lambda_2)$

$$\begin{aligned} A(\mu)B(\lambda_1)B(\lambda_2)\left|0\right\rangle &= \frac{(\lambda_1-\mu+\eta)}{(\lambda_1-\mu)}B(\lambda_1)A(\mu)B(\lambda_2)\left|0\right\rangle - \frac{\eta}{(\lambda_1-\mu)}B(\mu)A(\lambda_1)B(\lambda_2)\left|0\right\rangle \\ &= \frac{(\lambda_1-\mu+\eta)}{(\lambda_1-\mu)}B(\lambda_1)\left(\frac{(\lambda_2-\mu+\eta)}{(\lambda_2-\mu)}B(\lambda_2)A(\mu)\left|0\right\rangle - \frac{\eta}{(\lambda_2-\mu)}B(\mu)A(\lambda_2)\left|0\right\rangle\right) \\ &\quad - \frac{\eta}{(\lambda_1-\mu)}B(\mu)\left(\frac{(\lambda_2-\lambda_1+\eta)}{(\lambda_2-\lambda_1)}B(\lambda_2)A(\lambda_1)\left|0\right\rangle - \frac{\eta}{(\lambda_2-\lambda_1)}B(\lambda_1)A(\lambda_2)\left|0\right\rangle\right) \\ &= a(\mu)\Lambda_A B(\lambda_1)B(\lambda_2)\left|0\right\rangle + \sum_{j=1}^{2} a(\lambda_j)\Lambda_{A_j}B(\mu)\prod_{j\neq k}B(\lambda_k)\left|0\right\rangle \end{aligned} \tag{3.11}$$

here we have

$$\begin{aligned}
\Lambda_A &= \prod_{j=1}^{2} \frac{(\lambda_j - \mu + \eta)}{(\lambda_j - \mu)}, \\
\Lambda_{A_j} &= \frac{\eta}{(\mu - \lambda_j)} \prod_{j \neq k} \frac{(\lambda_k - \lambda_j + \eta)}{(\lambda_k - \lambda_j)}
\end{aligned} \tag{3.12}$$

So we can generalize this for 'N' as

$$\begin{aligned}
A(\mu) \prod_{j=1}^{N} B(\lambda_j) |0\rangle &= a(\mu)\Lambda_A \prod_{j=1}^{N} B(\lambda_j) |0\rangle - \sum_{j=1}^{N} a(\lambda_j)\Lambda_{A_j} B(\mu) \prod_{j \neq k} B(\lambda_k) |0\rangle \quad (3.13) \\
\Lambda_A &= \prod_{j=1}^{N} \frac{(\lambda_j - \mu + \eta)}{(\lambda_j - \mu)}, \qquad \Lambda_{A_j} = \frac{\eta}{(\mu - \lambda_j)} \prod_{j \neq k} \frac{(\lambda_k - \lambda_j + \eta)}{(\lambda_k - \lambda_j)}
\end{aligned}$$

Now we need to do similiar calculation with $D(\mu)$ instead of $A(\mu)$. So we start with $D(\mu)B(\lambda)$, we use the Eqn.(3.4)

$$\begin{aligned}
D(\mu)B(\lambda) |0\rangle &= \frac{(\mu - \lambda + \eta)}{(\mu - \lambda)} B(\lambda)D(\mu) |0\rangle - \frac{\eta}{(\mu - \lambda)} B(\mu)D(\lambda) |0\rangle \\
&= d(\mu)\frac{(\mu - \lambda + \eta)}{(\mu - \lambda)} B(\lambda) |0\rangle - d(\lambda)\frac{\eta}{(\mu - \lambda)} B(\mu) |0\rangle
\end{aligned} \tag{3.14}$$

Then $D(\mu)B(\lambda_1)B(\lambda_2)$ is

$$\begin{aligned}
D(\mu)B(\lambda_1)B(\lambda_2) |0\rangle &= \frac{(\mu - \lambda_1 + \eta)}{(\mu - \lambda_1)} B(\lambda_1)D(\mu)B(\lambda_2) |0\rangle - \frac{\eta}{(\mu - \lambda_1)} B(\mu)D(\lambda_1)B(\lambda_2) |0\rangle \\
&= \frac{(\mu - \lambda_1 + \eta)}{(\mu - \lambda_1)} B(\lambda_1) \left(\frac{(\mu - \lambda_2 + \eta)}{(\mu - \lambda_2)} B(\lambda_2)D(\mu) |0\rangle - \frac{\eta}{(\mu - \lambda_2)} B(\mu)D(\lambda_2) |0\rangle \right) \\
&\quad - \frac{\eta}{(\mu - \lambda_1)} B(\mu) \left(\frac{(\lambda_1 - \lambda_2 + \eta)}{(\lambda_1 - \lambda_2)} B(\lambda_2)D(\lambda_1) |0\rangle - \frac{\eta}{(\lambda_1 - \lambda_2)} B(\lambda_1)D(\lambda_2) |0\rangle \right) \\
&= d(\mu)\Lambda_D B(\lambda_1)B(\lambda_2) |0\rangle + \sum_{j=1}^{2} d(\lambda_j)\Lambda_{D_j} B(\mu) \sum_{j \neq k} B(\lambda_k) |0\rangle
\end{aligned} \tag{3.15}$$

So we can generalize this for 'N' as

$$
\begin{aligned}
D(\mu)\prod_{j=1}^{N} B(\lambda_j)\left|0\right\rangle &= d(\mu)\Lambda_D \prod_{j=1}^{N} B(\lambda_j)\left|0\right\rangle + \sum_{j=1}^{N} d(\lambda_j)\Lambda_{D_j} B(\mu)\prod_{j\neq k} B(\lambda_k)\left|0\right\rangle \\
\Lambda_D &= \prod_{j=1}^{N}\frac{(\mu-\lambda_j+\eta)}{(\mu-\lambda_j)}, \qquad \Lambda_{D_j} = \frac{\eta}{(\lambda_j-\mu)}\prod_{j\neq k}\frac{(\lambda_j-\lambda_k+\eta)}{(\lambda_j-\lambda_k)}
\end{aligned}
\tag{3.16}
$$

Finally we can write

$$
\begin{aligned}
\mathcal{T}(\mu)\prod_{j=1}^{N} B(\lambda_j)\left|0\right\rangle &= (A(\mu)+D(\mu))\prod_{j=1}^{N} B(\lambda_j)\left|0\right\rangle \\
&= (a(\mu)\Lambda_A + d(\mu)\Lambda_D)\prod_{j=1}^{N} B(\lambda_j)\left|0\right\rangle \\
&\quad + \sum_{j=1}^{N}(a(\lambda_j)\Lambda_{A_j} + d(\lambda_j)\Lambda_{D_j})B(\mu)\prod_{j\neq k} B(\lambda_k)\left|0\right\rangle
\end{aligned}
\tag{3.17}
$$

If the second term of the RHS is zero, then the vector $B(\lambda_1)B(\lambda_2)\dots B(\lambda_N)\left|0\right\rangle$ is an eigenvector of $\mathcal{T}(\mu)$ with eigenvalue $a(\mu)\Lambda_A + d(\mu)\Lambda_D$. For that we need the condition

$$
\begin{aligned}
a(\lambda_j)\Lambda_{A_j} + d(\lambda_j)\Lambda_{D_j} &= a(\lambda_j)\frac{\eta}{(\mu-\lambda_j)}\prod_{j\neq k}\frac{(\lambda_k-\lambda_j+\eta)}{(\lambda_k-\lambda_j)} + d(\lambda_j)\frac{\eta}{(\lambda_j-\mu)}\prod_{j\neq k}\frac{(\lambda_j-\lambda_k+\eta)}{(\lambda_j-\lambda_k)} \\
&= \frac{\eta}{(\mu-\lambda_j)}\left(a(\lambda_j)\prod_{j\neq k}(\lambda_k-\lambda_j+\eta) + d(\lambda_j)\prod_{j\neq k}(\lambda_j-\lambda_k+\eta)\right)\frac{1}{(\lambda_k-\lambda_j)}
\end{aligned}
$$

putting this to zero for $j = 1, 2, \dots, N \implies$

$$
\begin{aligned}
a(\lambda_j)\prod_{j\neq k}(\lambda_k-\lambda_j+\eta) &+ d(\lambda_j)\prod_{j\neq k}(\lambda_j-\lambda_k+\eta) = 0 \\
a(\lambda_j)\prod_{j\neq k}(\lambda_k-\lambda_j+\eta) &= -d(\lambda_j)\prod_{j\neq k}(\lambda_j-\lambda_k+\eta) \\
\implies \frac{a(\lambda_j)}{d(\lambda_j)} &= -\prod_{j\neq k}\frac{(\lambda_j-\lambda_k+\eta)}{(\lambda_k-\lambda_j+\eta)}
\end{aligned}
\tag{3.18}
$$

When this condition is statified we get

$$(A(\mu)+D(\mu))\prod_{j=1}^{N} B(\lambda_j)\,|0\rangle = (a(\mu)\Lambda_A + d(\mu)\Lambda_D)\prod_{j=1}^{N} B(\lambda_j)\,|0\rangle \tag{3.19}$$

3.2.1 Action on Dual space

Now we consider how these operaors acts in a dual space $\mathcal{H}^*$ with a dual vacuum vector $\langle 0|$. The properties we need are

$$\begin{aligned}
\langle 0| &= |0\rangle^\dagger, \\
\langle 0|0\rangle &= 1, \\
\langle 0|\,A(\lambda) &= a(\lambda)\,\langle 0|, \\
\langle 0|\,D(\lambda) &= d(\lambda)\,\langle 0|, \\
\langle 0|\,B(\lambda) &= 0
\end{aligned} \tag{3.20}$$

The function $a(\lambda)$ and $d(\lambda)$ are the same function which were in the case of $|0\rangle$. In this dual space $B(\lambda)$ acts as annihilation operator and action of $C(\lambda)$ on $\langle 0|$ will produce some other dual vector, hence we say $C(\lambda)$ is the creation operator in this dual space $\mathcal{H}^*$.

Same as what we did above, consider XXX-spin chain and take the vacuum state with all spin up, that is

$$\langle 0| = \langle \uparrow\uparrow \ldots \uparrow| = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \begin{pmatrix}1\\0\end{pmatrix}_2 \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}_N \tag{3.21}$$

Now if we did the same computation above using Eqn.(3.4) for the action of transfer matrix on the dual space, we get

$$\begin{aligned}
\langle 0| \left(\prod_{j=1}^{N} C(\lambda_j)\right) \mathcal{T}(\mu) &= \langle 0| \left(\prod_{j=1}^{N} C(\lambda_j)\right) (A(\mu) + D(\mu)) \\
&= (a(\mu)\Lambda'_A + d(\mu)\Lambda'_D) \langle 0| \left(\prod_{j=1}^{N} C(\lambda_j)\right) \\
&\quad + \sum_{j=1}^{N} (a(\lambda_j)\Lambda_{A'_j} + d(\lambda_j)\Lambda_{D'_j}) \langle 0| \left(C(\mu)\prod_{j\neq k} C(\lambda_k)\right) \qquad (3.22)
\end{aligned}$$

where

$$\Lambda_A = \Lambda'_A = \prod_{j=1}^{N} \frac{(\lambda_j - \mu + \eta)}{(\lambda_j - \mu)}, \qquad \Lambda_{A_j} = \Lambda_{A'_j} = \frac{\eta}{(\mu - \lambda_j)} \prod_{j\neq k} \frac{(\lambda_k - \lambda_j + \eta)}{(\lambda_k - \lambda_j)} \qquad (3.23)$$

$$\Lambda_D = \Lambda'_D = \prod_{j=1}^{N} \frac{(\mu - \lambda_j + \eta)}{(\mu - \lambda_j)}, \qquad \Lambda_{D_j} = \Lambda_{D'_j} = \frac{\eta}{(\lambda_j - \mu)} \prod_{j\neq k} \frac{(\lambda_j - \lambda_k + \eta)}{(\lambda_j - \lambda_k)} \qquad (3.24)$$

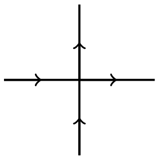
Here, if the second term of the RHS is zero, then the vector $\langle 0| C(\lambda_N) \ldots C(\lambda_1)$ is a eigen vector of $\mathcal{T}(\lambda)$ with eigenvalue $a(\mu)\Lambda_A + d(\mu)\Lambda_D$. For that we need the condition Eqn.(3.18).

Chapter 4

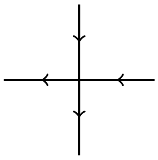
Six-Vertex Model

The vertex model is a representation of particles or atoms by vertices in a type of statistical mechanical model where the vertex is associated with the Boltzmann weights[11, 12]. This contrasts with a nearest-neighbour model, such as the Ising model, in which the energy, and thus the Boltzmann weight of a statistical microstate is attributed to the bonds connecting two neighbouring particles. The energy associated with a vertex in the lattice of particles is thus dependent on the state of the bonds which connect it to adjacent vertices. It turns out that every solution of the Yang–Baxter equation with spectral parameters in a tensor product of vector spaces $V \otimes V$ yields an exactly-solvable vertex model.

Six-vertex model is a square lattice model which is vertex model with two incoming and two outgoing arrows are each vertices (that is, number of incoming and outgoing arrows are equall for each vertices.)[13]. Hence the possible lattice vertices are



a_1-type



a_2-type

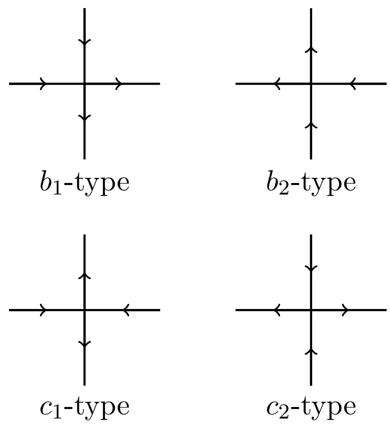


FIGURE 4.1: Types of vertices in 6-vertex model.

Here we can see that the net arrow at any vertex is zero. From this we can draw a 6-vertex model corresponding to a general 4×4 square lattice as

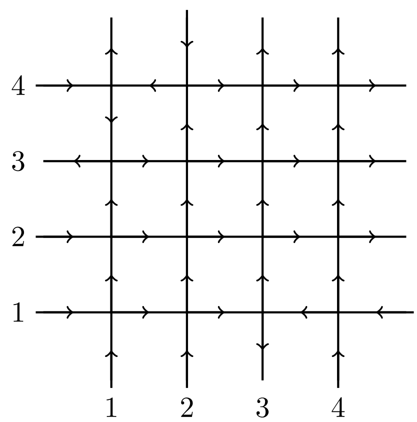


FIGURE 4.2: 4×4 lattices of 6-vertex model.

The six-vertex model is not only closely related to the XXZ and XXX Heisenberg chains, but in essence this is the same thing.

4.1 Boundary Condition

We also impose boundary conditions, the most commonly encountered types are:

• Fixed boundary conditions, in which we fix an arrangement of thick and thin lines around the edge of the grid and consider only interior configurations that are consistent with it.

• Periodic boundary conditions, in which we identify edges at opposite sides of the grid and require identified edges to be in the same state.

Periodic boundary conditions is same as that we put the lattice on a torus. In that case we can say there is no boundary or the boundary is periodic. Figure of periodic boundary condition is given as

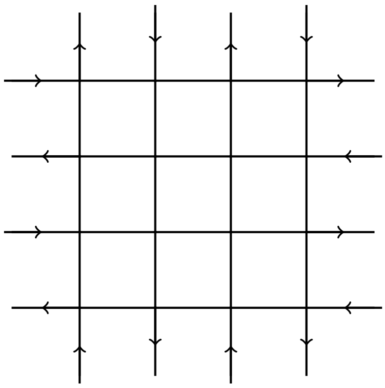


FIGURE 4.3: 6-vertex model with periodic boundary condition.

4.1.1 Domain Wall Boundary Condition

Domain Wall Boundary Conditions (DWBC) is a space case of fixed boundary condition for a six-vertex model where all incoming arrows in the right and left faces of the boundary, and outgoing arrows in the up and bottom faces of the boundary[14]. Figure showing six-vertex model with DWBC is given below:

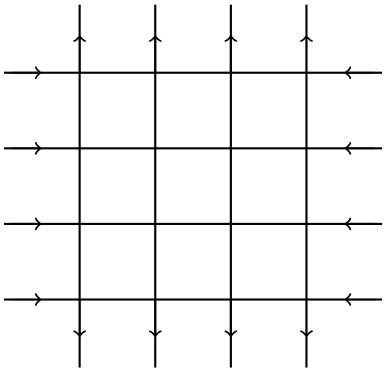


FIGURE 4.4: 6-vertex model with DWBC.

Here, two domains are separated from the lattice : one of them consists of all 'spins' up and another consists of all 'spins' down. The lattice itself plays the role of the domain wall. We can see this by match the orthogonal plane by two values to arrows in the lattice. If we now look at the lattice under some angle, then on the bottom and right faces all the arrows will be directed downwards, and on the left and the upper faces all the arrows will point up.

4.2 Partition Function

The statistical weight of the vertices are given by the complex parameters a, b and c. The first two configuration of vertex in the Figure(4.1) have the weight a, the next two have the weight b and the last to have the weight c.

Now we defined the weight of a configuration X to be the product of statistical weight of all the vertices in the lattice.

$$W(X) = \prod_{vertices} W_{ij} \tag{4.1}$$

where W_{ij} denotes the statistical weight of the vertex at i^{th} row (horizontal line) and j^{th} column (vertical line). It can be a, b or c.

The partition function associated to the lattice is the sum of all possible configurations of the weight of configurations and is given as

$$Z_N = \sum_{configurations} W(X) = \sum_{configurations} \prod_{vertices} W_{ij} \tag{4.2}$$

The weight $W(X)$ of a configuration X is physically interpreted as the relative probability of the occurrence of X. The true probability of X is therefore $W(X)/Z$. Thus the partition function is a fundamental quantity in statistical mechanics, and the first step in the analysis of a statistical system is to calculate it.

Z_N may be quite computicated for our case. Still the asymptoics of partition function as $N \to \infty$ is simpler. The notion of transfer matrix is used from this simplification.

Here we recall the $R-$matrix as the matrices of the statistical weights. So, we can denote R-matrix as

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{4.3}$$

We see that, by construction, the matrix of statistical weights R acts in the tensor product of two spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$. The first space $\mathbb{C}^2$ is called a horizontal space (since the indices α and β are on the horizontal edges of the vertex), while the second space $\mathbb{C}^2$ is a vertical space (since the indices j and k are on the vertical edges of the vertex).Hence statistical weight for a vertex is $R^{\alpha\beta,jk}$.

To calculate the partition function of a lattice consisting of two neighbouring vertices. First we must compute the product of the statistical weights of both vertices for each configuration, and then sum up the result over all possible configurations. We assume that the indices on the external edges are fixed. That is there is a boundary condition. Then there are only two configurations: in the first the intermediate horizontal edge has the index $\alpha = 1$, in the second configuration it has the index $\alpha = 2$. In the first case, the product of the statistical weights is equal to $R^{\alpha 1,k_2k_2}R^{1\beta,k_1k_1}$ and for the second case it is $R^{\alpha 2,k_2k_2}R^{2\beta,k_1k_1}$. Hence the partition function is

$$Z = R^{\alpha\alpha',k_2k_2}R^{\alpha'\beta,k_1k_1} = (R_{02}R_{01})^{\alpha\beta,j_2k_2,j_1k_1} \tag{4.4}$$

We can generalise this for N vertices in a horizontal line and it will give us the monodromy matrix. That is, for a space $V_0, V_1 \dots V_N$, we get

$$Z = (R_{0N}\dots R_{01})^{\alpha\beta,j_Nk_N,\dots,j_1k_1} = T^{\alpha\beta} \tag{4.5}$$

where T is the monodromy matrix.

For the case $\alpha = \beta$, we will get the trace of the monodromy matrix which is equals to transfer matrix $\mathcal{T}$. $\mathcal{T}$ is a $2^N \times 2^N$ matrix which acts on the space $V_1 \otimes \dots V_N$. From the point of view of the 6-vertex model, it describes the transition from vertical edges adjacent to the bottom of one horizontal line to the edges adjacent to the same horizontal line from above. Therefore, it is natural to call $\mathcal{T}$ the transfer matrix of the horizontal line. In order to calculate the partition function of two adjacent horizontal lines, it is enough to multiply the corresponding matrices $\mathcal{T}$. Continuing this process and taking into account periodic boundary conditions we obtain

$$Z_N = \mathrm{Tr}(\mathcal{T}^N) = \sum_{k=1}^{2^N} \Lambda_k^N, \text{if } \mathcal{T} \text{ is diagonalize} \tag{4.6}$$

and Λ_k is the eigenvalue of the transfer matrix. Thus, the problem of calculating the partition function reduces to the finding the eigenvalues of the XXZ chain transfer matrix. In its turn, the latter problem can be solved by the algebraic Bethe ansatz.

4.2.1 Inhomogeneous model

Let us assume that some field acts in parallel to the plane of the lattice. Then the energy of each vertex (and, accordingly, its statistical weight) can depend not only on the type of the vertex, but also on its position on the lattice, that is, $W = W_{ij}$, where i and j are the indices of the horizontal and vertical lines. Such a model is called inhomogeneous. Generally the partition function of the inhomogeneous model cannot be calculated exactly. However, this can be done, if statistical weights depend on the position of the vertex in a special way.

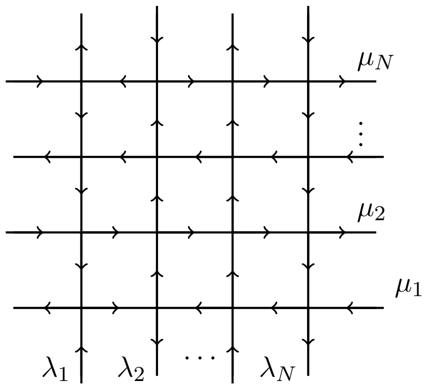


FIGURE 4.5: Inhomogenous 6-vertex model.

Let a parameter λ_i match to the ith horizontal line and a parameter μ_j match to the jth vertical line, as shown on Figure. Let the matrix of statistical weights at each vertex have the form

$$R(\lambda_i, \mu_j) = \gamma \begin{pmatrix} f(\lambda_i, \mu_j) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda_i, \mu_j) & 0 \\ 0 & g(\lambda_i, \mu_j) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda_i, \mu_j) \end{pmatrix} \tag{4.7}$$

Then the partition function of each horizontal line is the transfer matrix of the inhomogeneous XXZ-chain

$$\mathcal{T}(\lambda_i) = \mathcal{T}(\lambda_i|\bar{\mu}) = \mathrm{Tr}_0(R_{0N}(\lambda_i, \mu_N) \dots R_{01}(\lambda_i, \mu_1)) \tag{4.8}$$

$\mathcal{T}$ can be simultaneously diagonalizable and whole the partition function is given by

$$Z_N(\bar{\lambda}|\bar{\mu}) = \mathrm{Tr}(\mathcal{T}(\lambda_N) \dots \mathcal{T}(\lambda_2)\mathcal{T}(\lambda_1)) \tag{4.9}$$

where the trace is taken over the space $V_1 \otimes \dots \otimes V_N$. We know that the transfer matrices of the inhomogeneous XXZ-chain commute with each other for arbitrary values of the arguments λ_k. Therefore, they are all simultaneously diagonalizable. Thus, the problem is again reduced to the finding of the transfer matrix eigenvalues

4.2.2 Domain Wall Partition Function

The domain wall partition function DWPF, is the partition function of 6-vertex model under DWBC.

Assume the matrix of the statistical weights of the vertices at the intersection ith and jth lines of the inhomogeneous model which acts on the space $V_{i'} \otimes V_j$. Also set the weight of $b = 1$, that means $\gamma = 1$ in Eqn.(4.7). We denote this DWPF as $K_N(\bar{\lambda}|\bar{\mu})$. Then the monodromy matrix for the inhomogeneous XXZ chain is

$$T(\lambda_i|\bar{\mu}) = R_{i'N}(\lambda_{i'}, \mu_N) \dots R_{i'2}(\lambda_{i'}, \mu_2)R_{i'1}(\lambda_{i'}, \mu_1) \tag{4.10}$$

DWPF plays important role in the computation of the scalar products of Bethe vectors. To calculate this we will not take the trace of the monodromy matrix like in the case of periodic boundary condition. Instead we take $T^{12}(\lambda_i)$, that is the operator $B(\lambda_i)$. Also the commutation relations of the monodromy matrix entries can be also written with the help of the DWPF.

To calculate the partition function it is necessary to multiply the contributions from each horizontal line, that is, to take the product $B(\lambda_N) \dots B(\lambda_1)$. Due to the boundary conditions on the upper and lower faces of the lattice, the DWPF is equal to the following matrix element of the resulting product

$$K_N(\bar{\lambda}|\bar{\mu}) = (B(\lambda_N)\dots B(\lambda_1))^{21,\dots,21} \tag{4.11}$$

The corresponding monodromy matrix for the vertical line can be written as the product of $R-$matrices along the vertical line

$$\tilde{T}(\bar{a}|\mu_j) = R_{N'j}(\lambda_{N'},\mu_j)\dots R_{2'j}(\lambda_{2'},\mu_j)R_{1'j}(\lambda_{1'},\mu_j) = \begin{pmatrix} \tilde{A}(\mu_j) & \tilde{B}(\mu_j) \\ \tilde{C}(\mu_j) & \tilde{D}(\mu_j) \end{pmatrix} \tag{4.12}$$

Due to the boundary conditions on the upper and lower faces, the contribution from one vertical line to the partition function is equal to $\tilde{T}^{21}(\mu_j) = \tilde{C}(\mu_j)$. Therefore, in complete analogy to the Eqn.(4.11), we obtain another presentation for the DWPF

$$K_N(\bar{\lambda}|\bar{\mu}) = (\tilde{C}(\mu_N)\dots \tilde{C}(\mu_1))^{12,\dots,12} \tag{4.13}$$

This partition function for the six-vertex model with DWBC can be written as a determinent

4.2.3 Properties of Partition Function

The properties which uniquely determine the partion function[14] are:

1. $K(\{\lambda\},\{\mu\})$ is a symmetric function of λ and a symmetric function of μ separately.
2. $K_1(\lambda,\zeta) = c(\lambda-\zeta) = \frac{\eta}{\lambda-\zeta+\eta}$ for a R-matrix

$$R(\lambda,\zeta) = \frac{1}{\lambda-\zeta+\eta}\begin{pmatrix} \lambda-\zeta+\eta & 0 & 0 & 0 \\ 0 & \lambda-\zeta & \eta & 0 \\ 0 & \eta & \lambda-\zeta & 0 \\ 0 & 0 & 0 & \lambda-\zeta+\eta \end{pmatrix} \tag{4.14}$$

3. $B(\lambda)$ is a polynomial of degree (N-1) in all λ. That is

$$B(\lambda) = \lambda^{N-1} B_{N-1} + \lambda^{N-2} B_{N-2} + \cdots + \lambda B_1 + B_0. \tag{4.15}$$

   Also, $K_N(\{\lambda\},\{\mu\})$ is a polynomial of degree (N-1) in both λ and μ.

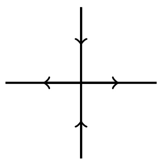
4.

$$K_N(\{\lambda\},\{\zeta\})\Big|_{\lambda_k=\zeta_k} = kK_{N-1}(\{\lambda,\hat{\lambda}_k\},\{\mu,\hat{\zeta}_k\}) \tag{4.16}$$

   where $\hat{\lambda}_k = \{\lambda\}/\lambda_k$ and $\hat{\zeta}_k = \{\zeta\}/\zeta_k$

The first property is trivially satisfied in Eqn.(3.4) by $\big[B(\lambda), B(\mu)\big] = 0$. We can also show this from RTT-relation as Eqn.(2.20).

By imposing Domain Wall Boundary Condition on the 6-vertex model for a single vertex we get the c-type vertex.



c-type

FIGURE 4.6: 6-vertex model for a single vertex.

The second property is the partition function of c-type vertex, that is

$$K_1(\lambda,\zeta) = R^{21,12} = \frac{\eta}{\lambda-\zeta+\eta} = c(\lambda-\zeta) \tag{4.17}$$

For the third property, we need to consider Eqns.(2.14,2.15,2.19). Then

$$T_\alpha(\lambda) = \prod_{k=1}^{N} L_{\alpha k}(\lambda_\alpha - \mu_k) \tag{4.18}$$

$$L_{\alpha k}(\lambda_\alpha - \mu_k) = \lambda_1\alpha - \mu_k - i\frac{\eta}{2}\sigma_\alpha^z\sigma_k^z - i\eta(\sigma_\alpha^-\sigma_k^+ + \sigma_\alpha^+\sigma_k^-) \tag{4.19}$$

$$\Longrightarrow \frac{\partial^N}{\partial\lambda^N} T(\lambda) = \begin{pmatrix} N! & 0 \\ 0 & N! \end{pmatrix} = N!\mathbb{I} \tag{4.20}$$

That is

$$\frac{\partial^N}{\partial \lambda^N} B(\lambda) = 0 \tag{4.21}$$

So, $B(\lambda)$ is a polynolmial of degree $(N-1)$ in all λ.

Similarly we get

$$\frac{\partial^N}{\partial \lambda^N} C(\lambda) = 0 \tag{4.22}$$

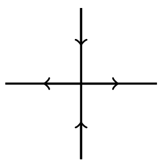
So, $C(\lambda)$ is a polynolmial of degree $(N-1)$ in all λ.

We have $K_N(\{\lambda\}, \{\mu\}) = \langle 0'| \, B(\lambda_1) \ldots B(\lambda_N) \, |0\rangle$, then

$$\frac{\partial^N}{\partial \lambda^N} K_N = 0, \qquad \frac{\partial^N}{\partial \mu^N} K_N = 0. \tag{4.23}$$

That means, $K_N(\{\lambda\}, \{\mu\})$ is a polynommial of degree (N-1) in both λ and μ.

The forth property can be easily shown will diagrams.



c-type $\implies$ $K_1(\{\lambda\}, \{\zeta\}) = c(\lambda - \zeta)$

FIGURE 4.7: 6-vertex model for a single vertex with DWBC.

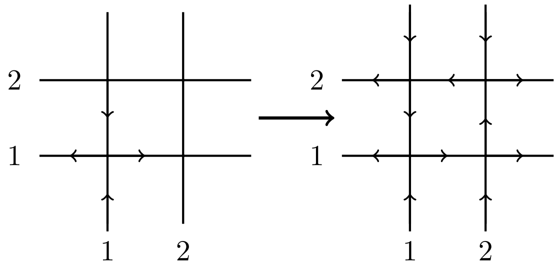


FIGURE 4.8: 2×2 lattices of 6-vertex model with DWBC.

$$\implies K_2(\{\lambda\}, \{\zeta\}) = c(\lambda_1 - \zeta_1)c(\lambda_2 - \zeta_2) = c(\lambda_1 - \zeta_1)K_1(\{\lambda\}, \{\zeta\}) \tag{4.24}$$

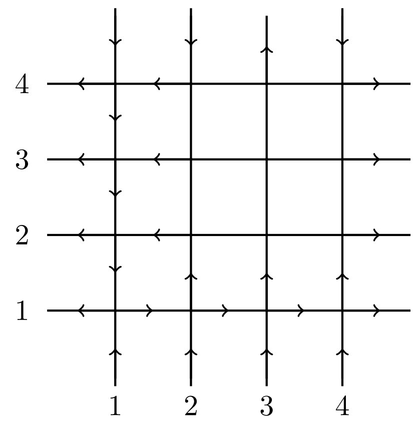


FIGURE 4.9: 4×4 lattices of 6-vertex model.

$$\Longrightarrow K_4(\{\lambda\},\{\zeta\})\Big|_{\lambda_1=\zeta_1} = c(\lambda_1-\zeta_1)K_3(\{\lambda\},\{\zeta\}) \tag{4.25}$$

by induction

$$K_N(\{\lambda\},\{\zeta\})\Big|_{\lambda_1=\zeta_1} = c(\lambda_1-\zeta_1)K_{N-1}(\{\lambda\},\{\zeta\}) \tag{4.26}$$

Generalizing this for any arbitrary k, we get Eqn.(4.16).

4.3 Determinant Representation of Partition Function

The partition function of a 6-vertex model with DWBC can be written in terms of determinant[15]. We will see this for the partition function of the six-vertex model with domain wall boundary conditions on the square lattice. QISM is used to formulate the partition function and determine its recursive properties. The recursion relation for the partition function is solved by a determinant formula. Here are going to formulate the determinant formula.

The statistical weight can be written in terms of the $L-$operator defined in Eqn.(2.15, 4.19, 2.14). In this case each spin variable takes only two values, which is $+1$ or -1. We denote $+1$ by $\uparrow$ and -1 by $\downarrow$.

Then the partition function $K(\{\lambda_\alpha\},\{\zeta_k\})$ can be written as

$$\begin{aligned} K(\{\lambda_\alpha\},\{\zeta_k\}) &= \sum_{configurations} \prod_{vertices} L_{\alpha k}(\lambda_\alpha - \mu_k) \\ &= \left(\prod_{\beta=1}^{N} \uparrow_\beta \prod_{j=1}^{N} \downarrow_j\right)\left(\prod_{\alpha=1}^{N}\prod_{k=1}^{N} L_{\alpha k}(\lambda_\alpha - \mu_k)\right)\left(\prod_{\beta=1}^{N} \downarrow_\beta \prod_{j=1}^{N} \uparrow_j\right) \end{aligned} \tag{4.27}$$

Here the double product shall be taken as space ordered

$$prod_{\alpha=1}^{N} \prod_{k=1}^{N} L_{\alpha k} = (L_{NN}\dots L_{N2}L_{N1})\dots(L_{2N}\dots L_{22}L_{21})(L_{2N}\dots L_{22}L_{21}) \tag{4.28}$$

Hence using Eqn.(2.14) to change the partition function in terms of mondromy matrix and further into the creation operators. That is

$$\begin{aligned} K_N(\{\lambda_\alpha\},\{\zeta_k\}) &= \left(\prod_{j=1}^{N} \downarrow_j\right)\left(\prod_{\alpha=1}^{N} \uparrow_\alpha T_\alpha(\lambda_\alpha) \prod_{\alpha=1}^{N} \downarrow_\alpha\right)\left(\prod_{j=1}^{N} \uparrow_j\right) \\ &= \left(\prod_{j=1}^{N} \downarrow_j\right)\left(\prod_{\alpha=1}^{N} B(\lambda_\alpha)\right)\left(\prod_{j=1}^{N} \uparrow_j\right) \end{aligned} \tag{4.29}$$

Let's use this for XXX model.

$$\begin{aligned} K_N(\{\lambda_\alpha\},\{\zeta_k\})\Big|_{\lambda_1-\zeta_1=i\eta} &= -i\eta \prod_{k=2}^{N}(\lambda_1 - \zeta_k - \frac{i}{2}\eta)\prod_{\alpha=2}^{N}(\lambda_\alpha - \zeta_1 - \frac{i}{2}\eta) K_{N-1}(\{\hat{\lambda}_\alpha\},\{\hat{\zeta}_k\}) \\ &= (-1)^N \frac{\prod_{\alpha=2}^{N}\prod_{k=2}^{N}(\lambda_\alpha - \zeta_k - \frac{i}{2}\eta)(\lambda_\alpha - \zeta_k + \frac{i}{2}\eta)}{\prod_{1\le\alpha<\beta\le N}(\lambda_\alpha - \lambda_\beta)\prod_{1\le k<j\le N}(\zeta_j - \zeta_k)} \det(\mathcal{M}) \end{aligned} \tag{4.30}$$

where

$$\mathcal{M} = \frac{i\eta}{(\lambda_\alpha - \zeta_k - \frac{i}{2}\eta)(\lambda_\alpha - \zeta_k + \frac{i}{2}\eta)} \tag{4.31}$$

we can get this using the operator C also. For that we just need to use

$$K_N(\{\lambda_\alpha\},\{\zeta_k\}) = \left(\prod_{\beta=1}^{N} \uparrow_\beta\right)\left(\prod_{j=1}^{N} C(\zeta_j)\right)\left(\prod_{\beta=1}^{N} \downarrow_\beta\right) \tag{4.32}$$

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