# ADVANCED ENGINEERING MATHEMATICS

# CYBER SEC. ( III SEM)

# CLASSICAL OPTIMIZATION TECHNIQUES


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## INTRODUCTION

The classical optimization techniques are used to obtain an optimal solution of certain types of problems involving continuous and differentiable functions. These techniques are analytical in nature and make use of differential calculus to find points of maxima and minima for both unconstrained and constrained continuous objective functions.

In this chapter we shall discuss the necessary and sufficient conditions for obtaining an optimal solution of

(*i*) Unconstrained single and multiple variable optimization problems and

(*ii*) Constrained multivariable optimization problems with equality and inequality constraints.

## CONDITIONS FOR LOCAL MINIMUM AND MAXIMUM VALUE

**Theorem** : *(Necessary Condition)*

A necessary condition for a point $x_0$ to be the local extrema (local maximum and minimum) of a function $y = f(x)$ defined in the interval $a \leq x \leq b$ is that the first derivative of $f(x)$ exists as a finite quantity at $x = x_0$ and $f'(x_0) = 0$.

**Theorem** : ***(Sufficient Condition)***

If at an extreme point $x = x_0$ of $f(x)$, the first $(n-1)$ derivatives of it become zero, *i.e.*, $f'(x_0) = 0 = f''(x_0) = \ldots = f^{(n-1)}(x_0)$ and $f^{(n)}(x_0) \neq 0$, then

(*i*) local maximum of $f(x)$ occurs at $x = x_0$ if $f^{(n)}(x_0) < 0$, for $n$ even,

(*ii*) local minimum of $f(x)$ occurs at $x = x_0$ if $f^{(n)}(x_0) > 0$ for $n$ even.

(*iii*) point of inflection occurs at $x = x_0$, if $f^{(n)}(x_0) \neq 0$ for $n$ odd.

*Example* *Determine the maximum and minimum values of the function.*

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

***Solution :*** $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x-3)(x-1)$

$f'(x) = 0$ gives $x = 1, 3, 0$

$f''(x) = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$

At $x = 1$, $f''(x) = 10(2 - 6 + 3) = -10 < 0$, hence $x = 1$ is a relative maximum and $f_{max} = f(x = 1) = 0$

At $x = 3$. $f''(x) = 30(18 - 18 + 3) = 90 > 0$. Hence $x = 3$ is a relative minimum

and $f_{min} = f(x = 3) = 243 - 405 + 135 - 1 = -28$

At $x = 0$, $f''(x) = 0$. So we must investigate the next derivative. $f'''(x) = 60x^2 - 120x + 30 = 30$ at $x = 0$. Hence $x = 0$ is neither a maximum nor a minimum point.

*Example*

*Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of C km. / hr. is $\frac{3C}{2}$ km. / hr.*

**Solution :**

Let the velocity of the motor boat be $v$ km./hr, then the velocity of the motor boat relative to the current will be $(v - c)$km/hr.

Let the total distance covered by boat be $d$ kms.

Then, total time taken ($t$) to cover the distance of $d$ kms. will be $\frac{d}{(v-c)}$ hr.

According to problem,

Petrol burnt in one hour $\alpha$ $v^3$

or $$P \,\alpha\, v^3$$

or $$P = \lambda v^3 \text{ where } \lambda \text{ is some positive constant.}$$

If $R$ be the petrol burnt for $d$ kms. Then $R = Pt$

or $$R = \lambda v^3 \frac{d}{(v-c)}$$

$$R = \lambda v^3 \; \frac{d}{(v - c)}$$

On differentiation,

$$\frac{dR}{dv} = \lambda d \left[ \frac{3(v-c)\,v^2 - v^3}{(v-c)^2} \right]$$

$$\frac{dR}{dv} = \frac{\lambda d(2v^3 - 3cv^2)}{(v-c)^2}$$

$$\frac{dR}{dv} = 0 \qquad \Rightarrow \frac{\lambda d(2v^3 - 3cv^2)}{(v-c)^2} = 0$$

$$v = 0,\; \frac{3c}{2}$$

Now,

$$\frac{d^2R}{dv^2} = \lambda d \left[ \frac{(v-c)^2\,(6v^2 - 6cv) - 2(2v^3 - 3cv^2)\,(v-c)}{(v-c)^4} \right]$$

$$= \lambda d \left[ \frac{6v(v-c)^3 - 2v^2(v-c)\,(2v-3c)}{(v-c)^4} \right]$$

$$= \frac{\lambda d}{(v-c)^3}\;[6v(v-c)^2 - 2v^2(2v-3c)]$$

If $v = 0$ then $R = 0$

It means that no petrol is burnt.

If $v = \dfrac{3c}{2}$ then

$$\frac{d^2R}{dv^2} = \frac{\lambda d}{\left(\frac{c}{2}\right)^3}\left[6\left(\frac{3c}{2}\right)\left(\frac{c}{2}\right)^2 - 2\left(\frac{3c}{2}\right)^2 . 0\right]$$

$$= 18\lambda d > 0$$

Hence $v = \dfrac{3c}{2}$ is a point of minima.

Petrol burnt is minimum when $v = \dfrac{3c}{2}$ km/hr. (most economical speed)

## SOME IMPORTANT DEFINITIONS

**(i) Leading Minors of $A$ Matrix :**

Consider a square matrix $A$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & \cdots\cdots & a_{3n} \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

Then $$A_1 = |a_{11}|, A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are said to be leading minors of matrix $A$.

**(ii) Positive Definite Matrix :** The matrix $A$ is said to be positive definite if all its leading minors are positive.

**(iii) Negative Definite Matrix :** The matrix $A$ is said to be negative definite if $A_1$, $A_3$, $A_5$ .... are negative and $A_2$, $A_4$, $A_6$ .... are positive.

**(iv) Positive Semi Definite Matrix :** If some of the leading minors are positive and remaining are zero, then the matrix $A$ is said to be positive semi definite matrix.

**(v) Negative Semi Definite Matrix :** If some of the leading minors $Aj$ are of the sign $(-1)^j$ and remaining $Aj$ are zero, then the matrix $A$ is said to be negative semi definite matrix.

*Example : Determine the nature (positive definite, negative definite or indefinite) of the following matrices*

$$(i)\ A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix} \qquad (ii)\ B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \end{bmatrix}$$

$$(iii)\ C = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 0 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

***Solution :***

(i) $A_1 = |3| = 3;\quad A_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8,\ A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{vmatrix} = 36$

$\therefore$ A is positive definite

(ii) $B_1 = |-1| = -1,\ B_2 = \begin{vmatrix} -1 & -1 \\ -1 & -2 \end{vmatrix} = 2,\ B_3 = \begin{vmatrix} -1 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \end{vmatrix} = -1$

$\therefore$ B is negative definite.

(iii) $C_1 = |4| = 4,\ C_2 = \begin{vmatrix} 4 & -3 \\ -3 & 0 \end{vmatrix} = -9;\ C_3 = \begin{vmatrix} 4 & -3 & 0 \\ -3 & 0 & 4 \\ 0 & 4 & 2 \end{vmatrix} = -82$

C is indefinite.

## MULTI-VARIABLE OPTIMIZATION WITH NO CONSTRAINTS

In this section we consider the necessary and sufficient conditions for the minimum or maximum of an unconstrained function of several variables.

**Theorem** **(Necessary Condition) :** If $f(X)$ has an extreme point (maximum or minimum) at $X = X'$ and if the first partial derivatives of $f(X)$ exist at $X'$, then

$$\frac{\partial f(X')}{\partial x_1} = \frac{\partial f(X')}{\partial x_2} = \ldots . = \frac{\partial f(X')}{\partial x_n} = 0$$

where $X = (x_1, x_2 \ldots . x_n)$ and $X' = (x_1 = a, x_2 = b \ldots . x_n = k)$

**Theorem** **(Sufficient Condition) :** A sufficient condition for a stationary point $X'$ to be an extreme point is that the matrix of second order partial derivatives

$$H = \left[ \left. \frac{\partial^2 f}{\partial x_i \, \partial x_j} \right|_{X = X'} \right]$$

evaluated at $X'$ is ($a$) positive definite when $X'$ is a local minimum point, ($b$) negative definite when $X'$ is a local maximum point.

*Example* *Find the extreme points of the function*

$$f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$$

*Solution :*

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 = x_1 (3x_1 + 4)$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = x_2 (3x_2 + 8)$$

At an extreme point $\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0$

These equations are satisfied at (0, 0), (0, –8/3), (–4/3, 0) and (–4/3, –8/3).

These are the stationary points of f :

$$\text{The Hessian matrix } J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

$$J_1 = |6x_1 + 4| = 6x_1 + 4 \ ; \ J_2 = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix} = (6x_1 + 4)(6x_2 + 8)$$

**Nature of stationary points**

(i) Point (0, 0); $J_1 = 4$ ; $J_2 = 32 \Rightarrow J|_{(0,\,0)}$ is positive definite

$\Rightarrow$ (0, 0) is a minimum and f(x) = 6

(ii) Point (0, –8/3) ; $J_1 = 4$ ; $J_2 = -32 \Rightarrow J|_{(0,\,-8/3)}$ is indefinite

$\Rightarrow$ (0, –8/3) is a saddle point and f(x) = 418/27

(iii) Point (–4/3, 0); $J_1 = -4$, $J_2 = -32 \Rightarrow J|_{(-4/3,\,0)}$ is indefinite

$\Rightarrow$ (–4/3, 0) is a saddle point and f(x) = 194/27

(iv) Point (–4/3, –8/3) ; $J_1 = -4$, $J_2 = 32 \Rightarrow J|_{(-4/3,\,-8/3)}$ is negative definite

$\Rightarrow$ (–4/3, –8/3) is a maximum and f(x) = 50/3.

# MULTIVARIABLE OPTIMIZATION WITH EQUALITY CONSTRAINTS

(*i*) **Direct Substitution Method :** In the problem (2.7) we have $n$ variables with $m$ equality constraints and $m < n$. It is theoretically possible to solve simultaneously the $m$ equality constraints and express any set of $m$ variables in terms of the remaining $n - m$ variables. We substitute these $m$ variables in the objective function. The new objective function so obtained is a function of $(n - m)$ variables and is not subjected to any constraints and hence its optimum value can be obtained by the method discussed in the previous article.

(*ii*) **Lagrange Multipliers Method :** In Lagrange multipliers method an additional variable in each of the given constraints is added. If the problem has $n$ variables and m equality constraints, then $m$ additional variables are to be added so that the problem will have $m + n$ variables.

*Example* *Find the values of x, y and z that maximize the function.*

$$f(x,y,z) = \frac{6xyz}{x+2y+2z}$$

where x, y and z are restricted by the relation $xyz = 16$

***Solution :*** $x = \dfrac{16}{yz}$

$$\Rightarrow f(x, y, z) = \frac{16xyz}{x+2y+2z} = \frac{6 \times 16}{\frac{16}{yz}+2y+2z}$$

$$= \frac{48yz}{8+y^2z+yz^2} = f(y, z)$$

$$\frac{\partial f}{\partial y} = 48z\left[\frac{1}{8+y^2+yz^2} - \frac{y(2yz+z^2)}{(8+y^2z+yz^2)^2}\right]$$

$$= \frac{48z(8-y^2z)}{(8+y^2z+yz^2)^2}$$

$$\frac{\partial f}{\partial z} = \frac{48y(8-zy^2)}{(8+y^2z+yz^2)^2}$$

$\dfrac{\partial f}{\partial y} = 0$, $\quad \dfrac{\partial f}{\partial z} = 0$ gives $y = z \Rightarrow y^* = z^* = 2$

therefore $x^* = 4$

$\therefore \quad (x^*, y^*, z^*) = (4, 2, 2)$ is the stationary point.

Now to check the sufficient condition.

$$\frac{\partial^2 f}{\partial y^2} = 48z\left[-\frac{2yz}{(8+y^2z+yz^2)^2} - \frac{2(8-y^2z)(2yz+z^2)}{(8+y^2z+yz^2)^3}\right]$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(y^*,z^*)} = -\frac{4}{3}$$

similarly $\left.\frac{\partial^2 f}{\partial Z^2}\right|_{(y^*,Z^*)} = -\frac{4}{3}$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 48\left[\frac{8-2y^2z}{(8+y^2z+yz^2)^2} - \frac{2z(8-y^2z)(y^2+2yz)}{(8+y^2z+yz^2)^3}\right]$$

$$\left.\frac{\partial^2 f}{\partial y \partial z}\right|_{(y^*,z^*)} = -\frac{2}{3} = \left.\frac{\partial^2 f}{\partial z \partial y}\right|_{(y^*,z^*)}$$

$$\therefore \quad J|_{(y^*,z^*)} = \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix}, \; J_1 = |-4/3| = -4/3$$

$$J_2 = \begin{vmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{vmatrix} = \frac{12}{9} = 4/3$$

$\therefore$ Hessian matrix is negative definite, hence, $(x^*, y^*, z^*) = (4,2,2)$ corresponds to the maximum of $f(x, y, z)$.

*Example* . Minimize $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

s.t. $g_1(x) = x_1 - x_2 = 0$

and $g_2(x) = x_1 + x_2 + x_3 - 1 = 0$

by Lagrange's multipliers method.

**Solution :**

Define the Lagrangian function

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1(x_1 - x_2) + \lambda_2(x_1 + x_2 + x_3 - 1) \quad ...(1)$$

At the extreme points, we have $\frac{\partial L}{\partial x_i} = 0$ for $i = 1, 2, 3$ and $\frac{\partial L}{\partial \lambda_i} = 0$ for $i = 1, 2$

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + \lambda_2 = 0 \quad ...(2)$$

$$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2 = 0 \quad ...(3)$$

$$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2 = 0 \quad ...(4)$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 - x_2 = 0 \quad ...(5)$$

$$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + x_3 - 1 = 0 \quad ...(6)$$

On Solving equations (2), (3), (4), (5) and (6), we get

$$x_1 = \frac{1}{3} = x_2 = x_3,\ \lambda_1 = 0,\ \lambda_2 = \frac{-1}{3}$$

The second order partial derivatives of $L$ and first order partial derivatives of $g$ are given by

$$\frac{\partial^2 L}{\partial x_1^2} = 1, \quad \frac{\partial^2 L}{\partial x_1\, \partial x_2} = \frac{\partial^2 L}{\partial x_1\, \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_2\, \partial x_1} = 0, \quad \frac{\partial^2 L}{\partial x_2^2} = 1, \quad \frac{\partial^2 L}{\partial x_2\, \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_3\, \partial x_1} = 0, \quad \frac{\partial^2 L}{\partial x_3\, \partial x_2} = 0, \quad \frac{\partial^2 L}{\partial x_3^2} = 1$$

$$\frac{\partial g_1}{\partial x_1} = 1, \quad \frac{\partial g_1}{\partial x_2} = -1, \quad \frac{\partial g_1}{\partial x_3} = 0$$

$$\frac{\partial g_2}{\partial x_1} = 1, \quad \frac{\partial g_2}{\partial x_2} = 1, \quad \frac{\partial g_2}{\partial x_3} = 1$$

The sufficient condition for maxima or minima is that all the eigen values of the Hessian matrix should be of the same sign.

So, we find the root of

$$H = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - k & \frac{\partial^2 L}{\partial x_1\, \partial x_2} & \frac{\partial^2 L}{\partial x_1\, \partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2\, \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - k & \frac{\partial^2 L}{\partial x_2\, \partial x_3} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial^2 L}{\partial x_3\, \partial x_1} & \frac{\partial^2 L}{\partial x_3\, \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} - k & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & 0 & 0 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1-k & 0 & 0 & 1 & 1 \\ 0 & 1-k & 0 & -1 & 1 \\ 0 & 0 & 1-k & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

$$(1-k)\begin{vmatrix} 1-k & 0 & -1 & 1 \\ 0 & 1-k & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 1-k & 0 & 1 \\ 0 & 0 & 1-k & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1-k & 0 & -1 \\ 0 & 0 & 1-k & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$$6(1-k)^3 = 0 \qquad \Rightarrow k = 1, 1, 1$$

All the values of $k$ are of same sign. As $k \geq 0$, hence the solution $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a point of minima.

# Multivariable Optimization with inequality constraints (using Kuhn-Tucker conditions)

If Min $f(X)$ s.t. $g_i(X) \geq 0$ then necessary conditions are

$$\frac{\partial f(X)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(X)}{\partial x_j} = 0$$

$$\lambda_i g_i(X) = 0$$

$$g_i(X) \geq 0$$

$$\lambda_i \leq 0.$$

If Min $f(X)$ s.t. $g(X) \leq 0$ then necessary conditions are

$$\frac{\partial f(X)}{\partial x_j} + \sum_{i=1}^{m} \frac{\lambda_i \partial g_i(X)}{\partial x_j} = 0$$

$$\lambda_i g_i(X) = 0$$

$$-g_i(X) \leq 0$$

$$\lambda_i \geq 0.$$

**EXAMPLE** *State Kuhn – Tucker conditions and use $k - T$ conditions to solve*

*Min.* $f(x, y, z) = x^2 + y^2 + z^2 + 20x + 10y$

$$s.t.\ x \geq 40$$

$$x + y \geq 80$$

$$x + y + z \geq 120$$

**Solution :** ($i$) Kuhn-tucker conditions for the problem are

$$\frac{\partial f(X)}{\partial x_j} + \sum_{i=1}^{3} \lambda_i \frac{\partial g_i(X)}{\partial x_j} = 0,$$

$$\lambda_i g_i(X) = 0,$$

$$g_i(X) \geq 0$$

and

$$\lambda_i \leq 0$$

Define the Lagrangian function

$$L(X, S, \lambda) = x^2 + y^2 + z^2 + 20x + 10y + \lambda_1(x - 40 + S_1^2)$$

$$+ \lambda_2 (x + y - 80 + S_2^2) + \lambda_3 (x + y + z - 120 + S_3^2) \quad \ldots(1)$$

The $K - T$ conditions for minimization problem are

$$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial z} = 0$$

$$\lambda_1 g_1 = 0, \lambda_2 g_2 = 0, \lambda_3 g_3 = 0, g_i(X) \geq 0$$

and

$$\lambda_1, \lambda_2, \lambda_3 \leq 0.$$

Now
$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x + 20 + \lambda_1 + \lambda_2 + \lambda_3 = 0 \quad \text{...(2)}$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y + 10 + \lambda_2 + \lambda_3 = 0 \quad \text{...(3)}$$
$$\frac{\partial L}{\partial z} = 0 \Rightarrow 2z + \lambda_3 = 0 \quad \text{...(4)}$$
$$\lambda_1 g_1 = 0 \Rightarrow \lambda_1(x - 40) = 0 \quad \text{...(5)}$$
$$\lambda_2 g_2 = 0 \Rightarrow \lambda_2(x + y - 80) = 0 \quad \text{...(6)}$$
$$\lambda_3 g_3 = 0 \Rightarrow \lambda_3 (x + y + z - 120) = 0 \quad \text{...(7)}$$
$$x \geq 40 \quad \text{...(8)}$$
$$x + y \geq 80 \quad \text{...(9)}$$
$$x + y + z \geq 120 \quad \text{...(10)}$$
$$\lambda_1, \lambda_2, \lambda_3 \leq 0 \quad \text{...(11)}$$

If $\lambda_1, \lambda_2, \lambda_3 \neq 0$, then from equation (5), (6) and (7) we have
$$x = 40, y = 40, z = 40$$

On putting the values of $x, y$ and $z$ into equations (2), (3) and (4) we can get values of $\lambda_1$, $\lambda_2$ and $\lambda_3$.
$$\lambda_1 = -10, \lambda_2 = -10, \lambda_3 = -80$$

The conditions $\lambda_i \leq 0$ is satisfied here

Hence, optimal solution is
$$x_1 = x_2 = x_3 = 40$$
$$\text{Min. } f = (40)^2 + (40)^2 + (40)^2 + 20(40) + 10(40)$$
$$= 6000.$$

**EXAMPLE** *Solve the following problem :*

$$\text{Min.} \quad f(X) = x_1^2 + x_2^2 + x_3^2$$

$$\text{s.t.} \quad g_1(X) = 2x_1 + x_2 - 5 \le 0$$

$$g_2(X) = x_1 + x_3 - 2 \le 0$$

$$g_3(X) = 1 - x_1 \le 0$$

$$g_4(X) = 2 - x_2 \le 0$$

$$g_5(X) = - x_3 \le 0$$

**Solution** Define the Lagrangian function

$$L(X, S, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda_1(2x_1 + x_2 - 5 + S_1^2) + \lambda_2 (x_1 + x_3 - 2 + S_2^2) + \lambda_3(1 - x_1 + S_3^2) + \lambda_4(2 - x_2 + S_4^2) + \lambda_5(- x_3 + S_5^2) \quad ...(1)$$

The $K$-$T$ conditions for minimization problem are

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0$$

$$\lambda_1 g_1 = \lambda_2 g_2 = \lambda_3 g_3 = \lambda_4 g_4 = \lambda_5 g_5 = 0$$

$$g_i(X) \le 0 \quad \text{and} \quad \lambda_i \ge 0$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 = 0 \quad ...(2)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + \lambda_1 - \lambda_4 = 0 \quad ...(3)$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 + \lambda_2 - \lambda_5 = 0 \quad ...(4)$$

$$\lambda_1 g_1 = 0 \Rightarrow \lambda_1(2x_1 + x_2 - 5) = 0 \quad ...(5)$$

$$\lambda_2 g_2 = 0 \Rightarrow \lambda_2(x_1 + x_3 - 2) = 0 \quad ...(6)$$

$$\lambda_3 g_3 = 0 \Rightarrow \lambda_3 (1 - x_1) = 0 \quad ...(7)$$

$$\lambda_4 g_4 = 0 \Rightarrow \lambda_4(2 - x_2) = 0 \quad ...(8)$$

$$\lambda_5 g_5 = 0 \Rightarrow - \lambda_5 x_3 = 0 \quad ...(9)$$

$$g_i(X) \le 0 \Rightarrow 2x_1 + x_2 - 5 \le 0 \quad ...(10)$$

$$x_1 + x_3 - 2 \le 0 \quad \text{...(11)}$$
$$1 - x_1 \le 0 \quad \text{...(12)}$$
$$2 - x_2 \le 0 \quad \text{...(13)}$$
$$-x_3 \le 0 \quad \text{...(14)}$$

and $\lambda_i \ge 0$ ; $i = 1, 2, 3, 4, 5$ Equations (12) and (13) provide

$$x_1 \ne 0, x_2 \ne 0 \quad \text{...(15)}$$

If $\lambda_3 \ne 0$, then equation (7) gives

$$x_1 = 1 \quad \text{...(16)}$$

If $\lambda_4 \ne 0$, then equation (8) gives

$$x_2 = 2 \quad \text{...(17)}$$

Equation (9) gives either $\lambda_5 = 0$ or $x_3 = 0$

Let
$$x_3 = 0 \quad \text{...(18)}$$

Inequalities (10) – (14) are all satisfied through equation (16) – (18).

From equations (5) and (6)

We have $2x_1 + x_2 - 5 \ne 0$ and $x_1 + x_3 - 2 \ne 0$

So that
$$\lambda_1 = 0, \lambda_2 = 0$$

Equation (2) $\Rightarrow 2 \times 1 + 2 \times 0 + 0 - \lambda_3 = 0$

or
$$\lambda_3 = 2$$

Equation (3) $\Rightarrow 2 \times 2 + 0 - \lambda_4 = 0$

or
$$\lambda_4 = 4$$

Equation (4) $\Rightarrow 2 \times 0 + 0 - \lambda_5 = 0$

or
$$\lambda_5 = 0$$

Hence the optimal solution is

$$x_1 = 1, x_2 = 2, x_3 = 0, \lambda_1 = 0 = \lambda_2 = \lambda_5$$
$$\lambda_3 = 2, \lambda_4 = 4$$
$$\text{max. } f = x_1^2 + x_2^2 + x_3^2$$
$$= 1 + 4 + 0 = 5$$

# Exercise

1. Determine the maximum and minimum values of the function
   $x^5 - 5x^4 + 5x^3 - 1$
2. The profit per acre of a farm is given by $f(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$, where $x_1$ and $x_2$ denote the labour cost and the fertilizer cost. Find the values of $x_1$ and $x_2$ to maximize the profit.
3. Find the maximum or minimum point of the function
   $f(x_1, x_2) = x_1 + 2x_2 + x_1x_2 - x_1^2 - x_2^2$
4. Find the extreme points of the function
   $f(x, y, z) = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 16yz$
5. Minimize $f(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)$
   subject to $g_1(x) = x_1 + 2x_2 + 3x_3 + 5x_4 - 10 = 0$
   $g_2(x) = x_1 + 2x_2 + 5x_3 + 6x_4 - 15 = 0$

6. Consider the following optimization problem:

Maximize $f = -x_1 - x_2$

subject to $x_1^2 + x_2 \geq 2$

$4 \leq x_1 + 3x_2$

$x_1 + x_2^4 \leq 30$

(a) Find whether the vector $x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies the Kuhn-Tucker conditions for above problem?

(b) What are the values of the Lagrange multipliers for $x^*$?

7. Find the solution of the following problem using K – T conditions:

Minimize $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to $x_1 + 3x_2 \leq 6$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

8. Find the solution of the following problem using K – T conditions:

Maximize $z = 2x_1 - x_1^2 - x_2$

subject to $2x_1 + 3x_2 \leq 6$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

9. Find the solution of the following problem using K – T conditions:

Maximize $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to $x_1 + x_2 \leq 2$

$2x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0$

# Answers

1. $x = 0$ is a saddle point, $x = 1$ is a maxima with 0 as maximum value of the function and $x = 3$ is a minima with - 28 as minimum value of the function.
2. $x_1 = 7,\ x_2 = 9$
3. (4/3,5/3) is point of maxima.
4. (0,0,0) is a saddle point.
5. (-5/74,-5/37,155/74,30/37) is a point of minima.
6. (a) Satisfies (b) $\lambda_1 = 2/5,\ \lambda_2 = 1/5\ \lambda_3 = 0$
7. $x_1 = 2/3,\ x_2 = 14/9$ and minimum value of $z = 22/9$
8. $x_1 = 2/3,\ x_2 = 14/9$ and maximum value of $z = 22/9$
9. $x_1 = 1/2,\ x_2 = 3/2,\ x_3 = 0$ and maximum value of $z = 17/2$

THANKS