ADVANCED ENGINEERING MATHEMATICS

IT (III SEM)

Course Objectives:

CO1 To prepare many optimization tools for applying them into different research areas as per the requirement.

CO2 To prepare important strategies of linear programming for applying them into solving the problems of transportation and assignment.

CO3 To establish different theorems and properties of random variables for understanding expectation, moments, moment generating function etc.

CO4 To analyze the various discrete and continuous distributions with their appropriate applications.

TRANSPORTATION PROBLEMS (CO2)

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INTRODUCTION OF TRANSPORTATION PROBLEM

The transportation problem is a particular case of linear programming problem and occurs very frequently in practical life. The objective of a transportation problem is to transport various amounts of a single homogenous commodity that are initially stored in various origins, to different destinations in such a way that the total transportation cost is minimum. In its standard form, the problem involves selecting the optimal way of distribution of goods from a number of sources (origins) to a number of locations (destinations).

The transportation problem described above is usually represented in a tabular form as follows :

Destinations / Origins	D_1	D_2	...	...	...	D_j	...	...	...	D_n	Origin Capacities
O_1	c_{11}	c_{12}	...	...	...	c_{1j}	...	...	...	c_{1n}	a_1
O_2	c_{21}	c_{22}	...	...	...	c_{2j}	...	...	...	c_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$				$\vdots$				$\vdots$	$\vdots$
O_i	c_{i1}	c_{i2}	...	...	...	c_{ij}	...	...	...	c_{in}	a_i
$\vdots$	$\vdots$	$\vdots$				$\vdots$				$\vdots$	$\vdots$
O_m	c_{m1}	c_{m2}	...	...	...	c_{mj}	...	...	...	c_{mn}	a_m
Destination requirements	b_1	b_2	...	...	...	b_j	...	...	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Feasible solution : By feasible solution it is meant a set of non negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes all the existing surpluses and satisfies all the existing deficiencies.

Basic feasible solution : A feasible solution is said to be basic if the number of +ve allocations equals $m + n - 1$.

Optimal Solution : A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

Remarks : (1) When the total capacity equals total requirements, the problem is called a balanced transportation problem, otherwise an unbalanced transportaiton problem. We can add a dummy row or a dummy column to convert an unbalanced transportation problem to a balanced one as per the requirement of the problem.

(2) When the number of positive allocations at any stage of the feasible solution is less than the number of independent constraint equations, *i.e.*, $(m + n - 1)$, the solution is said to be degenerate otherwise non-degenerate.

(3) Cells in the transportation table having positive allocation are known as occupied cells, otherwise non-occupied or empty cells.

OPTIMAL SOLUTION OF TRANSPORTATION PROBLEM

The method of finding an optimal solution of the given transportation problem consists of two main steps :

(i) Finding an initial basic feasible solution

(ii) Obtaining an optimal solution by making successive improvements in initial basic feasible solution

METHODS TO FIND AN INITIAL BASIC FEASIBLE SOLUTION

(i) North West Corner Rule (NWCR)

(ii) Lowest Cost Entry Method

(iii) Vogel's Approximation Method

NWCR

As per the north west corner rule, we start allocation from the north west corner i.e. (1, 1) cell and then we allocate the cells within first row moving from left to right till all possible allocations of first row are finished. Then, we repeat the same procedure in second row, third row, etc. i.e. in the increasing order of rows.

LEAST COST ENTRY METHOD

As per the least cost entry method, we start allocating the least cell value and then we allocate the cells with increasing values of cost. In case of a tie between least value of cell among different cells, we allocate the cell, which can be allocated larger amount.

VOGEL'S APPROXIMATION METHOD (UNIT COST PENALTY METHOD)

In comparison of north west corner rule and lowest cost entry method, the Vogel's method gives a very good initial feasible solution, which sometimes becomes the optimal solution.

Procedure:

(i) Set up the cost matrix

(ii) Write down the difference between the smallest and second smallest elements in each column below the corresponding column.

(iii) Write down the difference between the smallest and second smallest elements in each row to the right of the corresponding row. These individual differences (numbers) can be thought of as **penalties** for making allocations.

(iv) Select the row or column with the greatest (i.e. biggest penalty) difference and allocate as much as possible within the restrictions, to the lowest cost cell in the row or column selected. In case of a tie, choose any one of them.

(v) Cross out the row or column in which the supply or demand has been satisfied and **construct** the reduced matrix.

(vi) Continue the process on the reduced matrices till all allocations have been made.

Optimality Test through Modified Distribution (MODI) algorithm

1. Find the initial basic feasible solution by any of the methods given above and examine the number of independent allocations (occupied cells) which must be equal to $m + n - 1$.

2. Determine a set of values of u_i and v_j for rows and columns. To do so, any one of u_i's or v_j's is assigned the value zero.
It is better to assign zero for a particular u_i or v_j where there are maximum number of allocations in a row or a column respectively.
Complete the calculation of u_i's and v_j's for other rows and columns by using the relation

$$c_{ij} = u_i + v_j \quad \text{for occupied cells } (i, j)$$

3. Calculate the cell evaluations d_{ij} for each unoccupied cell (i, j) by using the relationship

$$d_{ij} = c_{ij} - (u_i + v_j)$$

4. Examine the matrix of cell evaluations :
   (a) If all $d_{ij} > 0$, then the current basic feasible solution is an optimal and unique solution.
   (b) If all $d_{ij} \geq 0$ with atleast one $d_{ij} = 0$, then the solution under test is optimal and an alternative optimal solution exists.
   (c) If atleast one $d_{ij} < 0$ then the solution is not optimal.

5. Construct a closed path or loop for the unoccupied cell with largest negative opportunity cost d_{ij}. Start the closed path with the selected unoccupied cell and mark a plus sign (+) in the cell, trace a path along the rows (or column) to an occupied cell, mark the corner with minus sign (–) and continue down the column (or row) to an occupied cell and mark the corner with plus sign (+) and minus sign (–) alternatively. Choose the path back to the selected unoccupied cell.

6. Choose the smallest quantity amongst the cells marked with minus sign on the corners of closed path. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract it from the occupied cells marked with minus signs.

7. Find a new improved solution by allocating units to the unoccupied cell according to step 6.

8. The revised solution is to be checked for optimality. The procedure ends when all $d_{ij} \geq 0$, for unoccupied cells and then calculate the optimal total transportation cost.

SPECIAL CASES

DEGENERACY AND ITS RESOLUTION IN TRANSPORTATION PROBLEM

A basic feasible solution for transportation problem of m origins and n destinations mostly consist of exactly $m + n - 1$ positive allocations in independent positions, otherwise the solution degenerates. Thus, degeneracy occurs in the transportation problem when the number of allocations is less than $m + n - 1$.

To resolve degeneracy we allocate a very small quantity (say $\in$) close to zero to one or more (if needed) empty cells (generally lowest cost cells in case of minimization transportation problem) so as to get $m + n - 1$ number of occupied cells. This quantity $\in$ would not affect the total cost as well as supply and demand values. The cell containing this extremely small allocation is then treated like an occupied cell.

MAXIMIZATION TRANSPORTATION PROBLEM

A maximization transportation problem can be converted into the usual minimization transportation problem by subtracting all the costs from the highest cost involved in the problem. Now minimization problem can be solved in the usual manner as discussed earlier.

UNBALANCED TRANSPORTATION PROBLEM

If in a transportation problem, the sum of all available quantities is not equal to the sum of all requirements, that is $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$ then such problem is called unbalanced transportation problem.

Now there are two possibilities :

(*i*) If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, introduce a dummy destination in the problem. Assign a demand of $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ to this column and give the cost zero to each of the cells in this column.

(*ii*) If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, introduce a dummy source in the problem. Assign an availability of $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$ to this row and give the cost zero to each of the cells in this row.

After converting an unbalanced transportation problem to a balanced one by introducing a dummy row or dummy column, we can solve the above problem as discussed earlier.

(NWCR)

Q.1. *Find the initial feasible solution to the following transportation problem (in which the cells contain the transportation cost in rupees) using Norh-west corner rule. Also calculate the corresponding cost.*

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Demand	30	30	15	20	5	100

Solution.

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	$\boxed{30}$ 7	$\boxed{10}$ 6	4	5	9	40 10 0
F_2	8	$\boxed{20}$ 5	$\boxed{10}$ 6	7	8	30 10 0
F_3	6	8	$\boxed{5}$ 9	$\boxed{15}$ 6	5	20 15 0
F_4	5	7	7	$\boxed{5}$ 8	$\boxed{5}$ 6	10 5 0
Demand	30	30	15	20	5	100
	0	20	5	5	0	
		0	0	0		

Corresponding cost = Rs. $(30 \times 7 + 10 \times 6 + 20 \times 5 + 10 \times 6 + 5 \times 9 + 15 \times 6 + 5 \times 8 + 5 \times 6)$ = Rs. 635

Note: *This is just an initial feasible solution,*

(LOWEST COST METHOD)

Q.1. *Find the initial feasible solution to the following transportation problem (in which the cells contain the transportation cost in rupees) using* lowest cost method. *Also calculate the corresponding cost.*

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Demand	30	30	15	20	5	100

Solution.

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	$\boxed{5}$ 7	6	$\boxed{15}$ 4	$\boxed{20}$ 5	9	40 25 5 0
F_2	8	$\boxed{30}$ 5	6	7	8	30 0
F_3	$\boxed{15}$ 6	8	9	6	$\boxed{5}$ 5	20 15 0
F_4	$\boxed{10}$ 5	7	7	8	6	10 0
Demand	30 20 5 0	30 0	15 0	20 0	5 0	100

Corresponding cost = Rs. (35 + 60 + 100 + 150 + 90 + 25 + 50) = Rs. 510

Note: *This is just an initial feasible solution*

(VOGEL'S APPROXIMATION METHOD)

Q.1. Find the optimal solution of following tranportation problem.

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Demand	30	30	15	20	5	100

Sol. Writing penalties for each row and column, we get the following matrix :

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	7	6	4	5	9	40 (1)
F_2	8	5	6	7	8	30 (1)
F_3	6	8	9	6	5	20 (1)
F_4	5	7	7	8	6	10 (1)
Demand	30 (1)	30 (1)	15 (2)	20 (1)	5 (1)	100

Since the maximum penalty (2) is associated with the column 3, we allocate the maximum possible amount 15 to cell (1,3) and cross this column 3.

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	7	6	$\boxed{15}$ 4	5	9	40 (1)
F_2	8	5	6	7	8	30 (1)
F_3	6	8	9	6	5	20 (1)
F_4	5	7	7	8	6	10 (1)
Demand	30 (1)	30 (1)	15 (2)	20 (1)	5 (1)	100

The reduced matrix after leaving column 3 with remaining demand and supply and renewed penalties is as follows:

	W_1	W_2	W_4	W_5	Supply
F_1	7	6	5	9	25 (1)
F_2	8	5	7	8	30 (2)
F_3	6	8	6	5	20 (1)
F_4	5	7	8	6	10 (1)
Demand	30 (1)	30 (1)	20 (1)	5 (1)	

Since the maximum penalty (2) is associated with the row 2, we allocate the maximum possible amount 30 to cell (2,2) and cross row 2 amd column 2.

	W_1	W_2	W_4	W_5	Supply
F_1	7	6	5	9	25 (1)
F_2	8	30 5	7	8	30 (2)
F_3	6	8	6	5	20 (1)
F_4	5	7	8	6	10 (1)
Demand	30 (1)	30 (1)	20 (1)	5 (1)	

The reduced matrix after leaving row 2 and column 2 with remaining demand and supply and nenewed penalties is as follows :

	W_1	W_4	W_5	
F_1	7	5	9	25 (2)
F_3	6	6	5	20 (1)
F_4	5	8	6	10 (1)
	30 (1)	20 (1)	5 (1)	

Since the maximum penalty (2) is associated with the column 4, we allocate the maximum possible amount 20 to cell (1,4) and cross column 4.

	W_1	W_4	W_5		
F_1	7	20 5	9	25	(2)
F_3	6	6	5	20	(1)
F_4	5	8	6	10	(1)
	30	20	5		
	(1)	(1)	(1)		

The reduced matrix after leaving column 4 with remaining demand and supply and nenewed penalties is as follows :

	W_1	W_5		
F_1	7	9	5	(2)
F_3	6	5	20	(1)
F_4	5	6	10	(1)
	30	5		
	(1)	(1)		

The maximum penalty now is in row 1, we allocate the maximum possible amount 5 to cell (1,1) with lowest cost and cross out the row 1.

	W_1	W_5		
F_1	$\boxed{5}$ 7	9	5	(2)
F_3	6	5	20	(1)
F_4	5	6	10	(1)
	30	5		
	(1)	(1)		

The reduced matrix after leaving row 1 with remaining demand and supply and nenewed penalties is as follows :

	W_1	W_5		
F_3	6	5	20	(1)
F_4	5	6	10	(1)
	25	5		
	(1)	(1)		

The maximum penalty now is (1), which is in each row and column, we allocate the maximum possible amount 10 to cell (4,1) with lowest cost and cross out the row 4.

	W_1	W_5		
F_3	6	5	20	(1)
F_4	$\boxed{10}$ 5	6	10	(1)
	25	5		
	(1)	(1)		

The reduced matrix after leaving row 4 with remaining demand and supply is as follows :

	W_1	W_5	
F_3	6	5	20
	15	5	

Now, finally allocating the remaining values to remaining cells, we get

	W_1	W_5
F_3	$\boxed{15}$ 6	$\boxed{5}$ 5

Hence the initial table with the complete allocations by VAM is as follows :

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	$\boxed{5}$ 7	6	$\boxed{15}$ 4	$\boxed{20}$ 5	9	40
F_2	8	$\boxed{30}$ 5	6	7	8	30
F_3	$\boxed{15}$ 6	8	9	6	$\boxed{5}$ 5	20
F_4	$\boxed{10}$ 5	7	7	8	6	10
Demand	30	30	15	20	5	100

Here, m + n - 1 = 4 + 5 - 1 = 8 and the number of positive allocations = 7 i.e. number of positive allocations < m + n - 1. So, the above solution is degenerate and for removing this degeneracy, we have to allocate $\in$ as amount to any non-occupied cell with minimum cost. Here, 6 is the minimum cost in non-occupied cells at three places, out of which any place can be chosen. So, we allocate $\in$ to cell (1,2).

Hence the initial table with the complete allocations by VAM is as follows :

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	[5] 7	[$\in$] 6	[15] 4	[20] 5	9	40
F_2	8	[30] 5	6	7	8	30
F_3	[15] 6	8	9	6	[5] 5	20
F_4	[10] 5	7	7	8	6	10
Demand	30	30	15	20	5	100

Now, the table of u_i and v_j values for the occupied cells is given below :

						$v_j \downarrow$
	7	6	4	5	.	0
	.	5	.	.	.	−1
	6	.	.	.	5	−1
	5	.	.	.	.	−2
$u_i \rightarrow$	7	6	4	5	6	

Cell evaluations for each unoccupied cell can be obtained by using the relationship

$$[d_{ij}] = [c_{ij}] - [u_i + v_j]$$

$$= \begin{bmatrix} . & . & . & . & 9 \\ 8 & . & 6 & 7 & 8 \\ . & 8 & 9 & 6 & . \\ . & 7 & 7 & 8 & 6 \end{bmatrix} - \begin{bmatrix} . & . & . & . & 6 \\ 6 & . & 3 & 4 & 5 \\ . & 5 & 3 & 4 & . \\ . & 4 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} . & . & . & . & 3 \\ 2 & . & 3 & 3 & 3 \\ . & 3 & 6 & 2 & . \\ . & 3 & 5 & 5 & 2 \end{bmatrix}$$

Since all d_{ij} for all empty cells are positive, the current initial basic feasible solution is optimal.

Hence, the minimum transportation cost = 35 + 0 + 60 + 100 + 150 + 90 + 25 + 50 = ₹ 510

Q.2. *A company has four factories F_1, F_2, F_3, F_4, from which it supplies to three warehouses W_1, W_2, W_3. Determine the optimal transportation plan from the following data giving the factories to warehouses shifting costs, quantities available at each factory and quantities required at each warehouse.*

		Factories				
		F_1	F_2	F_3	F_4	Required at warehouses
warehouses	W_1	6	4	1	5	14
	W_2	8	9	2	7	16
	W_3	4	3	6	2	5
Available at factory		6	10	15	4	35

Sol. By Vogel's Approximation method (VAM), initial B.F.S. is prepared as :

		Factories				Required at warehouses					
		F_1	F_2	F_3	F_4						
warehouses	W_1	$\boxed{4}$ 6	$\boxed{10}$ 4	1	5	14	(3)	14 (1)	14 (2)	4	0
	W_2	$\boxed{1}$ 8	9	$\boxed{15}$ 2	7	16	(5)	1 (1)	1 (1)	1	0
	W_3	$\boxed{1}$ 4	3	6	$\boxed{4}$ 2	5	(1)	5 (1)	1 (1)	1	0
Available at factory		6	10	15	4	35					
		(2)	(1)	(1)	(3)						
		0	0	0	0						

Now, initial basic feasible solution is given as :

		Factories				
		F_1	F_2	F_3	F_4	Required at warehouses
warehouses	W_1	$\boxed{4}$ 6	$\boxed{10}$ 4	1	5	14
	W_2	$\boxed{1}$ 8	9	$\boxed{15}$ 2	7	16
	W_3	$\boxed{1}$ 4	3	6	$\boxed{4}$ 2	5
Available at factory		6	10	15	4	35

Here the number of independent allocations

$$= m + n - 1 = 3 + 4 - 1 = 6.$$

Hence, the condition of optimality test is satisfied.

Now, the table of u_i and v_j values for the occupied cells is given below :

					$v_j \downarrow$
	6	4	.	.	6
	8	.	2	.	8
	4	.	.	2	4
$u_i \rightarrow$	0	–2	–6	–2	

Cell evaluations for each unoccupied cell can be obtained by using the relationship

$$[d_{ij}] = [c_{ij}] - [u_i + v_j]$$

$$= \begin{bmatrix} \cdot & \cdot & 1 & 5 \\ \cdot & 9 & \cdot & 7 \\ \cdot & 3 & 6 & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & 0 & 4 \\ \cdot & 6 & \cdot & 6 \\ \cdot & 2 & -2 & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot & 1 & 1 \\ \cdot & 3 & \cdot & 1 \\ \cdot & 1 & 8 & \cdot \end{bmatrix}$$

Since all d_{ij} for all empty cells are positive, the current initial basic feasible solution is optimal.

Hence, the minimum transportation cost = 24 + 40 + 8 + 30 + 4 + 8 = ₹ 114

Q.3. *Solve the following transportation problem to maximize the profit.*

From \ To (origin / destination)	1	2	3	4	Supply
A	90	90	100	110	200
B	50	70	130	85	100
Demand	75	100	100	30	300 \ 305

Sol. After balancing the problem by introducing one **dummy** origin C with supply 5, the new transportation matrix is

	1	2	3	4	
A	90	90	100	110	200
B	50	70	130	85	100
C	0	0	0	0	5
	75	100	100	30	

Above transportation problem of maximization is converted into transportation problem of minimization by subtracting all cell entries from the maximum value of cell entry, which is 130. Then transportation problem of minimization is shown as :

	1	2	3	4	
A	40	40	30	20	200
B	80	60	0	45	100
C	130	130	130	130	5
	75	100	100	30	

By Vogel's Approximation method (VAM), initial B.F.S. is **prepared as :**

	1	2	3	4				
A	$\boxed{70}$ 40	$\boxed{100}$ 40	$\boxed{\in}$ 30	$\boxed{30}$ 20	200 (10)	200(20)	170 (0)	70 | 0
B	80	60	$\boxed{100}$ 0	45	100 (45)	0		
C	$\boxed{5}$ 130	130	130	130	5 (0)	5 (0)	5 (0)	5 | 0
	75 (40)	100 (20)	100 (30)	30 (25)				
	75 (90)	100 (90)	0	30 (110)				
	0	0		0				

Here, m + n - 1 = 3 + 4 - 1 = 6 and the number of positive allocations = 5 i.e. number of positive allocations < m + n - 1. So, the above solution is degenerate and for removing this degeneracy, we have to allocate $\in$ as amount to any non-occupied cell with minimum cost. Here, 30 is the minimum cost in non-occupied cells at (1,3) position. So, we allocate $\in$ to cell (1,3).

Now, initial basic feasible solution is given as :

	1	2	3	4	
A	$\boxed{70}$ 40	$\boxed{100}$ 40	$\boxed{\epsilon}$ 30	$\boxed{30}$ 20	200
B	80	60	$\boxed{100}$ 0	45	100
C	$\boxed{5}$ 130	130	130	130	5
	75	100	100	30	

Now, the table of u_i and v_j values for the occupied cells is given below :

					$v_j \downarrow$
	40	40	30	20	0
	·	·	0	·	- 30
	130	·	·	·	90
$u_i \rightarrow$	40	40	30	20	

Cell evaluations for each unoccupied cell can be obtained by using the relationship

$$[d_{ij}] = [c_{ij}] - [u_i + v_j]$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 80 & 60 & \cdot & 45 \\ \cdot & 130 & 130 & 130 \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 10 & 10 & \cdot & -10 \\ \cdot & 130 & 120 & 110 \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 70 & 50 & \cdot & 55 \\ \cdot & 0 & 10 & 20 \end{bmatrix}$$

Since all d_{ij} for all empty cells are ≥ 0 the current initial basic feasible solution is optimal.
Hence, the maximum profit = (70x90) + (100x90) + (30x110) + (100x130) + (5x0)
= 6300 + 9000 + 3300 + 13000 + 0 = Rs. 31600

Q.4. Find the optimal solution of following tranportation problem.

		warehouses				
		W_1	W_2	W_3	W_4	*Supply*
factory	F_1	19	30	50	10	7
	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
Demand		5	8	7	14	34

Sol. **Now, initial basic feasible solution through Vogel's approximation method is given as :**

		warehouses				
		W_1	W_2	W_3	W_4	*Supply*
factory	F_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7
	F_2	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9
	F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	18
Demand		5	8	7	14	34

Here the number of independent allocations

$$= m + n - 1 = 3 + 4 - 1 = 6.$$

Hence, the condition of optimality test is satisfied.

Now, the table of u_i and v_j values for the occupied cells is given below :

					$v_j \downarrow$
	19	·	·	10	0
	·	·	40	60	50
	·	8	·	20	10
$u_i \rightarrow$	19	-2	-10	10	

Cell evaluations for each unoccupied cell can be obtained by using the relationship

$$[d_{ij}] = [c_{ij}] - [u_i + v_j]$$

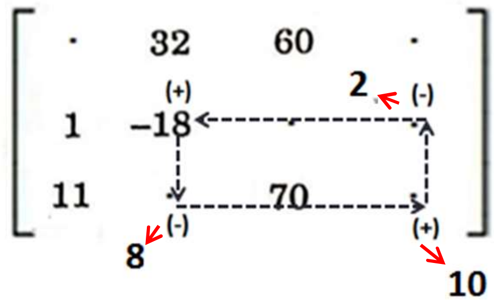
$$= \begin{bmatrix} \cdot & 30 & 50 & \cdot \\ 70 & 30 & \cdot & \cdot \\ 40 & \cdot & 70 & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & -2 & -10 & \cdot \\ 69 & 48 & \cdot & \cdot \\ 29 & \cdot & 0 & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & 32 & 60 & \cdot \\ 1 & -18 & \cdot & \cdot \\ 11 & \cdot & 70 & \cdot \end{bmatrix}$$

Since one of the d_{ij} [at cell (2,2)] for empty cells is negative, the current initial basic solution is not optimal. Now, we make a closed path as shown in the diagram below starting with empty cell (2,2) by covering allocated cells as corner points.

Also, we assign + sign to cell (2,2) and then alternative signs for corners of the closed path.

$$\begin{bmatrix} \cdot & 32 & 60 & \cdot \\ 1 & \overset{(+)}{-18} & \cdot & \overset{(-)}{\cdot} \\ 11 & \underset{(-)}{\cdot} & 70 & \underset{(+)}{\cdot} \end{bmatrix}$$



Now, the values of allocations of already allocated cells as corners of above closed path are 8, 10 and 2 respectively as shown before in initial basic feasible solution, out of which 2 is the minimum value.

So, we add 2 as allocation with (+) sign cells and subtract 2 from allocation with (-) sign cells and get the new improved basic feasible solution shown as :

		warehouses				
		W_1	W_2	W_3	W_4	*Supply*
factory	F_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7
	F_2	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	9
	F_3	40	$\boxed{6}$ 8	70	$\boxed{12}$ 20	18
Demand		5	8	7	14	34

Here the number of independent allocations

$$= m + n - 1 = 3 + 4 - 1 = 6.$$

Hence, the condition of optimality test is satisfied.

Now, the table of u_i and v_j values for the occupied cells is given below :

					$v_j \downarrow$
	19	·	·	10	0
	·	30	40	·	32
	·	8	·	20	10
$u_i \rightarrow$	19	-2	8	10	

Cell evaluations for each unoccupied cell can be obtained by using the relationship

$$[d_{ij}] = [c_{ij}] - [u_i + v_j]$$

$$= \begin{bmatrix} \cdot & 30 & 50 & \cdot \\ 70 & \cdot & \cdot & 60 \\ 40 & \cdot & 70 & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & -2 & 8 & \cdot \\ 51 & \cdot & \cdot & 42 \\ 29 & \cdot & 18 & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & 32 & 42 & \cdot \\ 19 & \cdot & \cdot & 18 \\ 11 & \cdot & 52 & \cdot \end{bmatrix}$$

Since all d_{ij} for all empty cells are ≥ 0 the current initial basic feasible solution is optimal.
Hence, the maximum profit = (19x5) + (10x2) + (30x2) + (40x7) + (8x6) + (20x12)
= 95 + 20 + 60 + 280 + 48 + 240 = Rs. 743

Exercise

1. Find the optimal solution for the following transportation problem:

Destinations → **Origins ↓**	D_1	D_2	D_3	**Supply**
O_1	4	14	8	5
O_2	6	6	14	8
O_3	10	8	2	7
O_4	2	12	4	14
Demand	7	9	18	34

2. Find the optimal solution for the following transportation problem:

Destinations → **Origins ↓**	D_1	D_2	D_3	D_4	**Supply**
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

THANKS