Chapter 8

FINITE DIFFERENCES AND INTERPOLATION

8.1 INTRODUCTION

The calculus of finite differences deals with the changes that take place in the value of the function (dependent variable), due to finite changes in the independent variable. Through this, we also study the relations that exists between the values assumed by the function whenever the independent variable changes by finite jumps whether equal or unequal. On the other hand, in an infinitesimal calculus we learn those changes of the function which occur when the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in the function when the independent variable changes by equal intervals.

8.2 FINITE DIFFERENCES

In this chapter we introduce variations in the functions, which are known as forward, backward and central differences. These differences are three standard examples of finite differences.

8.3 FORWARD DIFFERENCE

Let $y = f(x)$ be a function of equal interval of the independent variable x. Let $x = a, a + h, a + 2h, \ldots x_0 + nh$ giving $y = f(a), f(a + h), f(a + 2h), \ldots\ldots f(a + nh)$. Then the difference $f(a + h) - f(a)$ is known as the first difference of the function $f(x)$ at $x = a$ and is denoted with first difference operator $\Delta f(a)$. Similarly $f(a + 2h) - f(a + h)$ is known as first difference at the point $x = a + h$ and is denoted with $\Delta f(a + h)$ and so on. Similarly second difference is $\Delta f(a + h) - \Delta f(a)$ of the function $f(x)$ at the point $x = a$ and denoted with second difference operator $\Delta^2 f(a)$ and so on

i.e.

$$\Delta f(a) = f(a+h) - f(a) \quad \text{or} \quad \Delta y_1 = y_2 - y_1$$

$$\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a) \qquad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots$$

$$\Delta^n f(a) = \Delta^{n-1} f(a+h) - \Delta^{n-1} f(a) \qquad \Delta^n y_1 = \Delta^{n-1} y_2 - \Delta^{n-1} y_1$$

Here the notation Δ is known as forward difference operator for a given function $f(x)$.

These differences are systematically arranged in table 1.1.

In difference table x is called the argument and y or $f(x)$ the function. y_0, the first entry is called the leading term and $\Delta y_0\ \Delta^2 y_0\ \Delta^3 y_0 \ldots \Delta^n y_0$ are called the leading differences.

Table 8.1. Forward difference table

Value of x	Value of y	Differences I	II	III	......n^{th} diff.
x_0	y_0				
		Δy_0			
$x_0 + h$	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
$x_0 + 2h$	y_2		$\Delta^2 y_1$	$\vdots$	......$\Delta^n y_0$
		Δy_2	$\vdots$	$\Delta^3 y_{n-3}$	
$x_0 + 3h$	y_3	$\vdots$	$\Delta^2 y_{n-2}$		
$\vdots$		Δy_{n-1}			
$x_0 + nh$	y_n				

8.4 BACKWARD DIFFERENCE

As mentioned earlier, let $y_0\, y_1\, y_2 \ldots y_n$ be the values of a function $y = f(x)$, corresponding to the values $x_0\, x_1\, x_2 \ldots x_n$ of x respectively which are of intervaled with step length h. Then

$$\nabla f(a) = f(a) - f(a-h) \text{ or } \nabla y_1 = y_1 - y_0$$

which is known as first backward difference.

Similary second backward difference is $\nabla f(a) - \nabla f(a-h)$ of the function $f(x)$ at the point $x = a$ and is denoted with second difference operator $\nabla^2 f(a)$ and so on.

i.e.

$$\nabla f(a) = f(a) - f(a-h) \quad \text{or} \quad \nabla y_1 = y_1 - y_0$$

$$\nabla^2 f(a) = \nabla f(a) - \nabla f(a-h) \qquad \nabla^2 y_1 = \nabla y_1 - \nabla y_0$$

$$\vdots \qquad\qquad \vdots$$

$$\nabla^n f(a) = \nabla^{n-1} f(a) - \nabla^{n-1} f(a-h) \qquad \nabla^n y_1 = \nabla^{n-1} y_1 - \nabla^{n-1} y_0$$

Here notation ∇ is known as backward difference operator for a given function $f(x)$.

Table 8.2. Backward difference table

Value of x	*Value of y*	*I*	*II*	*III*	……..*n*^th^ *diff.*
x_0	y_0				
		∇y_1			
$x_0 + h$	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
$x_0 + 2h$	y_2		$\nabla^2 y_3$	$\vdots$	………$\nabla^n y_n$
		∇y_3	$\vdots$	$\nabla^2 y_n$	
$x_0 + 3h$	y_3	$\vdots$	$\nabla^2 y_n$		
$\vdots$	$\vdots$	∇y_n			
$x_0 + nh$	y_n				

8.5 CENTRAL DIFFERENCE

As mentioned earlier let $y_0 y_1 \ldots\ldots y_n$ be the values of the function $y = f(x)$, corresponding to the values $x_0 x_1 x_2 \ldots\ldots x_n$ of x respectively which are of intervaled with step length h, then

$$\delta f(a) = f\left(a + \frac{h}{2}\right) - f\left(a - \frac{h}{2}\right)$$

or

$$\delta y_{1/2} = y_1 - y_0 \qquad \delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$

$$\delta y_{3/2} = y_2 - y_1 \qquad \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$$

$$\vdots \qquad \vdots$$

$$\delta y_{n+\frac{1}{2}} = y_{n+1} - y_n \qquad \delta^2 y_n = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$$

which are first and second central differences where δ is known as first central difference and δ^2 is known as second central difference operator.

Similarly any central difference operator is defined

i.e.

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$

$$\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$$

$$\vdots \quad \vdots \quad \vdots$$

and so on.

These differences are shown in table 8.3.

Table 8.3. Central difference table

Value of x	Value of y	I	II	III	......
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$	$\vdots$	....
		$\delta y_{5/2}$	$\vdots$	$\delta^3 y_{n-3/2}$	
x_3	y_3	$\vdots$	$\delta^2 y_{n-1}$		
$\vdots$	$\vdots$	$\delta y_{n-\frac{1}{2}}$			
x_n	y_n				

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same colomn of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \frac{1}{2}(\delta y_{1/2} + \delta y_{3/2}), \mu\delta^2 y_{3/2} = \frac{1}{2}(\delta^2 y_1 + \delta^2 y_2)$$

and so on.

Remark: Reader should note that it is only the notation which changes note the difference *e.g.* $y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$.

8.6 OTHER OPERATORS

(*i*) **Shift operator (*E*).**

The shift operator is defined as

$$E[f(x)] = f(x + h)$$

or

$$Ey_x = y_{x+h}.$$

i.e. shift operator mean that it shift the value of $f(x)$ by one interval to the next higher value.

Second and higher shift operators can be defined as

$$E^2[f(x)] = f(x + 2h), E^3[f(x)] = f(x + 3h) \ldots\ldots E^n[f(x)] = f(x + nh)$$

where n is real number and h is the interval of differencing.

(*ii*) **Inverse operator (E^{-1}).**

The inverse operator (E^{-1}) is defined as

$$E^{-1}[f(x)] = f(x-h)$$

Properties of shift operator

(*a*) $$E[Cf(x)] = CE[f(x)] = Cf(x+h)$$

(*b*) $$E[f(x)+g(x)] = E[f(x)] + E[g(x)]$$

(*c*) $$E[C_1f_1(x)+C_2f_2(x)] = C_1E[f_1(x)] + C_2E[f_2(x)]$$

(*iii*) **Average operator (μ)**

The average operator is defined as

$$\mu[f(x)] = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

(*iv*) **Difference operators** (Δ, ∇, δ) are also defined as

(*a*) $$\Delta[f(x)] = f(x+h) - f(x)$$

(*b*) $$\nabla[f(x)] = f(x) - f(x-h)$$

(*c*) $$\delta[f(x)] = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

8.7 SOME IMPORTANT IDENTITIES OF OPERATORS

(*i*) $$\Delta = E - I$$

(*ii*) $$\nabla = I - E^{-1}$$

(*iii*) $$E = e^{hD} = I + \Delta$$ [*Raj. 2004*]

(*iv*) $$\delta = E^{1/2} - E^{-1/2}$$

(*v*) $$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

(*vi*) $$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$$

Proof.

(*i*) $$\Delta = E - I \qquad ...(8.1)$$

$$\Delta[f(x)] = f(x+h) - f(x)$$
$$= E[f(x)] - f(x)$$
$$= (E-I)f(x)$$

Hence, $$\Delta = E - I$$

(*ii*) $$\nabla = I - E^{-1} \qquad ...(8.2)$$

$$\nabla[f(x)] = f(x) - f(x-h)$$
$$= f(x) - E^{-1}[f(x)]$$
$$= (I - E^{-1})f(x)$$

Hence, $$\nabla = I - E^{-1}$$

(*iii*) $$E = e^{hD} \qquad ...(8.3)$$

$$E[f(x)] = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + ...$$

$$= f(x) + hD\,[f(x)] + \frac{h^2}{\lfloor 2}\,D^2\,[f(x)] + \ldots\ldots$$

$$= \left[1 + hD + \frac{h^2}{\lfloor 2}\,D^2 + \ldots\ldots\right] f(x)$$

$$= e^{hD} f(x).$$

Hence, $E = e^{hD} = I + \Delta$ [using equation (8.1)]

(iv) $\delta = E^{1/2} - E^{-1/2}$ ...(8.4)

$$\delta\,[f(x)] = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$= [E^{1/2} - E^{-1/2}]\, f(x)$$

Hence, $\delta = E^{1/2} - E^{-1/2}$

(v) $\mu = \frac{1}{2}\,(E^{1/2} + E^{-1/2})$ ...(8.5)

$$\mu\,[f(x)] = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\,[E^{1/2}\,[f(x)] + E^{-1/2}\,[f(x)]]$$

$$= \frac{1}{2}\,[E^{1/2} + E^{-1/2}]\, f(x).$$

Hence, $\mu = \frac{1}{2}\,[E^{1/2} + E^{-1/2}]$

(vi) $\Delta = \frac{1}{2}\,\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$ ...(8.6)

$$\frac{1}{2}\,\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} = \frac{1}{2}\,(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= \frac{1}{2}\,(E - 2 + E^{-1}) + \frac{(E^{1/2} - E^{-1/2})}{2}\sqrt{4 + (E - 2 + E^{-1})}$$

$$= \frac{1}{2}\,(E - 2 + E^{-1}) + \frac{(E^{1/2} - E^{-1/2})}{2}\sqrt{(E + 2 + E^{-1})}$$

$$= \frac{1}{2}\,[E - 2 + E^{-1} + E - E^{-1}] = E - I = \Delta$$

8.8 FACTORIAL FUNCTION

A product of the form $n(n-1)(n-2) \ldots\ldots (n-r+1)$ is denoted by $x^{(r)}$ and called a factorial.

In particular $x^{(1)} = x$; $x^{(2)} = x(x-1)$; $x^{(3)} = x(x-1)(x-2)$; etc.

In general $x^{(n)} = x(x-1)(x-2) \ldots\ldots (x-n+1)$.

In case, the interval of differencing is h, then $x^{(n)} = x(x-h)(x-2h) \ldots\ldots (x-\overline{n-1}\,h)$. Also, $x^{(0)} = 1$.

The factorial notation has special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation. Similary given any difference of a function in factorial notation we can find the corresponding function by simple integration.

The result of differentiating $x^{(r)}$ is analogous to that of differentiating x^r.

To show that

$$\Delta^n x^{(n)} = \underline{|n}\, h^n \text{ and } \Delta^{n+1} x^{(n)} = 0$$

We have,

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)(x+h-h)(x+h-2h) \ldots\ldots (x+h-\overline{n-1}\,h)$$

$$- x(x-h)(x-2h) \ldots\ldots (x-\overline{n-1}\,h)$$

$$= x(x-h) \ldots\ldots (x-\overline{n-2}\,h)\,[x+n-(x-nh+n)]$$

$$= nh\, x^{(n-1)} \qquad \ldots(8.7)$$

Similarly $\Delta^2 x^{(n)} = \Delta\,[nh\, x^{(n-1)}] = nh\, \Delta x^{(n-1)}$

replacing n by $(n-1)$ in (8.7), we get

$$\Delta^2 x^{(n)} = nh\,(n-1)\,h\, x^{(n-2)} = n(n-1)\,h^2 x^{(n-2)}$$

Proceeding in this way, we obtain

$$\Delta^n x^{(n)} = n(n-1) \ldots\ldots 2.1.h^n x^{(n-n)}$$

$$= n(n-1) \ldots\ldots 2.1\, h^n \qquad [\because\ x^{(0)} = 1]$$

$$= n!\, h^n \qquad \ldots(8.8)$$

also,

$$\Delta^{n+1} x^{(n)} = n!\, h^n - n!\, h^n = 0. \qquad \ldots(8.9)$$

In particular, when $h = 1$, we have

$$\Delta\, x^{(n)} = n\, x^{(n-1)} \text{ and } \Delta^n x^{(n)} = n!$$

Remark: Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice-versa.

8.9 CALCULATION OF MISSING TERM

If one or more values of $f(x)$ be missing from set of n observation values of $f(x)$ at regular interval h in the values of x. They can be calculated by the following procedure.

Let one value is missing from n observation then number of observations are $(n-1)$, so

$$\Delta^{n-1} f(x) = 0 \qquad \ldots(8.10)$$

or

$$(E-1)^{n-1} f(x) = 0 \qquad \ldots(8.11)$$

from the expansion, we get an equation. Taking $x = x_0$ we get the missing terms or using difference table and taking $(n-1)$ differences equal to zero we get the missing term.

ILLUSTRATIVE EXAMPLES

1.

(i) Prove the following identities with the usual notations :

(a) $\nabla = E^{-1}\Delta$

(b) $\Delta = (I - \nabla)^{-1} - I$

(c) $\Delta^r y_k = \nabla^r y_{k+r}$ [Raj. 1999]

(d) $\Delta - \nabla = \Delta\nabla = \delta^2$ [Raj. 2003]

(ii) Prove the following :

(a) $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$

(b) $\delta[f(x)\,.\,g(x)] = \mu[f(x)\,\delta[g(x)] + \mu[g(x)]\,\delta[f(x)]$ [Raj. 2003]

(c) $\mu[f(x)\,.\,g(x)] = \mu[f(x)]\,\mu[g(x)] + \dfrac{1}{4}\,\delta[f(x)]\,\delta[g(x)]$ [Raj. 1993, 94]

(iii) Evaluate the following considering 'h' as interval of differencing unless otherwise stated :

(a) $E^3\,(x \sin x)$

(b) $\Delta\,[\log f(x)]$ [Raj. 1998, 2006]

(c) $\Delta\left[\dfrac{2^x}{\lfloor x+1}\right]$; $h = 1$

(d) $\Delta\,(e^{ax} \log bx)$

(e) $\Delta^n\,(ax^n + bx^{n-1})$; $h = 1$ [Raj. 2005]

(f) $\Delta\,[\tan^{-1}(ax)]$; $h = 1$

(g) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$; $h = 1$

(h) $\Delta^n\,[\cos(ax+b)]$

(i) $\Delta^n\,x^{(n)}$; $h = 1$

(iv) Evaluate the following considering 'h' as interval of differencing unless otherwise stated :

(a) $\left(\dfrac{\Delta^2}{E}\right)x^3$; $h = 1$ [Raj. 2003]

(b) $(\Delta - \nabla)\,x^2$

(c) $\left(\dfrac{\Delta^2}{E}\right)e^x\,.\,\dfrac{E(e^x)}{\Delta^2(e^x)}$ [Raj. 2000]

(d) $(\nabla + \Delta)^2\,(x^2 + x)$; $h = 1$

(e) $\dfrac{\nabla^2(x^2)}{E\,(x + \log x)}$; $h = 1$

(v) *Find the function whose first difference is* $9x^2 + 11x + 5$

(vi) *Express* $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ *in factorial functions and hence evaluate* $\Delta^4 [f(x)]$ *and* $\frac{1}{\Delta^2} [f(x)]$

(vii) *Prove that* $x^{(-n)} = \dfrac{1}{(x + nh)^{(n)}}$ [Raj. 2006]

(viii) *Find the values of y for x = 15 and x = 25 using following table :*

x :	5	10	15	20	25	30
y :	7	10	–	17	–	28

(ix) *Prove the following :*

(a) $u_0 + \dfrac{u_1 x}{\lfloor 1} + \dfrac{u_2 x^2}{\lfloor 2} + \ldots\ldots = e^x \left[u_0 + x\Delta u_0 + \dfrac{x^2}{\lfloor 2} \Delta^2 u_0 + \ldots\ldots \right]$

[Raj. 1990, 2006]

(b) $u_{x+n} = u_n + {}^xC_1 \,\Delta u_{n-1} + {}^xC_2 \,\Delta^2 u_{n-2} + \ldots\ldots$

(c) $\Delta^n u_x = u_{x+n} - n_{c_1} u_{x+n-1} + n_{c_2} u_{x+n-2} - \ldots\ldots + (-1)^n u_x$

Solution. (*i*)

(*a*)
$$\begin{aligned}
\text{RHS} &= (E^{-1}\Delta) f(x) \\
&= E^{-1} [\Delta f(x)] = E^{-1} [f(x+h) - f(x)] \\
&= E^{-1} [f(x+h)] - E^{-1} [f(x)] \\
&= f(x) - f(x-h) \\
&= \nabla . = \text{LHS}
\end{aligned}$$

(*b*)
$$\begin{aligned}
\text{RHS} &= (1 - \nabla)^{-1} - I \\
&= (E^{-1})^{-1} - 1 \\
&= E - 1 \\
&= \Delta . = \text{LHS}
\end{aligned}$$

(*c*)
$$\begin{aligned}
\text{RHS} &= (1 - E^{-1})^r y_{k+r} \\
&= (1 - {}^rc_1 E^{-1} + {}^rc_2 E^{-2} + \ldots\ldots + (-1)^r E^r] \, y_{k+r} \\
&= y_{k+r} - {}^rc_1 E^{-1} y_{k+r} + {}^rc_2 E^{-2} y_{k+r} + \ldots + (-1)^r E^r y_{k+r} \\
&= y_{k+r} - {}^rc_1 y_{k+r-1} + {}^rc_2 y_{k+r-2} + \ldots + (-1)^r y_k \\
&= E^r y_k - {}^rc_1 E^{r-1} y_k + {}^rc_2 E^{r-2} y_k + \ldots + (-1)^r y_k \\
&= [E^r - {}^rc_1 E^{r-1} + {}^rc_2 E^{r-2} + \ldots + (-1)^r] \, y_k \\
&= (E - 1)^r y_k \\
&= \Delta^r y_k = \text{LHS}
\end{aligned}$$

(*d*)
$$\begin{aligned}
(\Delta - \nabla) f(x) &= f(x+h) - f(x) - f(x) + f(x-h) \\
&= [f(x+h) - f(x+n-h)] - [f(x) + f(x-h)] \\
&= \nabla f(x+h) - \nabla f(x) \\
&= \nabla [f(x+h) - f(x)] \\
&= \nabla \Delta
\end{aligned}$$

$$\delta^2 f(x) = \delta[\delta(f(x))]$$

$$= \delta\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right]$$

$$= \delta\left[f\left(x+\frac{h}{2}\right)\right]-\delta\left[f\left(x-\frac{h}{2}\right)\right]$$

$$= \left\{f\left(x+\frac{h}{2}+\frac{h}{2}\right)-f\left(x+\frac{h}{2}-\frac{h}{2}\right)\right\}$$

$$-\left\{f\left(x-\frac{h}{2}+\frac{h}{2}\right)-f\left(x-\frac{h}{2}-\frac{h}{2}\right)\right\}$$

$$= \{f(x+h)-f(x)\}-\{f(x)-f(x-h)\}$$

$$= \Delta - \nabla$$

(ii) (a)

$$\text{LHS} = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)\,g(x)-g(x+h)f(x)}{g(x+h)\,g(x)}$$

$$= \frac{f(x+h)\,g(x)-g(x+h)f(x)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,[f(x+h)-f(x)]-f(x)[g(x+h)-g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x)-f(x)\,\Delta g(x)}{g(x+h)\,g(x)} = \text{RHS}$$

(b)

$$\text{RHS} = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)-g\left(x-\frac{h}{2}\right)\right]$$

$$+\frac{1}{2}\left[g\left(x+\frac{h}{2}\right)+g\left(x-\frac{h}{2}\right)\right]\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)-f\left(x+\frac{h}{2}\right)g\left(x-\frac{h}{2}\right)\right.$$

$$\left.+f\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)g\left(x-\frac{h}{2}\right)\right]$$

$$+\frac{1}{2}\left[f\left(x+\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)\right.$$

$$\left.+g\left(x-\frac{h}{2}\right)f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)g\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[2f\left(x+\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)-2f\left(x-\frac{h}{2}\right)g\left(x-\frac{h}{2}\right)\right]$$

$$= f\left(x+\frac{h}{2}\right) g\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) g\left(x-\frac{h}{2}\right)$$

$$= \delta[f(x)\, g(x)] = \text{LHS}$$

(c)
$$\text{RHS} = \frac{1}{4}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)+g\left(x-\frac{h}{2}\right)\right]$$

$$+\frac{1}{4}\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)-g\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) g\left(x-\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right) g\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}}\, f\, E^{\frac{1}{2}}\, g + E^{-\frac{1}{2}}\, f\, E^{-\frac{1}{2}}\, g\right]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}}\,(fg) + E^{-\frac{1}{2}}\,(fg)\right]$$

$$= \mu\,\{f(x)\, g(x)\} = \text{LHS}$$

(*iii*) (*a*) $E^2\,[E(x \sin x)]$

$$= E\,[E((x+h)\sin(x+h))]$$

$$= E\,[(x+2h)\sin(x+2h)]$$

$$= (x+3h)\sin(x+3h)$$

(*b*) $\log\,[f(x+h)] - \log\,[f(x)]$

$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log\left[\frac{Ef(x)}{f(x)}\right]$$

$$= \log\left[\frac{(1+\Delta)\, f(x)}{f(x)}\right]$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(*c*)
$$\frac{2^{x+1}}{\underline{\left| x+1+1 \right.}} - \frac{2^x}{\underline{\left| x+1 \right.}}$$

$$= \frac{2^x}{\underline{\left| x+1 \right.}}\left[\frac{2}{x+2}-1\right]$$

$$= -\frac{x2^x}{\underline{\left| x+2 \right.}}$$

(*d*) We know that

$$\Delta\,[f(x)\,g(x)] = [f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)]$$

$$\therefore \quad \Delta f(x) = \Delta\,[e^{ax}] = e^{a(x+h)} - e^{ax}$$

$$= e^{ax}\,[e^{ah} - 1]$$

$$\Delta g(x) = \Delta\,[\log bx]$$

$$= \log\{b(x+h)\} - \log bx$$

$$= \log \frac{b(x+h)}{bx} = \log\left(1 + \frac{h}{x}\right)$$

$$\Delta\,[e^{ax}\log bx] = e^{a(x+h)}\log\left(1+\frac{h}{x}\right) + \log bx\,.\,e^{ax}\,(e^{ah}-1)$$

$$= e^{x}\left[e^{ah}\log\left(1+\frac{h}{x}\right) + \log bx\,.\,(e^{ah}-1)\right]$$

(*e*) $\Delta^n\,(ax^n + bx^{n-1})$

$$= a\Delta^n x^n \qquad \left[\because\ \Delta^n x^p = \begin{matrix} 0 & \text{if } n > p \\ p! & \text{if } n = p \end{matrix}\right]$$

$$= a\,.\,\underline{|n}$$

(*f*) $\tan^{-1}[a(x+1)] - \tan^{-1}[ax]$

$$= \tan^{-1}\left[\frac{a(x+1) - ax}{1 + a^2(x+1)x}\right]$$

$$= \tan^{-1}\left[\frac{a}{1 + a^2x + a^2x^2}\right]$$

(*g*) $\Delta^2\left[\dfrac{2}{x+2} + \dfrac{3}{x+3}\right]$

$$= \Delta\left[\Delta\left\{\frac{2}{x+2} + \frac{3}{x+3}\right\}\right]$$

$$= \Delta\left[2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right]$$

$$= \Delta\left[\frac{-2}{(x+2)(x+3)}\right] + \Delta\left[\frac{-3}{(x+3)(x+4)}\right]$$

$$= -2\left[\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right]$$

$$-3\left[\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right]$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

(*h*) We have

$$\Delta e^{(ax+b)i} = e^{(ax+b)i+ahi} - e^{(ax+b)i}$$
$$= (e^{ahi} - 1)\, e^{(ax+b)i}$$

Similarly $\Delta^2 e^{(ax+b)i} = (e^{ahi} - 1)^2 e^{(ax+b)i}$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta^n e^{(ax+b)i} = (e^{ahi} - 1)^n e^{(ax+b)i}$$

Now, Real part of $\Delta^n e^{(ax+b)i} = \Delta^n \cos(ax+b)$

$= \text{Real part of } (e^{ahi} - 1)^n e^{(ax+b)i}$

$= \text{Real part of } (\cos ah + i \sin ah - 1)^n (\cos(ax+b) + i\sin(ax+b))$

$$= \text{Real part of} \left(-2\sin^2 \frac{ah}{2} + 2i \sin \frac{ah}{2} \cos \frac{ah}{2}\right)^n (\cos(ax+b) + i\sin(ax+b))$$

$$= \text{Real part of} \left(2\sin\frac{ah}{2}\right)^n \left[-\sin\frac{ah}{2} + i\cos\frac{ah}{2}\right]^n (\cos(ax+b) + i\sin(ax+b))$$

$$= \text{Real part of} \left(2\sin\frac{ah}{2}\right)^n \left[\cos\left(\frac{ah+\pi}{2}\right) + i\sin\left(\frac{ah+\pi}{2}\right)\right]^n (\cos(ax+b) + i\sin(ax+b))$$

$$= \text{Real part of} \left(2\sin\frac{ah}{2}\right)^n \left[\cos\left\{(ax+b) + n\left(\frac{ah+\pi}{2}\right)\right\} + i\sin\left\{(ax+b) + n\left(\frac{ah+\pi}{2}\right)\right\}\right]$$

$$= \left(2\sin\frac{ah}{2}\right)^n \cos\left[(ax+b) + n\left(\frac{ah+\pi}{2}\right)\right]$$

(*i*)
$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$
$$= [(x+h)(x+h-h)(x+h-2h) \ldots\ldots (x+h-(n-1)h)] - [x\,.\,(x-h)(x-2h) \ldots\ldots (x-(n-1)h)]$$
$$= [(x+h)\,x(x-h) \ldots\ldots (x-(n-2)h)] - [x(x-h) \ldots\ldots (x-(n-2)h)(x-(n-1)h)]$$
$$= [x(x-h) \ldots\ldots (x-(n-2)h)]\,[x+h-x+(n-1)h]$$
$$= nh\,.\,x^{(n-1)}$$

Again,
$$\Delta^2 x^{(n)} = \Delta\,[\Delta x^{(n)}] = \Delta\,[nh\, x^{(n-1)}]$$
$$= nh\, \Delta x^{(n-1)}$$
$$= n(n-1)\, h^2\, x^{(n-2)}$$

Repeating the same process, we get

$$\Delta^n x^{(n)} = \lfloor n \; h^n$$

$$\Rightarrow \qquad \Delta^n x^{(n)} = \lfloor n \qquad [\because \; h = 1]$$

(iv)

(a) $$\left[\frac{(E-I)^2}{E}\right] x^3 = \frac{E^2-2E+1}{E} x^3$$
$$= (E - 2 + E^{-1})\, x^3$$
$$= (x+1)^3 - 2x^3 + (x-1)^3$$
$$= 6x$$

(b) $[(E-1)-(1-E^{-1})]\, x^2$
$$= [E + E^{-1} - 2]\, x^2$$
$$= [(x+h)^2 + (x-h)^2 - 2x^2]$$
$$= 2h^2$$

(c) $$Ee^x = e^{x+h} \qquad \text{...(i)}$$
$$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1) \qquad \text{...(ii)}$$
$$\Delta^2 e^x = e^x\,(e^h-1)^2 \qquad \text{...(iii)}$$

Again, $$\left(\frac{\Delta^2}{E}\right) e^x = (\Delta^2 E^{-1})\, e^x = \Delta^2\, [E^{-1} e^x]$$
$$= \Delta^2\, [e^{x-h}] = e^{-h}\, \Delta^2\, [e^x]$$
$$= e^{-h}\, e^x (e^h - 1)^2 \qquad \text{...(iv)}$$

So, $$= \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^{-h}\, e^x\,(e^h-1)^2 \cdot \frac{e^x e^h}{e^x (e^h-1)^2}$$
$$= e^x$$

(d) $(1 - E^{-1} + E - 1)^2\,(x^2 + x)\,;\, h = 1$
$$= (E^2 + E^{-2} - 2)\,(x^2 + x)$$
$$= (x+2)^2 + (x-2)^2 - 2x^2 + (x+2) + (x-2) - 2x = 8$$

(e) $$\frac{\nabla\,[\nabla x^2]}{E(x) + E\,(\log x)} = \frac{\nabla\,[x^2 - (x-1)^2]}{(x+1) + [\log\,(x+1)]}$$
$$= \frac{\nabla\,[2x-1)]}{[(x+1) + \log\,(x+1)]}$$
$$= \frac{(2x-1) - 2(x-1) + 1}{[(x+1) + \log\,(x+1)]}$$
$$= \frac{2}{[(x+1) + \log\,(x+1)]}$$

(v) Let $$\Delta f(x) = 9x^2 + 11x + 5$$
$$= 9x\,(x-1) + Ax + B$$

Putting $x = 0\,;\, B = 5$

Putting $x = 1\,;\, A = 20$

$\therefore$ $$\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5x^{(0)}$$

Integrating, we get

$$f(x) = \frac{9x^{(3)}}{3} + \frac{20x^{(2)}}{2} + 5x^{(1)} + Cx^{(0)}$$
$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + C$$
$$= 3x^3 + x^2 + x + C$$

where C is constant of integration.

Note: The coefficient of the highest power of x remains unchanged while transforming a polynomial to a factorial notation.

(*vi*) Let $Y = x^4 - 12x^3 + 42x^2 - 30x + 9$

$$= x(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + Dx + E$$

Putting $x = 0$; $E = 9$

Putting $x = 1$; $D = 1$

$x = 2$; $C = 13$

$x = 3$; $B = -6$

$\therefore$ $f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9x^{(0)}$

$\Rightarrow$ $\Delta f(x) = 4x^{(3)} - 18x^{(2)} + 26x^{(1)} + x^{(0)}$

$\Rightarrow$ $\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} + 26x^{(0)}$

$\Rightarrow$ $\Delta^3 f(x) = 24x^{(1)} - 36x^{(0)}$

$\Rightarrow$ $\Delta^4 f(x) = 24$

and $\frac{1}{\Delta}[f(x)] = \text{Integration of } f(x)$

$$f(x) = \frac{x^{(5)}}{5} - \frac{6x^{(4)}}{4} + \frac{13x^{(3)}}{3} + \frac{x^{(2)}}{2} + 9x^{(1)} + C_1 x^{(0)}$$

$$\Rightarrow \quad \frac{1}{\Delta}\left[\frac{1}{\Delta} f(x)\right] = \frac{x^{(6)}}{30} - \frac{6x^{(5)}}{20} + \frac{13x^{(4)}}{12} + \frac{x^{(3)}}{6} + \frac{9x^{(2)}}{2} + C_1 x^{(1)} + C_2 x^{(0)}$$

(*vii*) Factorial function $x^{(n)}$ for the interval of differencing h is defined as

$$x^{(n)} = x(x-h)(x-2h) \ldots\ldots (x-(n-1)h) \qquad \ldots(1)$$

From the definition of $x^{(n)}$, we also have

$$x^{(n)} = [x-(n-1)h]\, x^{(n-1)} \qquad \ldots(2)$$

Putting $n = 0$ in (2) and as $x^{(0)} = 1$, we get

$$1 = (x+h)\, x^{(-1)}$$

$$\Rightarrow \quad x^{(-1)} = \frac{1}{x+h}$$

Putting $n = -1$ in (2), we get

$$x^{(-1)} = (x+2h)\, x^{(-2)}$$

$$\Rightarrow \quad x^{(-2)} = \frac{1}{(x+2h)}\, x^{(-1)}$$

$$= \frac{1}{(x+2h)(x+h)}$$

Proceeding in the similar fashion, we get

$$x^{(-n)} = \frac{1}{(x+nh)\ldots\ldots(x+2h)(x+h)}$$

$$= \frac{1}{(x+nh)(x+nh-h)\ldots\ldots(x+h)}$$

$$= \frac{1}{(x+nh)^n}$$

(*viii*) We construct the following table

x	5	10	15	20	25	30
y	0	1	2	3	4	5
u_x	7	10	–	17	–	28

Since we are given four values so we assume u_x must be a polynomial of third degree, so that $\Delta^4 u_x = 0\ \forall\ x$.

$\because \quad \Delta^3 f(x) = \text{Constant.}$

Taking $\quad \Delta^4 u_x = (E-1)^4 u_x$

$$= (E^4 - 4E^3 + 6E^2 - 4E + 1)\, u_x$$

$$= E^4 u_x - 4E^3 u_x + 6E^2 u_x - 4E u_x + u_x \quad \ldots(i)$$

Putting $x = 0$, we get

$$\Delta^4 u_0 = E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 - 4E u_0 + u_0$$

$$= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0$$

$$0 = u_4 - 4(17) + 6u_2 - 4(10) + 7$$

$$u_4 + 6u_2 = 101 \quad \ldots(ii)$$

Putting $x = 1$ in (*i*), we get

$$\Delta^4 u_1 = E^4 u_1 - 4E^3 u_1 + 6E^2 u_1 - 4E_1 u_1 + u_1$$

$$= u_5 - 4u_4 + 6u_3 - 4u_2 + u_1$$

$$0 = 28 - 4u_4 + 6(17) - 4u_2 + 10$$

$$u_4 + u_2 = 35 \quad \ldots(iii)$$

Solving (2) and (3), we have

$$u_2 = 13.2 \text{ and } u_4 = 21.8.$$

i.e. The values of y for $x = 15$ and 25 are 13.2 and 21.8 or 13 and 22 respectively.

Alter. We construct the difference table from given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	7				
		3			
10	10		$y_2 - 13$		
		$y_2 - 10$		$40 - 3y_2$	
15	y_2		$27 - 2y_2$		$6y_2 + y_4 - 101$
		$17 - y_2$		$3y_2 + y_4 - 61$	
20	17		$y_2 + y_4 - 34$		$140 - 4y_2 - 4y_4$
		$y_4 - 17$		$79 - y_2 - 3y_4$	
25	y_4		$45 - 2y_4$		
		$28 - y_4$			
30	28				

Since $\Delta^4 y_0 = 0$

$\therefore$ $6y_2 + y_4 = 101$ and $y_2 + y_4 = 35$.

On solving these two equations we get $y_2 = 13.2$ and $y_4 = 21.8$

(ix) (a)
$$\text{LHS} = u_0 + x\frac{u_1}{1!} + x^2\frac{u_2}{2!} + x^3\frac{u_3}{3!} + \dots$$
$$= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \dots$$
$$= \left[1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \dots\right]u_0$$
$$= e^{xE}u_0 = e^{x(1+\Delta)}u_0 = e^x e^{x\Delta}u_0$$
$$= e^x\left(1 + \frac{x\Delta}{1!} + \frac{(x\Delta)^2}{2!} + \frac{(x\Delta)^3}{3!} + \dots\right)u_0$$
$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \dots\right)$$
$$= \text{RHS}$$

(b)
$$\text{LHS} = u_{x+n} = E^x u_n$$
$$= (1+\Delta)^x u_n$$
$$= (1 + {}^xC_1\Delta + {}^xC_2\Delta^2 + {}^xC_3\Delta^3 + \dots)u_n$$
$$= u_n + {}^xC_1\Delta u_{n-1} + {}^xC_2\Delta^2 u_{n-2} + \dots = \text{RHS}$$

(c)
$$\text{LHS} = \Delta^n u_x = (E-1)^n u_x$$
$$= [E^n - {}^nC_1E^{n-1} + {}^nC_2E^{n-2} + \dots + (-1)^n]u_x$$
$$= E^n u_x - {}^nC_1E^{n-1}u_x + {}^nC_2E^{n-2}u_x + \dots + (-1)^n u_x$$
$$= u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + \dots + (-1)^n u_x = \text{RHS}$$

EXERCISE 8.1

1. Prove the following identities with the usual notations :
   (a) $E^{-1} = I - \nabla$ [Raj. 2002]
   (b) $\Delta = E\nabla = \nabla E$
   (c) $(I+\Delta)(I-\nabla) = I$ [Raj. 2003]
   (d) $\nabla^r y_k = \Delta^r y_{k-r}$
   (e) $\nabla = I - (I+\Delta)^{-1}$
   (f) $\mu = \sqrt{1 + \dfrac{\delta^2}{4}}$
2. Prove the following :
   (a) $\Delta[f(x).g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$
   (b) $\delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\mu[g(x)]\,\delta[f(x) - \mu[f(x)]\,\delta[g(x)]}{g\left(x-\dfrac{h}{2}\right)g\left(x+\dfrac{h}{2}\right)}$ [Raj. 2003]

(c) $\mu\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\mu[f(x)]\,\mu[g(x)] - \frac{1}{4}\,\delta[f(x)]\,\delta[g(x)]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$ [Raj. 2001]

3. Evaluate the following considering 'h' as interval of differencing unless otherwise stated :

(a) $E^2(x^3)$; $h = 2$ (b) $E^n(e^x)$

(c) $\Delta^n\,[\log(ax)]$ (d) $\Delta\,[\sqrt{u_x}\,]$

(e) $\Delta^3\,(ax^3 + bx^2 + cx + d)$ (f) $\Delta(x + \cos x)$

(g) $\Delta^6\,(ax-1)\,(bx^2-1)\,(cx^3-1)$; $h = 1$ [Raj. 2003, 05]

(h) $\Delta\,[\cosh(ax+b)]$ (i) $\Delta(\lfloor x\,)$; $h = 1$

(j) $\Delta\left(\dfrac{x^2}{\cos 2x}\right)$ (k) $\Delta^n\,(e^{ax+b})$; $h = 1$

(l) $\Delta^n\,(ab^{cx})$; $h = 1$

4. Evaluate the following considering 'h' as interval of differencing unless otherwise stated :

(a) $\dfrac{\Delta^2 x^3}{Ex^3}$; $h = 1$ (b) $\Delta^2 E^3 x^2$; $h = 1$

(c) $(E+2)\,(E-1)\,(e^x + x)$; $h = 1$ (d) $\dfrac{\nabla^2 x^3}{\delta x^2}$; $h = 1$

(e) $\dfrac{\Delta^2}{E}\,[\sin(x+h)] + \dfrac{\Delta^2\,[\sin(x+h)]}{E\,[\sin(x+h)]}$ (f) $(\Delta+1)\,(2\Delta-1)\,(x+1)^2$

5. Find the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

6. Express $f(x) = x^3 - 3x + 1$ in factorial function and hence evaluate $\Delta^3\,[f(x)]$.

7. Express $f(x) = x^4 - 2x^3 + 4x^2 - 5x + 8$ in factorial function and hence evaluate $\Delta^2[f(x)]$.

8. Prove that $B(m+1, n) = (-1)^m\,\Delta^m\left(\dfrac{1}{n}\right)$. [Raj. 2006]

9. Find the first term of the following sequence 8, 3, 0, – 1, 0.

10. Find the missing term in the following table :

x :	1	2	3	4	5
y :	2	5	7	–	32

[Raj. 2004]

11. Prove the following ;

(a) $u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \dfrac{x}{1-x}\,u_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta u_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$

(b) $u_0 + u_1 + u_2 + \ldots + u_n = {}^{n+1}C_1\,u_0 + {}^{n+1}C_2\,\Delta u_0 + \ldots + \Delta^n u_n$

(c) $u_0 + {}^xC_1\,\Delta u_1 + {}^xC_2\,\Delta^2 u_2 + \ldots + \Delta^n u_x = u_x + {}^xC_1\,\Delta^2 u_{x-1} + {}^xC_2\,\Delta^4 u_{x-2} + \ldots$

(d) $u_0 + {}^nC_1\,u_1 x + {}^nC_2\,u_2 x^2 + \ldots + u_n x^n$
$= (1+x)^n\,u_0 + {}^nC_1\,(1+x)^{n-1}\,x\,\Delta u_0 + \ldots + x^n\,\Delta^n u_0$

(e) $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2\,u_{x-3} + \ldots + \Delta^{n-1}\,u_{x-n} + \Delta^n\,u_{x-n-1} + \ldots$

ANSWERS

3. (a) $(x+4)^3$ (b) e^{x+nh}

(c) $\log a(x+nh) - {}^nC_1 \log a(x+\overline{n-1}\,h) + \ldots$

(d) $\dfrac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}}$ (e) $6ah^3$

(f) $h - 2\sin\left(x + \dfrac{h}{2}\right)\sin\dfrac{h}{2}$ (g) $720\, abc$

(h) $-2\sinh\dfrac{ah}{2}\sinh\left(ax + b + \dfrac{ah}{2}\right)$ (i) $x\,\lfloor x$

(j) $\dfrac{h\,(2x+h)\cos 2x + 2x^2 \sin h \sin(2x+h)}{\cos 2(x+h)\cos 2x}$

(k) $(e^a - 1)^n e^{ax+b}$ (l) $(b^c - 1)^n ab^{cx}$

4. (a) $\dfrac{6}{(1+x)^2}$ (b) 2

(c) $e^x(e^2 + e - 2) + 3$ (d) $3\left(1 - \dfrac{1}{x}\right)$

(e) $2(\cos h - 1)\{\sin(x+h) + 1\}$ (f) $5h^2 + 2hx + 2h - x^2 - 2x - 1$

5. $\dfrac{1}{4}x^{(4)} + 2x^{(3)} + \dfrac{9}{2}x^{(2)} + 12x^{(1)} + cx^{(0)}$

6. $x^{(3)} + 3x^{(2)} - 2x^{(1)} + x^{(0)}$

7. $x^{(4)} + 4x^{(3)} + 5x^{(2)} - 2x^{(1)} + 8x^{(0)}$; $12x^{(2)} + 24x^{(1)} + 10x^{(0)}$

9. 15

10. 14

8.10 INTERPOLATION

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	x_2 ...... x_n
y :	y_0	y_1	y_2 ...... y_n

then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i)). In such cases, $f(x)$ is replaced by a similar function $g(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $g(x)$

which is known as the interpolating function or smoothing function. If $g(x)$ is a polynomial, then it is called the interpolating polynomial and the process is called the polynomial interpolation.

The study of interpolation is based on the calculus of finite differences. We begin by deriving two important interpolation formulae by means of forward and backward differences of a function. These formulae are often employed in engineering and scientific investigations.

8.11 NEWTON'S FORWARD INTEPOTATION FORMULA

[*Raj.* 2004]

Let the function $y = f(x)$ has the set of $(n + 1)$ values of y *i.e.* $y_0, y_1, y_2, \ldots\ldots,$ y_n corresponding to the values of x as $x_0, x_1 x_2, \ldots\ldots, x_n$ such that x is equi-spaced *i.e.* $x_i = x_0 + ih$ $(i = 0, 1, 2, 3 \ldots\ldots)$. Let $y_n(x)$ be a polynomial of n^{th} degree such that $y_n(x_i) = y_i$ and $y_n(x)$ is assumed as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots\ldots$$
$$+ a_n(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \qquad \ldots(8.12)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are $(n + 1)$ constants which are obtained by putting $x =$ $x_0, x_1, \ldots\ldots x_n$ sucessively in equation (8.12)

$$y_0 = a_0, y_1 = a_0 + a_1(x_1 - x_0),$$
$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \qquad \ldots(8.13)$$

and so on.

From these equations we find that

$$a_0 = y_0, \Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$$

or

$$a_1 = \frac{1}{h} \Delta Y_0$$

also,

$$\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$= a_1 h + a_2 . 2h . h = \Delta y_0 + 2h^2 a_2$$

or

$$a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{\underline{|2}\, h^2} \Delta^2 y_0$$

Similarly,
$$a_3 = \frac{1}{\underline{|3}\, h^3} \Delta^3 y_0 \text{ and so on.}$$

Substituting these values in equation (8.12), we get

$$y_n(x) = y_0 + \frac{1}{h} \Delta y_0 (x - x_0) + \frac{1}{\underline{|2}\, h^2} \Delta^2 y_0 (x - x_0)(x - x_1)$$
$$+ \ldots + \frac{1}{\underline{|n}\, h^n} \Delta^n y_0 (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \qquad \ldots(8.14)$$

By putting $x = x_0 + uh$, the above equation (8.14) becomes

$$y_n(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 y_0 + \ldots$$

$$+ \frac{u(u-1)(u-2)\ldots[u-(n-1)]}{\lfloor n} \Delta^n y_0 \qquad \ldots(8.15)$$

Remark: Above formula is used for intepolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (*i.e.* to the left) of y_0.

8.12 NEWTON'S BACKWARD INTERPOLATION FORMULA

Again let the function $y = f(x)$ has the set of $(n + 1)$ values of y *i.e.* $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values of x as $x_0, x_1, x_2, \ldots, x_n$ such that x is equispaced *i.e.* $x_i = x_0 + ih$ $(i = 0, 1, 2, 3, \ldots)$. Let $y_n(x)$ be a polynomial of n^{th} degree such that $y_n(x_i) = y_i$ and $y_n(x)$ is assumed as

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots$$
$$+ a_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \qquad \ldots(8.16)$$

where $a_0, a_1, a_2, \ldots, a_n$ are $(n + 1)$ constants which are obtained by putting $x = x_n, x_{n-1}, \ldots, x_0$ successively in eqn. (8.16) as

$$y_n = a_0\,;\; y_{n-1} = a_0 + a_1(x_{n-1} - x_n)\,;$$
$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) \qquad \ldots(8.17)$$

and so on.

From these equations we find that

$$a_0 = y_n\,;\; \nabla y_n = y_n - y_{n-1} = a_1(x_n - x_{n-1}) \text{ or } \frac{1}{h} \nabla y_n = a_1$$

Also,
$$\nabla y_{n-1} = y_{n-1} - y_{n-2}$$
$$= a_1(x_{n-1} - x_{n-2}) - a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$
$$= a_1 h - a_2(-2h)\,.\,(-h) = \nabla y_n - 2h^2 a_2$$

or
$$a_2 = \frac{1}{2h^2} (\nabla y_n - \nabla y_{n-1}) = \frac{1}{\lfloor 2\, h^2} \nabla^2 y_n$$

Similarly,
$$a_3 = \frac{1}{\lfloor 3\, h^3} \nabla^3 y_n \text{ and so on.}$$

Substituting these values in equation (8.16), we get

$$y_n(x) = y_n + \frac{1}{h} \nabla y_n (x - x_n) + \frac{1}{\lfloor 2\, h^2} \nabla^2 y_n (x - x_n)(x - x_{n-1})$$

$$+ \ldots + \frac{1}{\lfloor n\, h^n} \nabla^n y_n (x - x_n)(x - x_{n-1})(x - x_{n-2}) \ldots (x - x_1) \qquad \ldots(8.18)$$

By putting $x = x_n + uh$, the above equation (8.18) becomes

$$y_n (x_n + uh) = y_n + u\nabla y_n + \frac{u(u+1)}{\lfloor 2}\nabla^2 y_n + \ldots$$

$$+ \frac{u(u+1)\ldots(u+n-1)}{\lfloor n}\nabla^n y_n \qquad \ldots(8.19)$$

Remark: Above formula is used for interpolating the values of y near the end of a set of tabulated values and extrapolating values of y at a little forward (*i.e.* to the right) of y_n.

ILLUSTRATIVE EXAMPLES

2. (i) *If the area of a circle of diameter d is given for the following values of d*

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

then find the approximate value of the area of circle of diameter d = 82. [Raj. 2006]

(ii) *Use Newton's forward interpolation formula to find the value of $\log_{10} 3.1416$, given that $\log_{10} 3.141 = 0.49706794$, $\log_{10} 3.142 = 0.49720618$, $\log_{10} 3.143 = .49734438$, $\log_{10} 3.144 = 0.49748282$, $\log_{10} 3.145 = 0.49762065$ (correct upto six places of decimal).*

(iii) *If the number of alive people at the age of x years is represent by l_x in a life table as follows :*

x :	20	30	40	50
l_x :	512	439	346	243

then find l_x at $x = 47$.

(iv) *Use the following table to find the value of f(1963) :*

x :	1921	1931	1941	1951	1961
f(x)	19.96	39.65	58.81	77.21	94.61

[Raj. 1993, 2002, MNIT 2003]

(v) *Use the following table to find y at x = 25°*

x :	10°	20°	30°	40°	50°	60°	70°	80°
y :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

[Raj. 2003]

(vi) *Use the following table to find the number of workers getting wages between Rs. 10 and 15 :*

Wages (x) :	0 – 10	10 – 20	20 – 30	30 – 40
No. of Workers (y) :	9	30	35	42

(vii) *Use the following table to find the polynomial function y = f(x)*

x :	0	1	2	3	4
y :	0	7	26	63	124

(viii) *Use Newton's forward interpolation formula to find a second degree polynomial function passes through (0, 1,), (1, 3), (2, 7) and (3, 13).*

(ix) *If in the following table, value of y corresponding to $x = 3$ found with an error :*

x	0	1	2	3	4	5	6
y	0	3	14	45	84	155	258

then find the correct value of y at $x = 3$ using the fact that $y = f(x)$ is a polynomial function of degree 5.

Solution. (*i*) We construct the forward difference table as

d	A	ΔA	$\Delta^2 A$	$\Delta^3 A$	$\Delta^4 A$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

We know that the Newton's forward interpolation formula is

$$A(x) = A(x_0) + u\Delta A(x_0) + \frac{u(u-1)}{2!}\Delta^2 A(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 A(x_0) + \ldots$$

where $x = x_0 + uh$

For $x = 82$ $\qquad u = \dfrac{82-80}{5} = 0.4$

$$\therefore \quad A(82) = 5026 + (0.4)(648) + \frac{(.04)(0.4-1)}{2!} 40 + (0.4)(.4-1)\frac{(0.4-2)}{3!}(-2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(4)$$

$$= 5280.1056 \simeq 5280.$$

(*ii*) We construct the forward difference table for $y = \log_{10} x$ as

x :	3.141	3.142	3.143	3.144	3.145
y :	0.49706794	0.49720618	0.49734438	0.49748282	0.49762065

x	$y = \log_{10} x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3.141	0.49706794				
		0.00013824			
3.142	0.49720618		-0.0000004		
		0.0001382		0.00000064	
3.143	0.49734438		0.00000024		-0.00000151
		0.00013844		-0.00000087	
3.144	0.49748282		-0.00000063		
		0.00013781			
3.145	0.49762065				

We know that Newton's forward interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \ldots$$

where
$$u = \frac{x - x_0}{h}$$

$$= \frac{3.1416 - 3.141}{.001} = .6$$

$\therefore$
$$y(3.1416) = (0.49706794) + (0.6)(0.00013824)$$

$$+ \frac{(0.6)(0.6-1)}{2!}(-0.0000004) + \frac{(0.6)(0.6-1)(0.6-2)}{3!}(0.00000064)$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)(0.6-3)}{4!}(-0.00000151)$$

$$= 0.49715094 \text{ (approx.)}$$

(*iii*) We construct the backward difference table as

x	l_x	∇l_x	$\nabla^2 l_x$	$\nabla^3 l_x$
20	512			
		−73		
30	439		−20	
		−93		10
40	346		−10	
		−103		
50	243			

Applying Newton's backward interpolation formula

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \ldots$$

where
$$u = \frac{x - x_0}{h}$$

$$= \frac{47 - 50}{10} = -0.3$$

$$y(47) = 243 + (-0.3)(-103) + \frac{(-0.3)(-0.3+1)}{2!}(-10)$$

$$+ (-0.3)(-0.3+1) \cdot \frac{(-0.3+2)}{3!}(10)$$

$$= 274.355 \simeq 274 \text{ (approx.)}$$

(*iv*) We construct the backward difference table as

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1921	19.96				
		19.69			
1931	39.65		− 0.53		
		19.16		− 0.23	
1941	58.81		− 0.76		− 0.01
		18.40		− 0.24	
1951	77.21		− 1.00		
		17.40			
1961	94.61				

Using Newton's backward interpolation formula *i.e.*

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \ldots$$

where

$$u = \frac{x - x_0}{h} = \frac{1963 - 1961}{10} = .2$$

$$f(1963) = 94.61 + (0.2)(17.40) + \frac{(0.2)(1.2)}{2!}(-1.0)$$

$$+ \frac{(0.2)(1.2)(2.2)}{3!} \cdot (-0.24) + \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(-.01)$$

$$= 97.9481 \text{ (approx.)}$$

(*v*) We construct the forward and backward difference table as

x	y	Δy or ∇y	$\Delta^2 y$ or $\nabla^2 y$	$\Delta^3 y$ or $\nabla^3 y$	$\Delta^4 y$ or $\nabla^4 y$	$\Delta^5 y$ or $\nabla^5 y$	$\Delta^6 y$ or $\nabla^6 y$	$\Delta^7 y$ or $\nabla^7 y$
10	0.9848							
		– 0.0451						
20	0.9397		– 0.0286	0.0023				
		– 0.0737						
30	0.8660		– 0.0263		0.0008			
		– 0.1000		0.0031		– 0.0003		
40	0.7660		– 0.0232		0.0005		0.0006	
		– 0.1232		0.0036		0.0003		0.0012
50	0.6428		– 0.0196		0.0008		– 0.0006	
		– 0.1428		0.0044		– 0.0003		
60	0.5000		– 0.0152		0.0005			
		– 0.158		0.0049				
70	0.3420		– 0.0103					
		– 0.1683						
80	0.1737							

We know that the Newton's forward difference formula *i.e.*

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

where $$u = \frac{x - x_0}{h} = \frac{25-20}{10} = 0.5$$

$$\therefore \quad y(25) = 0.9397 + (0.5)(-0.0737) + \frac{(0.5)(-0.5)}{2!}(-0.0263) + \frac{(0.5)(-0.5)(-1.5)}{3!}(.0031)$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}(0.0005) + \frac{(0.5)(-0.5)(-1.5)(-2.5)(-3.5)(0.0003)}{5!} + \frac{(0.5)(-0.5)(-1.5)(-2.5)(-3.5)(-4.5)}{6!}(-0.0006)$$

$$= 0.90633 \text{ (approx.)}$$

(*vi*) We construct the forward difference table as

x (*Wages*)	y (*No. of workers*)	Δy	$\Delta^2 y$	$\Delta^3 y$
10	9			
		21		
20	30		–16	
		5		18
30	35		2	
		7		
40	42			

Here $$u = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$$

$$\therefore \quad y(10-15) = 9 + (0.5)(21) + \frac{(0.5)(-0.5)}{2!}(-16) + \frac{(0.5)(-0.5)(-1.5)}{3!}(18)$$

$$= 22.625 \text{ (approx.)}$$

(*vii*) We construct the forward difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		
		61			
4	124				

In the polynomial form,

$$y = P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots \qquad \dots(i)$$

We know that by Newton's forward difference formula

$$a_n = \frac{1}{n!\,h^n}\,\Delta^n y_0, \text{ we get}$$

$$a_0 = 0,\ a_1 = 7,\ a_2 = 6,\ a_3 = 1$$

Putting in (i) equation becomes

$$y = 7x + 6x^2 - 6x + x^3 - 3x^2 + 2x$$
$$= x^3 + 3x^2 + 3x$$

$(viii)$ We construct the forward difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
				0
		4		
2	7		2	
		6		
3	13			

We know that

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots$$

and

$$a_n = \frac{1}{n!\,h^n}\,\Delta^n y_0$$

$\therefore$

$$a_0 = 1,\ a_1 = 2,\ a_2 = 1$$
$$y = 1 + 2(x - 0) + 1(x - 0)(x - 1)$$
$$= x^2 + x + 1$$

(ix) We construct the forward difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	0						
		3					
1	3		8				
		11		$12 + \epsilon$			
2	14		$20 + \epsilon$		$-24 - 4\epsilon$		
		$31 + \epsilon$		$-12 - 3\epsilon$		$60 + 10\epsilon$	
3	$45 + \epsilon$		$8 - 2\epsilon$		$36 + 6\epsilon$		$-120 - 20\epsilon$
		$39 - \epsilon$		$24 + 3\epsilon$		$-60 - 10\epsilon$	
4	84		$32 + \epsilon$		$-24 - 4\epsilon$		
		71		$-\epsilon$			
5	155		32				
		103					
6	258						

Here we assume ϵ as error in the given value of y_3 *i.e.* $y_3 = 45 + \epsilon$ be the correct value of y_3. Since, it is given that y is a polynomial of degree 5 so $\Delta^5 y$ = constant or $\Delta^5 y$ should be same value everywhere.

$$\Rightarrow \quad 60 + 10\epsilon = -60 - 10\epsilon$$

$$\Rightarrow \quad \epsilon = -6$$

$$\therefore \quad y_3 = 45 - 6 = 39.$$

EXERCISE 8.2

1. Use following table to find the value of $f(3.62)$

x:	3.60	3.65	3.70	3.75
$f(x)$:	36.598	38.475	40.447	42.521

2. If the index number of 4 year is given as

Year (x):	1981	1983	1985	1987
Index number (y):	100	125	137	151

then find the index number for the year 1982.

3. Use Newton's forward interpolation formula to find the value of sin (0.1616) from the following table :

x:	0.160	0.161	0.162
$\sin x$:	0.15931821	0.16030535	0.16129234

4. If the population of a city is given as :

Year (x):	1891	1901	1911	1921	1931
Population (in thousand) (y):	46	66	81	93	101

then find the population for the year 1925. [Raj. 2005]

5. If the ordinates of the normal curve are given as :

x:	0.0	0.2	0.4	0.6	0.8
$f(x)$:	0.3989	0.3910	0.3683	0.3332	0.2897

then find the values of $f(0.25)$ and $f(0.62)$.

[Raj. 1993, 2001, 06, MREC (Auto) 2002, MNIT 2003]

6. Use following table to find the values of $f(0.23)$ and $f(0.29)$:

x:	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

7. Use Newton's backward interpolation formula to find the increase in the population of city during the period 1946 to 1948 from the following table :

Year (x):	1911	1921	1931	1941	1951	1961
Population (in thousands) (y):	12	15	20	27	39	52

8. If in an examination the number of students who obtained marks between certain limits are given as

Marks obtained :	0 – 19	20 – 39	40 – 59	60 – 79	80 – 99
No. of students :	41	62	65	50	17

then estimate the number of students who obtained less than 70 marks.

9. Use following table to find the number of students who obtained :
   (a) marks between 40 and 45
   (b) more than 45 marks.

Marks obtained :	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students :	31	42	51	35	31

10. Use following table to find the polynomial function $y = f(x)$:

x :	0	1	2	3	4	5
y :	0	3	8	15	24	35

11. Use following function to find the polynomial function $y = f(x)$:

x :	0	1	2	3
y :	1	0	1	10

Hence or otherwise find the value of $f(4)$.

12. Use newton's forward interpolation formula to find a third degree polynomial function passes through points $(0, -1)$, $(1, 1)$, $(2, 1)$ and $(3, -2)$.

13. If in the following table, one value of y is found with an error

x :	0	1	2	3	4	5	6	7
y :	25	21	18	18	27	45	76	123

then identify the incorrect value of y and correct it using the fact that $y = f(x)$ is a polynomial function of degree 3.

ANSWERS

1. 37.337
2. 115.062
3. 0.160700164
4. 96.6352
5. 0.3867, 0.32954
6. 1.6751, 1.7083
7. 2.706 (thousands)
8. 198.50
9. (a) 17 (b) 142
10. $y = x^2 + 2x$
11. $y = x^3 - 2x^2 + 1,\ f(4) = 33$
12. $y = -\dfrac{1}{6}x^3 - \dfrac{1}{2}x^2 + \dfrac{8x}{3} - 1$
13. y corresponding to $x = 3$ is incorrect and its correct value is 19.

8.13 STIRLING'S CENTRAL DIFFERENCE INTERPOLATION FORMULA FOR EQUAL INTERVALS

[Raj. 2004]

Stirling's interpolation formula is obtained by using the Newton's Forward interpolation formula as follows :

Newton's forward interpolation formula is given by

$$y_n = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots(8.20)$$

Now, we obtain the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ in terms of central difference as

$$\Delta y_0 = y_1 - y_0 = \delta y_{1/2}$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= \delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

Similarly $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = \delta^3 y_{3/2}$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ in equation (8.20), we get

$$y_n = y_0 + u\delta y_{1/2} + \frac{u(u-1)}{2!}\delta^2 y_1 + \frac{u(u-1)(u-2)}{3!}\delta^3 y_{3/2} + \ldots \quad \ldots(8.21)$$

In order to prove the interpolation formula, we introduce average operator μ for two adjoining values. Thus, we have

$$\mu\delta y_0 = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2})$$

$$\mu\delta^2 y_{1/2} = \frac{1}{2}(\delta^2 y_0 + \delta^2 y_1)$$

$$\mu\delta^3 y_0 = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2})$$

$$\mu\delta^4 y_{1/2} = \frac{1}{2}(\delta^4 y_0 + \delta^4 y_1)$$

$$\mu\delta^5 y_0 = \frac{1}{2}(\delta^5 y_{1/2} + \delta^5 y_{-1/2})$$

and $\delta y_{1/2}$, $\delta^2 y_1$ and $\delta^3 y_{3/2}$ can be obtained as

$$\delta y_{1/2} = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) + \frac{1}{2}(\delta y_{1/2} - \delta y_{-1/2})$$

$$= \mu\delta y_0 + \frac{1}{2}\delta^2 y_0$$

and

$$\delta^2 y_1 = \delta^2 y_1 - \delta^2 y_0 + \delta^2 y_0$$

$$= \delta^3 y_{1/2} + \delta^2 y_0$$

But,

$$\delta^3 y_{1/2} = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) + \frac{1}{2}(\delta^3 y_{1/2} - \delta^3 y_{-1/2})$$

$$= \mu\delta^3 y_0 + \frac{1}{2}\delta^4 y_0$$

$\therefore$

$$\delta^2 y_1 = \mu\delta^3 y_0 + \frac{1}{2}\delta^4 y_0 + \delta^2 y_0$$

Now,

$$\delta^3 y_{3/2} = \delta^3 y_{1/2} + \delta^3 y_{3/2} - \delta^3 y_{1/2}$$

$$= \delta^3 y_{1/2} + \delta^4 y_1$$

$$= \mu\delta^3 y_0 + \frac{1}{2}\delta^4 y_0 + \delta^4 y_1$$

Substituting all values in eqn. (8.21), we get

$$y_n = y_0 + u\left(\mu\delta y_0 + \frac{1}{2}\delta^2 y_0\right) + \frac{u(u-1)}{2!}\left[\mu\delta^3 y_0 + \frac{1}{2}\delta^4 y_0 + \delta^2 y_0\right]$$
$$+ \frac{u(u-1)(u-2)}{3!}\left[\mu\delta^3 y_0 + \frac{1}{2}\delta^4 y_0 + \delta^4 y_1\right] + \ldots$$

or $$y_n(x_0 + uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{2!}(\delta^2 y_0) + \frac{u(u^2-1^2)}{3!}(\mu\delta^3 y_0)$$
$$+ \frac{u^2(u^2-1^2)}{4!}(\delta^4 y_0) + \frac{u(u^2-1^2)(u^2-2^2)}{5!}(\mu\delta^5 y_0) + \ldots \quad (8.22)$$

Equation (8.22) represents Stirling's central difference interpolation formula.

Remark: We know that

$$\Delta y_0 = y_1 - y_0,$$
$$\nabla y_1 = y_1 - y_0$$

and $$\delta y_{1/2} = y_1 - y_0$$

Hence, we conclude that construction of difference table is same as we discussed earlier only notations are changed.

Illustrative Examples

3. (i) *Use Striling's central interpolation formula to find the value of $\log_{10} 337.5$ from the following table :*

x :	310	320	330	340	350	360
$y = \log_{10} x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

(ii) *Use stirling's central difference interpolation formula to find the value of y_{28}, given that*

$y_{20} = 49225, y_{25} = 48316, y_{30} = 47236, y_{35} = 45926$

and $y_{40} = 44306$. [Raj. 2001, 02, 04]

(iii) *Use the following table to find the value of $u_{12.2}$*

x :	10	11	12	13	14
$y = 10^5 u_x$:	23967	28060	31788	35209	38368

[Raj. 2001, 03; RTU 2009]

(iv) *Use stirling's central difference interpolation formula to find the polynomial function $y = f(x)$ from the following table :*

x :	1	2	3	4	5
y :	– 1	1	31	129	351

Solution. (*i*) We construct the central difference table as

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
310	2.4914					
		0.0138				
320	2.5052		−0.0005			
		0.0133		0.0002		
330	2.5185		−0.0003		−0.0003	
		0.0130		−0.0001		0.0004
340	2.5315		−0.0004		0.0001	
		0.0126		0.0000		
350	2.5441		−0.0004			
		0.0122				
360	2.5563					

Now, using stirling's central difference formula as

$$y(x_0 + uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{\lfloor 2}(\delta^2 y_0) + \frac{u(u^2-1^2)}{\lfloor 3}(\mu\delta^3 y_0)$$

$$+ \frac{u^2(u^2-1^2)}{\lfloor 4}(\delta^4 y_0) + \ldots$$

where $$u = \frac{337.5 - 335}{10} = 0.25$$

So, $$y(337.5) = 2.525 + (0.25)(0.0130) + \frac{(0.25)^2}{\lfloor 2}(-0.00035)$$

$$+ \frac{(0.25)(0.25^2 - 1^2)}{\lfloor 3}(0.0001)$$

$$+ \frac{(0.25)^2(0.25^2-1)}{\lfloor 4}(-0.0001) + \frac{(0.25)(0.25^2-1^2)(0.25^2-2^2)}{\lfloor 5}(0.0004)$$

$$= 2.525 + 0.00325 - 0.000010937 + 0.000003906$$

$$+ 0.000000244 + 0.000003076$$

$$= 2.52824 \text{ (approx.)}$$

(*ii*) We construct the central difference table as

x	y	δy	δy	$\delta^2 y$	$\delta^3 y$
20	49225				
		−909			
25	48316		−171		
		−1080		−59	
30	47236		−230		−21
		−1310		−80	
35	45926		−310		
		−1620			
40	44306				

Now, using stirling's central difference formula as

$$y\,(x_0 + uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{2!}\,(\delta^2 y_0) + \ldots$$

where
$$u = \frac{x - x_0}{h} = \frac{28 - 30}{5} = -\,0.4$$

So,
$$y(28) = 47236 + (-\,0.4)\,(-\,1195) + \frac{(-\,0.4)^2}{2!}\,(-\,230)$$

$$+\ \frac{(-\,0.4)\,((-\,0.4)^2 - 1)}{3!}\,.\,(-69.5) + \frac{(-\,0.4)^2\,((-\,0.4)^2 - 1^2)}{4!}\,(-\,21)$$

$$= 47236 + 478 - 18.4 + 3.892 + 0.1176$$

$$= 47691.8256$$

(*iii*) We construct the central difference table as

x	$y = 10^5 u_x$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
10	23967				
		4093			
11	28060		−365		
		3728		58	
12	31788		−307		−13
		3421		45	
13	35209		−262		
		3159			
14	38368				

Now, using Stirling's central difference formula as

$$y(x_0 + uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{2!}(\delta^2 y_0) + \frac{u(u^2 - 1^2)}{3!}(\mu\delta^3 y_0) + \ldots$$

where $$u = \frac{12.2 - 12}{1} = 0.2$$

So, $$y(12.2) = 31788 + (0.2)(3574.5) + \frac{(0.2)^2}{2!}(-307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1^2]}{3!} . (51.5) + \frac{(0.2)^2[(0.2)^2 - 1^2]}{4!}(-13)$$

$$= 32495.133 \text{ (approx.)} \quad \text{So,} \quad u_{12.2} = 0.3250 \text{ (approx.)}$$

(*iv*) We construct the central difference table as

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
1	– 1				
		2			
2	1		28		
		30		40	
3	31		68		16
		98		56	
4	129		124		
		222			
5	351				

By using Stirling's central difference formula as

$$y(x_0 + uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{2!}(\delta^2 y_0) + \frac{u(u^2 - 1^2)}{3!}(\mu\delta^3 y_0) + \ldots$$

where $$u = \frac{x - 3}{1} = x - 3$$

So, $$y(x) = 31 + (x - 3)(64) + \frac{(x - 3)^2}{2!}(68)$$

$$+ \frac{(x - 3)[(x - 3)^2 - 1^2]}{3!}(48) + \frac{(x - 3)^2[(x - 3)^2 - 1^2]}{4!}(16)$$

$$= 31 + (64x - 192) + (34x^2 - 204x + 306)$$

$$+ (8x^3 - 72x^2 + 216x - 216 - 8x - 24)$$

$$+ \frac{2}{3}(x^4 - 12x^3 + 54x^2 - 108x + 81 - x^2 + 6x - 9)$$

$$= \frac{2}{3}x^4 - \frac{8}{3}x^2 + 1 = \frac{1}{3}[2x^4 - 8x^2 + 3]$$

EXERCISE 8.3

1. If the area A of a circle of diameter d is given for the following values of d :

d:	80	85	90	95	100
A:	5062	5674	6362	7088	7854

then find approximate value of the area of circle of diameter 91. [Raj. 2006]

2. If the state of the firm for the last five years are given as :

Year	:	1946	1948	1950	1952	1954
Sales (in thousands)	:	40,	43	48	52	57

then estimate the sales for the year 1951. [MNIT 2004]

3. Use Stirling's formula to find the value of y_{35}, given that $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$ and $y_{50} = 243$.

4. Use Stirling's central difference interpolation formula to find the value of tan 16° from the following table :

x	:	0	5	10	15	20	25	30
$\tan x$	:	0.0000	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

5. Use the following table to find the value of u_{32}

x:	20	25	30	35	40	45
u_x:	14.035	13.674	13.275	12.734	12.089	11.309

[Raj. 2004, 05]

6. Use the following table to find the value of $f(1936)$

x:	1921	1931	1941	1951	1961
$f(x)$:	19.96	39.65	58.81	77.21	94.61

[Raj. 1993, MNIT 2003]

7. Use Stirling's central difference interpolation formula to find the value of $\sqrt{22.2}$, given that $\sqrt{20} = 4.472$, $\sqrt{21} = 4.583$, $\sqrt{22} = 4.690$, $\sqrt{23} = 4.796$ and $\sqrt{24} = 4.899$.

8. Use Stirling's central difference interpolation formula to find the polynomial function $y = f(x)$ from the following table

x:	1	2	3
y:	43	48	52

ANSWERS

1. 6504.672
2. 50.117
3. 395
4. 0.2867
5. 13.062
6. 49.30
7. 4.711
8. $y = \frac{1}{2}(-x^2 + 13x + 74)$

8.14 LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS

The various interpolation formula derived so for possess the disadvantage of being applicable only to equally intervaled values of the argument. It is, therefore, desirable to develop interpolation formula for unequally spaced values of x. Now we shall study such formula.

If $y = f(x)$ is a function of degree n such that $y_0, y_1, y_2, \ldots\ldots, y_n$ be the $(n + 1)$ values of the function corresponding to the argument $x = x_0, x_1, x_2, \ldots\ldots, x_n$ which are not equally intervaled, then the function $f(x)$ at any arbitrary point x may be written as

$$\begin{aligned} f(x) = {} & A_0(x - x_1)(x - x_2) \ldots\ldots (x - x_n) \\ & + A_1(x - x_0)(x - x_2) \ldots\ldots (x - x_n) \\ & + A_n(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \qquad \ldots(8.23) \end{aligned}$$

where $A_0, A_1, A_2, \ldots\ldots, A_n$ are $(n + 1)$ constants to be determined subject to the condition $f(x_i) = y_i$. Putting $x = x_0$ in equation (8.23), we get

$$f(x_0) = y_0 = A_0(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)$$

$$\Rightarrow \qquad A_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)}$$

Similarly by putting $x = x_1, x_2, \ldots\ldots, x_n$ in equation (8.23), we get

$$A_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)}$$

$$\vdots$$

$$A_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})}$$

Substituting the values of all constants $A_0, A_1, A_2, \ldots\ldots, A_n$ in equation (8.23), we get

$$\begin{aligned} f(x) = {} & \frac{(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)}\, y_0 \\ & + \frac{(x - x_0)(x - x_2)\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)}\, y_1 \\ & + \ldots\ldots + \frac{(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\ldots\ldots(x_n - x_{n-1})}\, y_n \qquad \ldots(8.24) \end{aligned}$$

The interpolation function denoted by eqn. (8.24) is known as Laugrauge's interpolation formula for unequal intervals.

8.15 LAGRANGE'S INVERSE INTERPOLATION FORMULA

So far, given a set of values of x and y, we have been finding the value of y corresponding to a certain value of x. On the otherhand, the process of estimating the value of x for a value of y (which is not in the table) is called inverse interpolation.

This procedure is similar to Lagrange's interpolation formula for unequal intervals, the only difference being that x is assumed to be expressible as function of y. *i.e.* if we interchange the role of x and y in Lagrange's interpolation formula (8.24), we get

$$f(y) = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}\, x_0 + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}\, x_1$$

$$+ \ldots\ldots + \frac{(y-y_0)(y-y_1)(y-y_2)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\ldots\ldots(y_n-y_{n-1})}\, x_n \qquad \ldots(8.25)$$

which is known as Langrauge's inverse interpolation formula.

ILLUSTRATIVE EXAMPLES

4. *(i) Use following table to find the value of y at x = 10*

x :	5	6	9	11
y :	12	13	14	16

(Raj. 2001, 06)

(ii) Use Lagrange's interpolation formula to find the polynomial function $y = f(x)$ from the following table

x :	0	1	4	5
y :	4	3	24	39

(Raj. 2001)

(iii) Use Langrange's interpolation formula to find the polynomial function $y = f(x)$ and hence value of y at $x = 6$ from the following table :

x :	3	7	9	10
y :	168	120	72	63

(Raj. 2003)

(iv) Use Langrange's inverse interpolation formula to find the value of x at $y = 7$ from the following table.

x :	1	3	4
y :	4	12	19

(Raj. 2003)

(v) Use Langrange's inverse interpolation formula to find a root of equation $f(x) = 0$; given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$ and $f(42) = 18$.

Solution. (*i*) Using Langrange's interpolation formula, we get

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\, y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

at $x = 10$, $y = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)$

$$+ \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = 14.66.$$

(*ii*) Using Langrange's interpolation formula, we get

$$y = f(x) = \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(4) + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24) + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$

$$= -\frac{1}{5}(x^3 - 10x^2 + 29x - 20)$$

$$+ \frac{1}{4}(x^3 - 9x^2 + 20x) - 2(x^3 - 6x^2 + 5x)$$

$$+ \frac{39}{20}(x^3 - 5x^2 + 4x)$$

$$= 2x^2 - 3x + 4.$$

(*iii*) Using Langranges interpolation formula, we get

$$y = f(x) = \frac{(x-7)(x-9)(x-10)}{(3-7)(3-9)(3-10)}(168)$$

$$+ \frac{(x-3)(x-9)(x-10)}{(7-3)(7-9)(7-10)}(120)$$

$$+ \frac{(x-3)(x-7)(x-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(x-3)(x-7)(x-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= -(x^3 - 26x^2 + 223x - 630) + 5(x^3 - 22x^2 + 147x - 270)$$

$$- 6(x^3 - 20x^2 + 121x - 210) + 3(x^3 - 19x^2 + 111x - 189)$$

$$= x^3 - 21x^2 + 119x - 27$$

at $x = 6$, $y = f(6) = 147$.

(*iv*) Using Langrange's inverse interpolation formula, we get

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}(x_0) + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}(x_1)$$

$$+ \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}(x_2)$$

at $y = 7$,

$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7} = 1.8571$$

(*v*) Using Langrange's inverse interpolation formula, we get one root of equation $f(x) = 0$ as

$$x = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}(34) + \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$= -0.78208 + 6.5323 + 33.6818 - 2.20161$$

$$= 37.2304$$

EXERCISE 8.4

1. Use following table to find the value of y at $x = 5$

x :	1	2	3	4	7
y :	2	4	8	16	128

(Raj. 2002, 04, 05)

2. Use following table to find the value of $f(4)$

x :	0	2	3	6
$f(x)$:	− 4	2	14	158

3. Use Lagrange's interpolation formula to find the value of y at $x = 2$ from the following table

x :	0	1	3	4
y :	5	6	50	105

(Raj. 2002)

4. Use Lagrange's interpolation formula to find the polynomial function $y = f(x)$ from the following table

x :	3	2	1	− 1
y :	3	12	15	− 21

5. Use Lagrange's interpolation formula to find the polynomial function $y = f(x)$ from the following table

x :	0	2	3	6
y :	659	705	729	804

6. Use Lagrange's interpolation formula to find the polynomial function $y = f(x)$ and hence value of y at $x = 3$ from the following table

x :	0	1	2	4	5
y :	0	16	48	88	0

(Raj. 2003)

7. Use Lagrange's inverse interpolation formula to find the value of x at $f(x) = 19$ from the following table

x :	0	1	2
$f(x)$:	0	1	20

(Raj. 2001)

8. Use Lagrange's inverse interpolation formula to find the value of x at $f(x) = 1300$ from the following table

x :	44	45	46	47
$f(x)$:	1340	1316	1293	1268

9. Use Lagrange's inverse interpolation formula to find the value of x at $y = 163$ from the following table

x :	80	82	84	86	88
y :	134	154	176	200	227

10. Use Lagrange's inverse interpolation formula to find the value of x at $y = 85$ from the following table

x :	2	5	8	14
y :	94.8	87.9	81.3	68.7

ANSWERS

1. 32.93
2. 40
3. 19
4. $y = x^3 - 9x^2 + 17x + 6$
5. $y = \frac{1}{72}(-x^3 + 29x^2 + 1602x + 47448)$
6. $y = -x^4 + 4x^3 + 3x^2 + 10x$, 84
7. 2.8
8. 45.69
9. 82.83
10. 6.30

MORE PROBLEMS ON FINITE DIFFERENCES AND INTERPOLATION

1. Prove that $\delta^n = \Delta^n E^{-n/2} = \nabla^n E^{n/2}$
2. Prove the following :

(i) $\delta = 2 \sin h \left(\frac{hD}{2}\right)$ (ii) $\mu = \cos h \left(\frac{hD}{2}\right)$ (iii) $\mu = E^{-1/2} + \frac{\delta}{2}$

3. Evaluate $\left(\frac{\Delta}{E}\right)^2 [f(x)]$, where h is the interval of differencing
4. Form a difference table for function $f(x) = x^3 - 3x^2 + 5x + 7$ with $x = 0, 2, 4, 6, 8$ and extend the table for the calculation of $f(10)$.

5. Use following table to find $\log_{10} 102$

x :	100	101	103	104
$y = \log_{10} x$:	2.0000	2.0043	2.0128	2.0170

6. Use following table to find value of p

x :	1	2	3	4	5
y :	7	p	13	21	37

7. If $u_0 = 580$, $u_1 = 556$, $u_2 = 520$ and $u_4 = 384$, then find u_3.

8. Find the missing terms in the following table

x :	1	2	3	4	5	6	7	8
y :	1	8	–	64	–	216	343	512

and hence find $f(1.5)$ and $f(7.5)$ (Raj. 2004, 06)

9. Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in factorial function form and hence find $\Delta^3 f(x)$.

10. Use following table to find the polynomial function $y = f(x)$

x :	0	1	2	3	4
y :	3	6	11	18	27

11. The following table provides the values of a polynomial $y = f(x)$ of degree 3. If one value of y is incorrect, then identify and correct it.

x :	0	1	2	3	4	5	6
y :	14	20	40	85	170	304	500

12. Apply Newton's forward interpolation formula to find the value of $\sqrt{5.5}$, given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$ and $\sqrt{8} = 2.828$.

13. Using following table find the value of an annuity at $5\frac{3}{8}\%$

Rate percent :	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
Annuity value :	17.2920	16.2889	15.3725	14.5338	13.7648

14. Apply stirling's central interpolation formula to find the value of $f(0.41)$, given that $f(0.30) = 0.1179$, $f(0.35) = 0.1368$, $f(0.40) = 0.1554$, $f(0.45) = 0.1736$ and $f(0.50) = 0.1915$

15. Use following table to find $f(0.43)$

x :	0.0	0.2	0.4	0.6	0.8
$f(x)$:	0.3989	0.3910	0.3683	0.3332	0.2897

[MREC (Auto) 2002]

16. Use following table to find y at $x = 32°$

x :	10°	20°	30°	40°	50°	60°	70°	80°
y :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

(Raj. 2003)

17. Use following table to find the value of $f(9.5)$ by Lagrange's interpolation formula

x :	7	8	9	10
$f(x)$:	3	1	1	9

(MNIT 2006)

18. Use Lagrange inverse interpolation formula to find the value of x at $f(x) = 13.6$ from the following table

x :	30	35	40	45	50
$f(x)$:	15.9	14.9	14.1	13.3	12.5

19. Use Lagrange's interpolation formula to find the polynomial function $y = f(x)$ from the following table

x :	0	1	2	5
y :	2	3	12	147

20. If $\sum_{x=1}^{10} f(x) = 500426$, $\sum_{x=4}^{10} f(x) = 329240$, $\sum_{x=7}^{10} f(x) = 175212$ and $f(10) = 40365$, then prove that $f(1) = 58843.4$

ANSWERS

3. $f(x) - 2f(x-h) + f(x-2h)$
4. $f(10) = 757$
5. 2.0086
6. $p = 9.5$
7. 465
8. 27, 125, $f(1.5) = 3.375$ and $f(7.5) = 421.875$
9. $f(x) = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10x^{(0)}$, $\Delta^3 f(x) = 12$
10. $f(x) = x^2 + 2x + 3$
11. y_3 is incorrect, correct $y_3 = 86$
12. 2.345
13. 14.7366
14. 0.1591
15. 0.3637
16. 0.8480
17. 3.625
18. 44.1
19. $f(x) = x^3 + x^2 - x + 2$

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