ADVANCED ENGINEERING MATHEMATICS

IT (III SEM)

MOMENTS (EXPECTATION, VARIANCE AND HIGHER ORDER), SKEWNESS AND KURTOSIS, CHEBYSHEV'S INEQUALITY

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MOMENTS (EXPECTATION, VARIANCE AND HIGHER ORDER)

MOMENTS ABOUT ORIGIN

Let X be any random variable and its r^{th} moment about origin, is denoted by μ'_r and is given by

$$\mu'_r = E[X^r] = \sum_{i=1}^{n} x^r_i \, p(x_i)$$

where $p(x_i)$ is $p.m.f.$ for discrete random variable X.

Similarly $$\mu'_r = E[X^r] = \int_a^b x^r f(x) dx$$

is r^{th} moment about origin of the continuous random variable X with $p.d.f.$ $f(x)$.

In particular, first and second moments about origin are given as :

The first moment about origin is

$$\mu'_1 = E(X) = \bar{x} = \text{Mean} = \sum_{i=1}^{n} x_i p(x_i) \text{ ; for discrete } r.v.$$

$$\mu'_1 = E(X) = \bar{x} = \text{Mean} = \int_a^b x f(x) dx \text{ ; for continuous } r.v.$$

The second moment about origin is

$$\mu'_2 = E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i) \quad ; \quad \text{for discrete } r.v.$$

$$\mu'_2 = E(X^2) = \int_a^b x^2 f(x) dx \quad ; \quad \text{for continuous } r.v.$$

MOMENTS ABOUT ANY POINT

The r^{th} moment about a point a is

$$\mu'_{ra} = \sum_{i=1}^{n} (x_i - a)^r \, p(x_i) \quad ; \quad \text{for discrete } r.v.$$

$$\mu'_{ra} = \int_a^b (x - a)^r \, f(x)\,dx \quad ; \quad \text{for continuous } r.v.$$

MOMENTS ABOUT MEAN OR CENTRAL MOMENTS

The r^{th} moment about mean ($\bar{x}$) is denoted by μ_r and is given by

$$\mu_r = \sum_{i=1}^{n} (x_i - \bar{x})^r \, p(x_i) \quad ; \quad \text{for discrete } r.v.$$

$$\mu_r = \int_a^b (x - \bar{x})^r \, f(x)\,dx \quad ; \quad \text{for continuous } r.v.$$

EXPECTATION

Expectation is first momemt about origin i.e.

Expectation of X $= E(X) = \mu'_1 = \bar{x} = \text{Mean} = \sum_{i=1}^{n} x_i p(x_i)$; for discrete $r.v.$

Expectation of X $= E(X) = \mu'_1 = \bar{x} = \text{Mean} = \int_a^b x f(x) dx$; for continuous $r.v.$

Note : Using above expressions, it is also clear that Expectation of X is same as mean of X i.e. $E(X) = \bar{x}$

VARIANCE

Variance is second moment about mean (second central moment) or difference between second and first moments about origin i.e.

Variance of *X* $= \sigma^2 = \mu_2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \, p(x_i)$; for discrete $r.v.$

Variance of *X* $= \sigma^2 = \mu_2 = \int_a^b (x - \bar{x})^2 \, f(x) \, dx$; for continuous $r.v.$

OR

Variance of *X* $= \sigma^2 = \mu_2' - \mu_1'^2 = \sum_{i=1}^{n} x_i^2 p(x_i) - \left[\sum_{i=1}^{n} x_i p(x_i) \right]^2$; for discretes $r.v.$

Variance of *X* $= \sigma^2 = \mu_2' - \mu_1'^2 = \int_a^b x^2 f(x) dx - \left[\int_a^b x f(x) dx \right]^2$; for continuous $r.v.$

Note : (1) Using above expressions it is clear that

Variance of *X* $= \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = E(X^2) - [E(X)]^2$

(2) Standard deviation of X $=$ $\sqrt{\text{Variance of X}} = \sigma$

RELATION BETWEEN MOMENTS

Central moments in terms of moments about origin / moments about any point

We have $$\mu_r = (\mu' - \mu'_1)^r$$

$$\Rightarrow \quad \mu_r = \mu'_r - {}^rC_1\, \mu'_{r-1}\mu'_1 + {}^rC_2\, \mu'_{r-2}\mu_1'^2 - {}^rC_3\, \mu'_{r-3}\mu_1'^3 + \ldots\ldots\ldots\ldots + (-1)^r \mu_1'^r \qquad \cdots (1)$$

[Using Binomial theorem]

Putting $r = 2$ in (1), we get

$$\mu_2 = \mu'_2 - {}^2C_1\, \mu_1'^2 + {}^2C_2\, \mu'_0 \mu_1'^2$$

$$\Rightarrow \quad \mu_2 = \mu'_2 - 2\mu_1'^2 + \mu_1'^2 \qquad [\text{since } \mu'_0 = 1]$$

$$\Rightarrow \quad \boxed{\mu_2 = \mu'_2 - \mu_1'^2}$$

Again putting $r = 3$ in (1), we get

$$\mu_3 = \mu'_3 - {}^3C_1\, \mu'_2\mu'_1 + {}^3C_2\, \mu'_1\mu_1'^2 - {}^3C_3\, \mu'_0\mu_1'^3$$

$$\Rightarrow \quad \mu_3 = \mu'_3 - 3\,\mu'_2\mu'_1 + 3\,\mu_1'^3 - \mu_1'^3 \qquad [\text{since } \mu'_0 = 1]$$

$$\Rightarrow \quad \boxed{\mu_3 = \mu'_3 - 3\,\mu'_2\mu'_1 + 2\mu_1'^3}$$

Similarly, putting $r = 4$ in (1), we get

$$\boxed{\mu_4 = \mu'_4 - 4\,\mu'_3\mu'_1 + 6\,\mu'_2\,\mu_1'^2 - 3\,\mu_1'^4}$$

and so on.

RELATION BETWEEN MOMENTS

Moments about origin / moments about any point in terms of central moments

We have $\mu'_r = (\mu + d)^r$ [where $d = \bar{x} - a$]

$$\Rightarrow \quad \mu'_r = \mu_r + {}^rC_1\, d\, \mu_{r-1} + {}^rC_2\, \mu_{r-2}\, d^2 + {}^rC_3\, \mu_{r-3}\, d^3 + \ldots + d^r \quad \ldots(2)$$

[Using Binomial Theorm]

Putting $r = 1$ in (2), we get

$$\mu'_1 = \mu_1 + {}^1C_1 \mu_0 d \qquad [\text{since } \mu_0 = 1\ ;\ \mu_1 = 0]$$

$$\Rightarrow \quad \boxed{\mu'_1 = d}$$

Putting $r = 2$ in (2), we get

$$\mu'_2 = \mu_2 + {}^2C_1\, \mu_1 d + {}^2C_2 \mu_0 d^2$$

$$\Rightarrow \quad \mu'_2 = \mu_2 + 2(0)d + d^2$$

$$\Rightarrow \quad \boxed{\mu'_2 = \mu_2 + d^2}$$

Similarly, putting $r = 3$ and $r = 4$ in (2), we get

$$\boxed{\mu'_3 = \mu_3 + 3\mu_2 d + d^3}$$

and

$$\boxed{\mu'_4 = \mu_4 + 4\mu_3\, d + 6\mu_2 d^2 + d^4}$$

and so on.

Q.1. *Two unbiased dice are thrown. Find the expected value of the sum of number of points on them.*

Sol. The probability distribution of $X = x$ (the sum of the numbers obtained on two dice) is

x	:	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then $$E(X) = \sum_{x=2}^{12} x\,p(x) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36}$$

$$+ 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= \frac{256}{36} = 7$$

Q.2. *In four tosses of a coin, let X be the number of heads. Tabulate the 16 possible outcomes with the corresponding values of X. By simple counting, derive the probability distribution of X and hence calculate the expected value of X.*

Sol. Let H represent a head, T a tail and X be the random variable denoting the number of heads.

S.No.	*Out comes*	*No. of heads (X)*	*S. No.*	*Out comes*	*No. of heads (X)*
1.	HHHH	4	9.	HTHT	2
2.	HHHT	3	10.	THTH	2
3.	HHTH	3	11.	THHT	2
4.	HTHH	3	12.	HTTT	1
5.	THHH	3	13.	THTT	1
6.	HHTT	2	14.	TTHT	1
7.	HTTH	2	15.	TTTH	1
8.	TTHH	2	16.	TTTT	0

The random variable X takes the values 0, 1, 2, 3 and 4. Since from the above table, we find that the number of cases favourable to the coming of 0, 1, 2, 3 and 4 heads are 1, 4, 6, 4 and 1 respectively, then

$$P(X=0)=\frac{1}{16},\quad P(X=1)=\frac{4}{16}=\frac{1}{4},\ P(X=2)=\frac{6}{16}=\frac{3}{8},$$

$$P(X=3)=\frac{4}{16}=\frac{1}{4},\ P(X=4)=\frac{1}{16}$$

The probability distribution of $X=x$ can be summarized as follows :

x	:	0	1	2	3	4
$p(x)$	:	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Now, $$E(X)=\sum_{x=0}^{4} xp(x) = 0\times\frac{1}{16}+1\times\frac{1}{4}+2\times\frac{3}{8}+3\times\frac{1}{4}+4\times\frac{1}{16}$$

$$=\frac{1}{4}+\frac{3}{4}+\frac{3}{4}+\frac{1}{4}=\frac{8}{4}=2$$

Q.3. *Let X be the total of the two dice in the experiment of tossing two dice. Find variance of X.*

Sol. The probability distribution of $X = x$ (the sum of the number obtained through two dice) is

x	:	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then, Mean $= E(X) = \mu'_1 = \bar{x} = \sum_{x=2}^{12} xp(x)$

$$= 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36}$$

$$+ 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= \frac{252}{36} = 7$$

Now, Variance $(X) = \sum_{x=2}^{12} (x - \bar{x})^2 p(x)$

$$= \sum_{x=2}^{12} (x-7)^2\, p(x) \qquad (\text{since } \bar{x} = 7)$$

$$= 25 \times \frac{1}{36} + 16 \times \frac{2}{36} + 9 \times \frac{3}{36} + 4 \times \frac{4}{36} + \frac{5}{36} + 0 + \frac{5}{36}$$

$$+ 4 \times \frac{4}{36} + 9 \times \frac{3}{36} + 16 \times \frac{2}{36} + 25 \times \frac{1}{36}$$

$$= \frac{210}{36} = \frac{35}{6}$$

Q.4. *A continuous random variable X has pdf given by*

$$f(x) = \begin{cases} K\,x\,e^{-\lambda x} & ; \quad x \geq 0,\ \lambda > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the constant K. Also find its mean and variance.

Sol. Since sum of total probability should be one

i.e. $\int_0^\infty f(x)dx = 1$ or $K\int_0^\infty xe^{-\lambda x}dx = 1$

or $K\int_0^\infty e^{-\lambda x}x^{2-1}dx = 1$ or $K\cdot\frac{\lceil 2}{\lambda^2} = 1$

$$\left\{\text{since} \int_0^\infty e^{-ax}x^{n-1}dx = \frac{\lceil n}{a^n}\right\}$$

or $\left(\frac{1}{\lambda^2}\right)K = 1$ or $K = \lambda^2$

Now, $$\text{mean} = \mu'_1 = \int_0^\infty x f(x)\,dx$$

$$= K\int_0^\infty x^2 e^{-\lambda x}\,dx \quad = \lambda^2 \int_0^\infty x^{3-1} e^{-\lambda x}\,dx \quad [\because K = \lambda^2]$$

$$= \lambda^2 \frac{\Gamma 3}{\lambda^3} = \frac{1}{\lambda}\lfloor 2 \qquad \left\{\text{since } \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}\right\}$$

or $$\mu'_1 = \frac{2}{\lambda} \qquad \ldots(1)$$

Then $$\text{variance} = \mu_2 = \mu'_2 - {\mu'_1}^2 \qquad \ldots(2)$$

Now, $$\mu'_2 = \int_0^\infty x^2 f(x)\,dx \qquad = K\int_0^\infty x^3 e^{-\lambda x}\,dx$$

$$= \lambda^2 \int_0^\infty x^{4-1} e^{-\lambda x}\,dx \quad = \lambda^2 \frac{\Gamma 4}{\lambda^4} = \frac{1}{\lambda^2}\lfloor 3$$

or $$\mu'_2 = \frac{6}{\lambda^2} \qquad \ldots(3)$$

Using (1) and (3) in (2), we get

$$\text{variance} = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

Q.5. *Calculate the first four moments about the mean of the following distribution*

Weekly earnings (in Rs.)	:	6	7	8	9	10	11	12
No. of persons	:	3	6	9	13	8	5	4

Sol. Given distribution is

x_i	:	6	7	8	9	10	11	12
f_i	:	3	6	9	13	8	5	4

We construct the following table for calculation of moments about mean = 9

Weekly earning (x_i)	No. of persons (f_i)	$f_i x_i$	$d = (x_i - 9)$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
6	3	18	−3	−9	27	−81	243
7	6	42	−2	−12	24	−48	96
8	9	72	−1	−9	9	−9	9
9	13	117	0	0	0	0	0
10	8	80	1	8	8	8	8
11	5	55	2	10	20	40	80
12	4	48	3	12	36	108	324
	$\Sigma f_i = 48$	$\Sigma f_i x_i = 432$		$\Sigma f_i d_i = 0$	$\Sigma f_i d_i^2 = 124$	$\Sigma f_i d_i^3 = 18$	$\Sigma f_i d_i^4 = 760$

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{432}{48} = 9$$

Hence, moments about mean = 9 are

$$\mu_1 = \frac{\sum f_i d_i}{\sum f_i} = 0$$

$$\mu_2 = \frac{\sum f_i d_i^2}{\sum f_i} = \frac{124}{48} = 2.58$$

$$\mu_3 = \frac{\sum f_i d_i^3}{\sum f_i} = \frac{18}{48} = 0.37$$

$$\mu_4 = \frac{\sum f_i d_i^4}{\sum f_i} = \frac{760}{48} = 15.83$$

Hence, $\mu_1 = 0$, $\mu_2 = 2.58$, $\mu_3 = 0.37$, $\mu_4 = 15.83$

Q.6. *The first four moments of a distribution about the value 4 of a variable are – 1.5, 17, –30 and 108*

Find : (a) Moments about mean (b) Moments about the point $x = 2$

Sol. Here the assumed mean $a = 4$

The moments about $a = 4$ are given by

$$\mu'_1 = -1.5,\ \mu'_2 = 17,\ \mu'_3 = -30,\ \mu'_4 = 108$$

(a) Moments about mean are given by :

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = (17) - (-1.5)^2 = 17 - 2.25 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu'_2\mu'_1 + 2\mu'^3_1 = (-30) - 3(17)(-1.5) + 2(-1.5)^3$$

$$= -30 + 76.5 - 6.75 = 76.5 - 36.75$$

$$= 39.75$$

$$\mu_4 = \mu'_4 - 4\mu_1'\mu_3' + 6\mu'^2_1\mu'_2 - 3\mu'^4_1$$

$$= (108) - 4(-1.5)(-30) + 6(-1.5)^2(17) - 3(-1.5)^4$$

$$= 108 - 180 + 229.5 - 15.1875 = 337.5 - 195.1875$$

$$= 142.3125 = 142.3$$

(b) Given μ'_1 at point 4 is -1.5

i.e. $\sum p_i (x_i - 4) = -1.5$ or $\sum p_i x_i - 4\sum p_i = -1.5$

or $\sum p_i x_i - 4 = -1.5$ or $\bar{x} = 2.5$

Here the assumed mean $a = 2$

So, $d = \bar{x} - a = 2.5 - 2 = 0.5$

Now, moments about point $x = 2$ are given by :

$$\mu_1' = \text{first moment about point } 2 = d = 0.5$$

$$\mu_2' = \mu_2 + d^2 = 14.75 + (0.5)^2 = 14.75 + 0.25 = 15$$

$$\mu_3' = \mu_3 - 3\mu_2 d + d^3 = 39.75 + 3(14.75)(0.5) + (0.5)^3$$

$$= 39.75 + 22.125 + 0.125 = 62$$

$$\mu_4' = \mu_4 - 4\mu_3 d + 6\mu_2 d^2 + d^4$$

$$= 142.3 + 4(39.75)(0.5) + 6(14.75)(0.5)^2 + (0.5)^4$$

$$= 142.3 + 79.5 + 22.125 + 0.0625 = 243.98$$

SKEWNESS AND KURTOSIS

KARL PEARSON β AND γ COEFFICIENTS

Karl pearson defined the following four coefficients based upon the first four central moments as follows :

β – coefficients	$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$; $\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$

γ – coefficients	$\boxed{\gamma_1 = \sqrt{\beta_1}}$; $\boxed{\gamma_2 = \beta_2 - 3}$

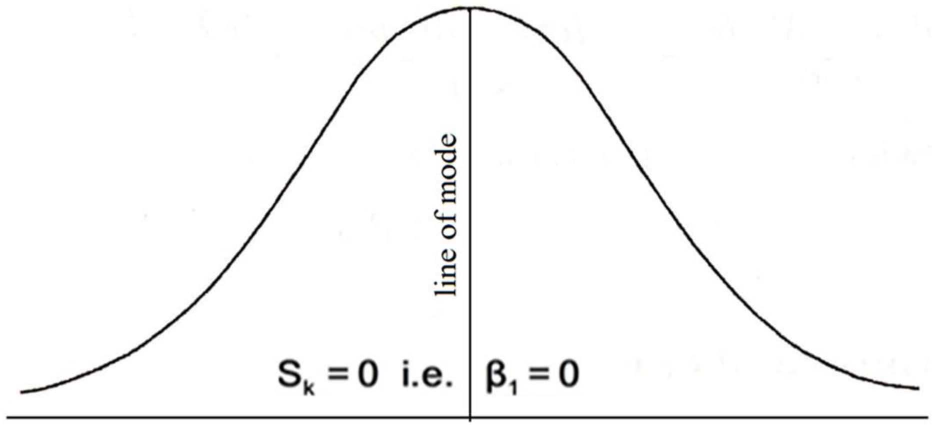
The uses of these coefficients is to measure skewness and kurtosis.

SKEWNESS

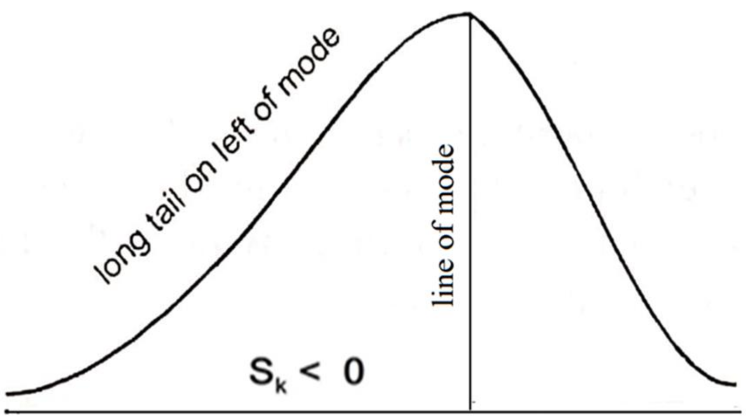
The term 'Skewness' is used to describe the symmetricity of a curve.

The formula for coefficient of skewness is

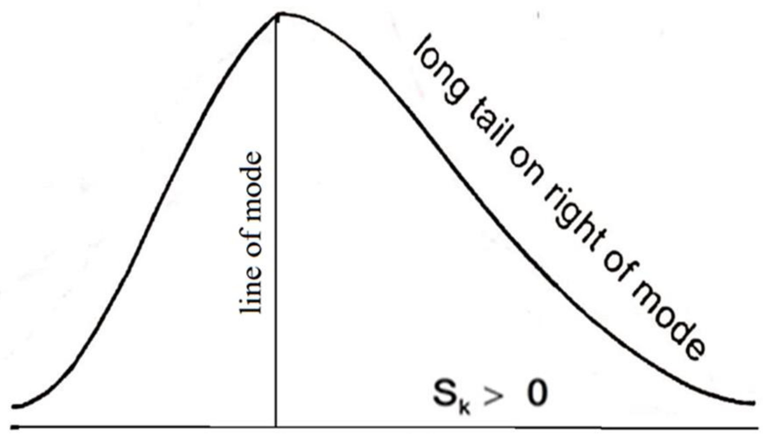
$$\boxed{S_k = \frac{\sqrt{\beta_1}\left(\beta_2 + 3\right)}{2\left(5\beta_2 - 6\beta_1 - 9\right)}}$$



Symmetrical Curve.



Negatively skewed curve



Positively skewed curve.

KURTOSIS

The term 'Kurtosis' is used to describe the peakedness of a curve.

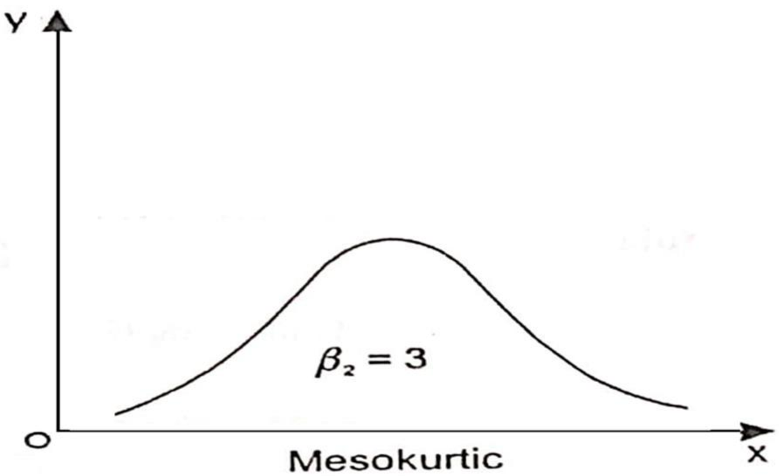
The measures of kurtosis are based upon Karl *Pearson's* β coefficient β_2.

We know that $\beta_2 = \frac{\mu_4}{\mu_2^2}$

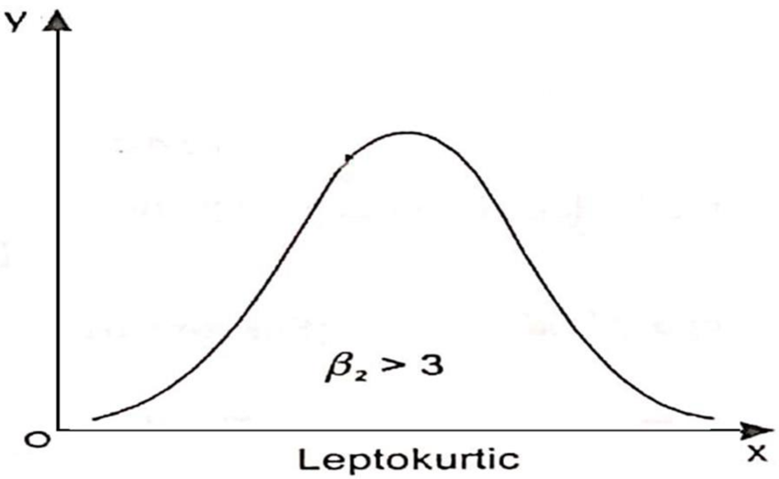
(I) If $\beta_2 = 3$, the distribution is said to be normal and the curve is a normal curve (mesokurtic).

(II) If $\beta_2 > 3$, the distribution is said to be more peaked and the curve is leptokurtic.

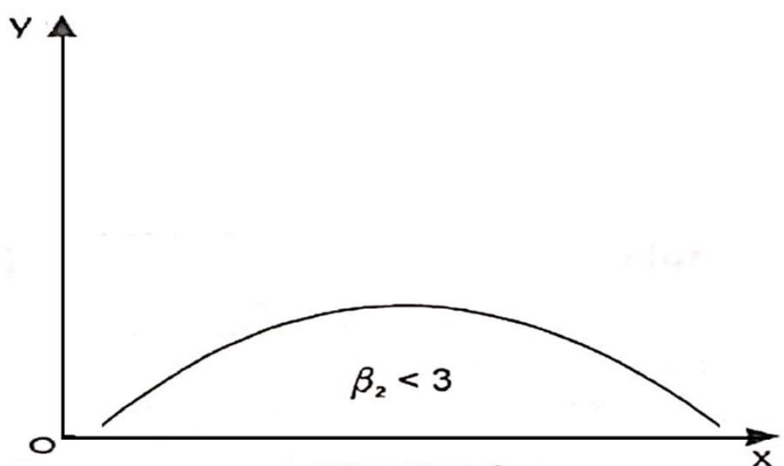
(III) If $\beta_2 < 3$, the distribution is said to be flat topped and the curve is platykurtic.



Mesokurtic



Leptokurtic



Platykurtic

Q.1. *The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Find the mean, variance, β_1 and β_2. Comment upon the nature of distribution.*

Sol. Here the assumed mean $a = 5$

The moments about $a = 5$ are given by

$\mu_1' = 2,\ \mu_2' = 20,\ \mu_3' = 40,\ \mu_4' = 50$

We know that $\mu_1' = \sum p_i (x_i - 5) = 2 \quad \Rightarrow \quad \sum p_i x_i - 5 \sum p_i = 2$

$\Rightarrow \quad \sum p_i x_i - 5 = 2 \quad \Rightarrow \quad mean = \bar{x} = 7$

Now, $variance = \mu_2 = \mu_2' - \mu_1'^2 = 20 - 4 = 16,$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = 40 - 3(2)(20) + 2(2)^3$$

$$= 40 - 120 + 16 = -64,$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

$$= 50 - 4(2)(40) + 6(2)^2(20) - 3(2)^4$$

$$= 50 - 320 + 480 - 48 = 530 - 368 = 162$$

Now, $$\beta_1 = \frac{\mu_3^{\ 2}}{\mu_2^{\ 3}} = \frac{(-64)^2}{(16)^3} = \frac{4096}{4096} = 1$$

and $$\beta_2 = \frac{\mu_4}{\mu_2^{\ 2}} = \frac{162}{(16)^2} = \frac{162}{256} = 0.6328$$

Here, $\beta_1 \neq 0$

Hence, curve is not symmetric.

Also, $$S_k = \frac{\sqrt{\beta_1}\left(\beta_2 + 3\right)}{2\left(5\beta_2 - 6\beta_1 - 9\right)} = \frac{\sqrt{1}\left(0.6328 + 3\right)}{2\left[5(0.6328) - 6(1) - 9\right]}$$

$$= \frac{3.6328}{-23.6720} = -0.1535$$

i.e. $S_k < 0$, which means the curve is negatively skewed and

$\beta_2 < 3$, which means the curve is a flat curve *i.e.* distribution is platykurtic.

CHEBYSHEV'S INEQUALITY

Statement of Chebyshev's Inequality

If X be a random variable (discrete or continuous) taking only non-negative values, possesses a finite mean μ and variance σ^2, then for $t > 0$,

$$\boxed{\left[P\left(|X-\mu|\right) \geq t\right] \leq \frac{\sigma^2}{t^2}}$$

<u>Note :</u> Chebyshev's inequality is also known as Tchebycheff's inequality.

Q.1 *For the number of points on symmetrical dice, prove that Tchebycheff's inequality gives*

$$P(|X - E(x)| > 2\cdot5) < 0\cdot47,$$

while the actual probability is zero.

Sol. For the given problem, we have

$$E(x) = \frac{1}{6} \sum_{i=1}^{6} x_i = 3\cdot5$$

and $$\sigma^2 = \frac{6^2 - 1}{12} = \frac{35}{12}$$

$$\left(\because \text{for first 'n' natural numbers, variance} = \sigma^2 = \frac{n^2 - 1}{12} \right)$$

Now, by Tchebycheff's inequality, *we have*

$$P(|X - E(X)| > t) < \frac{\sigma^2}{t^2}$$

so that $$P(|X - E(x)| > 2{\cdot}5) < \frac{35}{12 \times (2{\cdot}5)^2}$$

i.e. $$P(|X - E(x)| > 2{\cdot}5) < 0{\cdot}47$$

Since $E(X) = 3{\cdot}5$, there is no point on the dice which differs from $3{\cdot}5$ by more than $2{\cdot}5$ and hence the actual probability is zero.

Q.2. *A random variable x has the density function* e^{-x} *for* $x \geq 0$. *Show that Tchebycheff's inequality gives*

$$P\,[|X-1|>2] < \frac{1}{4}$$

and show that the actual probability is e^{-3}

Sol. *For the given problem, we have*

$$E(x) = \mu_1' = \int_0^\infty x.e^{-x}\,dx = \int_0^\infty x^{2-1} e^{-x}\,dx = \Gamma 2 = 1$$

and
$$\mu_2' = \int_0^\infty x^2 e^{-x}\,dx = \int_0^\infty x^{3-1} \cdot e^{-x} dx = \Gamma 3 = 2$$

$\therefore$
$$\sigma^2 = \mu_2' - \mu_1'^2 = 2 - 1 = 1$$

Now, by Tchebycheff's inequality, *we have*

$$P(|X - E(X)| > t) < \frac{\sigma^2}{t^2}$$

so that $$P[|X - 1| > 2] < \frac{1}{4}$$

Since $x \geq 0, |x - 1| > 2 \Rightarrow x - 1 > 2 \Rightarrow x > 3$

Hence, actual probability is given by

$$P = \int_3^\infty e^{-x}\,dx = \left[-e^{-x}\right]_3^\infty = e^{-3} - e^{-\infty}$$

$$= e^{-3} - 0 = e^{-3}$$

Exercise

1. Four balls are drawn simultaneously from a bag containing 6 White, 5 Black and 7 Red balls. If X denotes the number of White balls drawn, then find $E(X)$.

2. Find the standard deviation for the following distribution:

x	:	8	12	16	20	24
$p(x)$	:	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

3. A continuous random variable X has p.d.f. given as:

$$f(x)=\begin{cases} 2e^{-2x} & ; \quad x \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the mean and variance of X.

4. Calculate the first four moments about the mean for the following distribution:

x	:	0	1	2	3	4	5	6	7	8
f	:	1	8	28	56	70	56	28	8	1

Hence find β_1, β_2 and comment upon the nature of distribution.

5. The first four moments of a distribution about the value 5 of a variable are - 4, 22, - 117 and 560. Find moments about (a) mean (b) origin

Answers

1. $E(X) = 1.33$

2. $S.D. = 2\sqrt{5}$

3. mean $= \frac{1}{2}$, variance $= \frac{1}{4}$

4. $\mu_1 = 0, \mu_2 = 2, \mu_3 = 0, \mu_4 = 11; \beta_1 = 0, \beta_2 = 2.75;$

   distribution is symmetric and the curve is platykurtic.

5. (a) $\mu_1 = 0, \mu_2 = 6, \mu_3 = 19, \mu_4 = 32$

   (b) $\mu_1' = 1, \mu_2' = 7, \mu_3' = 38, \mu_4' = 145$

THANKS