Mathematical Logic

PROPOSITIONS

A declarative sentence (or assertion) which is true or false, but not both, is called a proposition (or statement). Sentences which are exclamatory, interrogative or imperative in nature are not propositions. Lower case letters such as p, q, r … are used to denote propositions.

For example, we consider the following sentences:

1. New Delhi is the capital city of India.
2. How beautiful is Rose?
3. $2 + 2 = 3$
4. What time is it?
5. $x + y = z$
6. Take a cup of coffee.

In the given statements, (2), (4) and (6) are obviously not propositions as they are not declarative in nature. (1) and (3) are propositions, but (5) is not, since (1) is true, (3) is false and (5) is neither true nor false as the values of x, y and z are not assigned.

Examples (i) Jaipur is the capital of Rajasthan

Answer It is true

(ii) Zero is a natural number.

Answer It is false as the set of natural members $N = \{1, 2, 3, \ldots\}$ does not include the number zero.

(iii) $\sqrt{2}$ is a rational number.

Answer It is false as $\sqrt{2}$ can not be expressed exactly in the p/q form.

(iv) $2 + 3 = 5$.

Answer It is a mathematical equation which is true.

(v) $x + 1 = 5$.

Answer It is not always true or false.

It is true if $x = 4$ and is false when $x \neq 4$. So it is not a proposition.

If a proposition is true, we say that the truth value of that proposition is true, denoted by T or 1. If a proposition is false, the truth value is said to be false, denoted by F or 0.

Propositions which do not contain any of the logical operators or connectives (to be introduced in the next section) are called atomic (primary or primitive) propositions.

Many mathematical statements which can be constructed by combining one or more atomic statements using connectives are called molecular or compound propositions.

The truth value of a compound proposition depends on those of sub propositions and the way in which they are combined using connectives.

The area of logic that deals with propositions is called propositional logic or propositional calculus.

Propositional Variables

In logic, it is required to draw conclusions from the given statements. Now, instead of writing the statements repeatedly, it is convenient to denote each of the statements by a unique variable, called *propositional variable*. These variables are usually denoted by an English alphabet p, q, r, . . ., etc., and can be replaced by statements. For example, we can write,

p : Delhi is the capital of India; q : It is raining.

COMPOUND STATEMENTS

Statements or propositional variables can be combined by means of logical connectives or operators to form a single statement called *compound statements* (or *compound propositions* or *molecular statement*).

Tautology and Fallacy Or Contradiction

If the truth values of a compound proposition involving two or more propositions be T in **all cases** then it is called a **tautology** and is **denoted by T** and if it is false **in all cases then** it is **called a fallacy** and is, **denoted by F**.

Contingency

A compound proposition which is neither a tautology nor a fallacy is called a **contingency**. It has truth values T and F both in some alternative cases.

LOGICAL EQUIVALENCES

Compound propositions that have the same truth values in all possible cases are called *logically equivalent*. The propositions p and q are logically equivalent written $p \equiv q$ if $p \leftrightarrow q$ is a tautology.

LOGICAL CONNECTIVES

There are five logical connectives as shown in the table given below, which are frequently used for this purpose

Table Logical Connectives

Symbol	Connective	Name
$\sim$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\rightarrow$	implies or if ... then	implication or conditional
$\leftrightarrow$	If and only if	equivalence or biconditional

Negation

If p denotes a statement, then the *negation* of p is the statement denoted by ~p and read as 'it is not the case that p'. So, it follows that if p is true then ~p is false, and if p is false then ~p is true.

Example If p : Ramanujan was a great mathematician.

then ~p : It is not the case that Ramanujan was a great mathematician.

or ~p : Ramanujan was not a great mathematician.

The truth value of ~p is relative to that of p and can be expressed in a tabular form, known as **truth table**, as shown below : **Table Truth table of ~p**

p	~p
T	F
F	T

Remark . Negation changes one statement into another, while other connectives combine two statements to form a third.

Remark . A truth table displays the relationships between the truth of propositions. It indicates the truth values of compound statements constructed from simpler statements.

Conjunction

If p and q are statements, then "p and q" is a compound statement, denoted as $p \wedge q$ and referred as the *conjunction* of p and q . The conjunction of p and q is true only when both p and q are true, otherwise, it is false.

Truth values of the statement $p \wedge q$ in terms of truth values of p and q are given in the truth table shown below

Truth table of $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example Form the conjunction of p and q for each of the following :

(i) p : It is snowing. q : I am cold.

Solution. (i) $p \wedge q$: It is snowing and I am cold. Since here both p and q are true so the conjunction $p \wedge q$ is true.

(ii) $p : 6 < 7$. $q : -3 > -4$.

Solution. (ii) $p \wedge q : 6 < 7$ and $-3 > -4$. Since here both p and q are true so the conjunction $p \wedge q$ is true.

(iii) p : It is raining. $q : 1 > 4$.

Solution. (iii) $p \wedge q$: It is raining and $1 > 4$. Since here q is false so the conjunction $p \wedge q$ is false.

Disjunction

If p and q are statements, then "p or q" is a compound statement, denoted as $p \vee q$ and referred as the *disjunction* of p and q. The disjunction of p and q is true whenever at least one of the two statements is true, and it is false only when both p and q are false.

The truth table for disjunction of p and q can be constructed as shown below :

Truth table of $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example Form the disjunction of p and q for each of the following :

(i) p : 4 is a positive integer. q : $\sqrt{5}$ is a rational number.

Solution. (i) $p \vee q$: 4 is a positive integer or $\sqrt{5}$ is a rational number. Since here p is true, so the disjunction $p \vee q$ is true, even though q is false.

(ii) p : $3 + 4 > 8$ q : Jaipur is the capital of Gujarat.

Solution. (ii) $p \vee q$: $3+4>8$ or Jaipur is the capital of Gujarat. Here $p \vee q$ is false as both p and q are false.

Remark Example (ii) depicts that in logic, we may connect two totally unrelated statements by the connective *or.*

Remark . In disjunction we have defined *or* in the *inclusive* sense, i.e., either p is true or q is true or both are true so this "or" could be known as *inclusive or*. But "or" can be used in the exclusive sense, also i.e., either p is true or q is true, but not both. In mathematics and computer science, we conventionally use the connective "or" in the inclusive manner.

Example . Make a truth table for each of the following

(i) $p \vee \sim q$ (ii) $(\sim p \wedge q) \vee p$ (iii) $(p \vee q) \vee \sim q$

(iv) $(p \vee q) \wedge r$ (v) $(\sim p \vee q) \wedge \sim r$ (vi) $\sim(\sim p)$.

Solution. (i) Truth table of $p \vee \sim q$

p	q	~q	p∨~q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ii) Truth table of $(\sim p \wedge q) \vee p$

p	q	~p	~p∧q	(~p∧q)∨p
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(iii) **Truth table of $(p \vee q) \vee \sim q$**

p	q	$p \vee q$	$\sim q$	$(p \vee q) \vee \sim q$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(iv) **Truth table of $(p \vee q) \wedge r$**

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(v) Truth table of $(\sim p \vee q) \wedge \sim r$

p	q	r	$\sim p$	$\sim p \vee q$	$\sim r$	$(\sim p \vee q) \wedge \sim r$
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
F	T	T	T	T	F	F
T	F	F	F	F	T	F
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	T	T	T	T

(vi) Truth table of $\sim(\sim p)$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Implication ($p \to q$)

Let p and q be two propositions then 'if p then q also called '**p implies q** is denoted as $p \to q$'. It is the proposition which is false only in one case when p is true but q is false and is true in the rest three cases.

In this implication, p is called the ***hypothesis*** and q is called the ***conclusion***

So its truth table is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication is an important connective which can be expressed in the language in many different ways as: (i) 'If p then q' $\equiv p \to q$

(ii) 'p if q' means p is true when q is true it is denoted by $q \rightarrow p$

(iii) 'p is sufficient for q' means a sufficiency condition for q to be true is that p should be true.

(iv) 'q if p' It means q is true whenever p is true

(v) 'q when p' means q is true when p is true

(vi) 'a necessary condition for p is q' means q follows from p, i.e., q is true p is true.

Example (. Write the implication $p \rightarrow q$ for each of the following :

(i) p : I am hungry. q : I will eat.

(ii) p : It is snowing. q : $3 + 8 = 11$

Solution. (i) $p \rightarrow q$: If I am hungry, then I will eat.

(ii) $p \rightarrow q$: If it is snowing, then $3 + 8 = 11$.

3.1-Tautology and Fallacy(Contradiction)

If the truth values of a compound proposition involving two or more propositions be T in **all cases** then it is called a **tautology** and is **denoted by T** and if it is false **in all cases then** it is **called a fallacy** and is, **denoted by F**.

3.2-Contingency

A compound proposition which is neither a tautology nor a fallacy is called a **contingency**. It has truth values T and F both in some alternative cases.

3.3- Equivalent

Compound propositions that have the same truth values in all possible cases are called *logically equivalent*. The propositions p and q are logically equivalent written $p \equiv q$ if $p \leftrightarrow q$ is a tautology.

Ex. Show that $p \vee \sim p \equiv T$ and $p \wedge \sim p \equiv F$

p	$\sim p$	$p \vee \sim p$		$p \wedge \sim p$
T	F	T	so $p \vee \sim p \equiv T$	F so
F	T	T		F $p \wedge \sim p \equiv F$

Example *Show that the statement $p \wedge q$ is a contingency where*

p : John is a bachelor

q : Smith is married

Also show that $p \vee \sim q$ is atautology and $P \wedge \sim q$ is a contrediction

Solution: Since

$p \wedge q$: John is a bachalor and Smith is married.

We prepare the truth table for $p \wedge q$ i.e.

p	$\sim p$	q	$\sim q$	$p\wedge q$	$p\vee\sim p$	$p\wedge\sim p$
T	F	T	F	T	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F

We see that from table that all values of $p \wedge q$ is meither T nor F. So $p \wedge q$ is a contingency.

Since all values of $p \vee \sim q$ is truth (T) so $p \vee \sim q$ is a tautology and all values of $P \wedge \sim q$ is false (F) $P \wedge \sim q$ is a contradiction.

Example · *Show that the following compound propositions are logically equivalent.*

(a) $\sim(p \vee q)$ and $\sim p \wedge \sim q$

Solution: Constructing truth table for the given statements, we have

(a)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Since, the truth values of the propositions $\sim(p \vee q)$ and $\sim p \wedge \sim q$ agree for all possible combinations of the truth values of p and q,

it follows $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ is a tautology and thus

$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$ [*De Morgan's law*] *(b)* $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$

Solution: Constructing truth table for the given statements, we have

(b)

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Since, the truth values of $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ agree for all possible combinations of the truth values of p and q, these propositions are logically equivalent.

(c) $p \to q$ and $\sim p \vee q$

Solution: Constructing truth table for the given statements, we have

(c)

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the truth values of $p \to q$ and $\sim p \vee q$ agree for all possible combinations of the truth values of p and q, $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology.

Thus, $(p \to q) \equiv (\sim p \vee q)$.

(d) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

Solution: Constructing truth table for the given statements, we have

(d)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F
F	T	F	F	F	T	T	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since, the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree.
Thus, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ [*Distributivity*].

Converse ;Contrapositive and Inverse

Converse

For a given implication $p \rightarrow q$, another implication $q \rightarrow p$ **is called its converse.** As can be seen by the following table that the two are not equivalent, as their truth tables are not identically the same.

p	q	$p \rightarrow q$		$q \rightarrow p$	
T	T	T	(i)	$T \rightarrow T = T$	(ii)
T	F	**F**		$F \rightarrow T =$ **T**	
F	T	**T**		$T \rightarrow F =$ **F**	
F	F	T		$F \rightarrow F = T$	

As the second and third rows in (i) and (ii) are opposite so **a conditional and its inverse are not equivalent,** i.e.

$$p \rightarrow q \neq q \rightarrow p$$

Contrapositive

For a given conditional $p \rightarrow q$, another conditional $\sim q \rightarrow \sim p$ is called its contrapositive. By truth tables it will be seen that these **two implications are equivalent.**

p	q	$p \rightarrow q$		$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	
T	T	**T**	(i)	F $\rightarrow$	F	= **T**	(ii)
T	F	**F**		T $\rightarrow$	F	= **F**	
F	T	**T**		F $\rightarrow$	T	= **T**	
F	F	**T**		T $\rightarrow$	T	= **T**	

By (i) and (ii) which are identical, we have the **equivalence**

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

Inverse

For an implication $p \rightarrow q$ ...(i)
another implication $\sim p \rightarrow \sim q$ is called the inverse of (i). The two are not equivalent

p	q	$p \rightarrow q$		$q \rightarrow p$	
T	T	T	(i)	$T \rightarrow T = T$	(ii)
T	F	**F**		$F \rightarrow T = \mathbf{T}$	
F	T	**T**		$T \rightarrow F = \mathbf{F}$	
F	F	T		$F \rightarrow F = T$	

As the second and the 3rd rows of (i) and (ii) are opposite. So $\mathbf{p \rightarrow q \neq \sim p \rightarrow \sim q}$ (Not equivalent)

If $p \rightarrow q$ is an implication, then
the **converse** of $p \rightarrow q$ is the implication $q \rightarrow p$,
the **contrapositive** of $p \rightarrow q$ is the implication $\sim q \rightarrow \sim p$,
and the **inverse** of $p \rightarrow q$ is the implication $\sim p \rightarrow \sim q$.

Truth table of $q \rightarrow p$ (converse)

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Truth table of $\sim q \rightarrow \sim p$ (contrapositive)

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Truth table of $\sim p \rightarrow \sim q$ (inverse)

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Example *Write the implication $p \to q$ and its converse, contrapositive and inverse where*

p : It is raining.

q : The home team wins.

Solution:

$p \to q$: If it is raining, then the home team wins.

Converse of this implication is

$q \to p$: If the home team wins, then it is raining.

Contrapositive is

$\sim q \to \sim p$: If the home team does not win then it is not raining.

and inverse is

$\sim p \to \sim q$: If it is not raining then the home team does not win.

Biconditional 'p ↔ q' for propositions p and q

For two propositions p and q the biconditional denoted by $p \leftrightarrow q$ is the proposition which is true when either p and q are **both true** or **both false** and is false when if only one of p and q is true and the other is false.

It can be seen that $$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \qquad ...(1)$$

p	q	$p \leftrightarrow q$		$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	**T**		~~F~~ T	~~F~~ T	**T**
T	F	**F** (i)		~~T~~ F	~~F~~ T	**F** (ii)
F	T	**F**		~~F~~ T	~~T~~ F	**F**
F	F	**T**		T	T	**T**

As the columns (i) and (ii) are identically same, equivalence (i) holds true.

Ex. . Show that the statements

$$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p) \text{ is a tautology} \equiv T$$

Sol.:

p	q	$p \rightarrow q$	$\sim q \rightarrow \sim p$	so $(p \leftrightarrow q) \rightarrow (\sim q \rightarrow \sim p)$
T	T	T	$F \rightarrow F = T$	$T \leftrightarrow T = T$
T	F	F	$T \rightarrow F = F$	$F \leftrightarrow F = T$
F	T	T	$F \rightarrow T = T$	$T \leftrightarrow T = T$
F	F	T	$T \rightarrow T = T$	$T \leftrightarrow T = T$

As the entries in the last column are all T so the given biconditional is a tautology.

Example

(a) p : You can take the flight.

q : You buy a ticket.

$p \leftrightarrow q$: You can take the flight if and only if you buy a ticket.

(b) Let A be a matrix.

p : Inverse of A exists.

q : Determinant $|A| \neq 0$.

$p \leftrightarrow q$: Inverse of A exists if and only if $|A| \neq 0$ or

The necessary and sufficient condition for inverse of A to exist is that $|A| \neq 0$.

Algebra of Propositions

As can be verified, the various laws true for union and intersection of sets are equally true for propositions for their disjunction and conjunction. The logical equivalences defined by these laws are as follows Their truth can be **verified by forming their truth tables.**

(i)	**Identity laws**	$p \wedge T \equiv p$ $p \vee F \equiv p$
(ii)	**Domination laws**	$p \vee T \equiv T$ $p \wedge F \equiv F$

(iii)	**Idempotent laws**	$p \vee p \equiv p$ $p \wedge p \equiv p$
(iv)	**Double negation laws**	$\sim(\sim p) \equiv p$
(v)	**Commutative laws**	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
(vi)	**Associative laws**	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
(vii)	**Distributive laws**	(i) left distributive laws $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (ii) right distributive laws $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$ $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$
(viii)	**De Morgan's law**	$\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$
(ix)	**Negation law**	$\sim(\sim p) \equiv p$ $p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$

Precedences to be followed while appliying logical operators

As there are precedences, which are observed while applying the operations of multiplication, division, addition on algebraic numbers as e.g.,

Sub.

$$a \div b \times c = \left(\frac{a}{b}\right) \times c$$

(Division is given precedence over multiplication)

$$\neq a \div bc$$

$$a \cdot b + c = (a \cdot b) + c$$

(Multiplication is given precedence over addition)

$$\neq a \cdot (b + c)$$

similarly there are orders of precedence to be followed while applying logical operations the precedence order number are given by the following table :

Table of precedence order of logical operators

Operator	Order of precedence	
Negation ($\sim$)	1^{st} preference is given to negation over all others	
Conjunction ($\wedge$)	II^{nd}	so $\sim p \wedge q = (\sim p) \wedge q$ and $\neq \sim (p \wedge q)$
Disjunction ($\vee$)	III^{rd}	so $p \wedge q \vee r = (p \wedge q) \vee r$ and not $p \wedge (q \vee r)$
Implication ($\rightarrow$)	4^{th}	so $p \vee q \rightarrow r \equiv (p \vee q) \rightarrow r$ and $\neq p \vee (q \rightarrow r)$
Biconditional ($\leftrightarrow$)	5^{th}	

Examples : By orders of precedences of the logical connectives, we note that :

(i) $\sim p \vee q \quad = (\sim p) \vee q$ and is $\neq \sim (p \vee q)$

(ii) $p \wedge q \vee r \quad = (p \wedge q) \vee r$ and is $\neq p \wedge (q \vee r)$

(iii) $p \wedge q \rightarrow r \quad = (p \wedge q) \rightarrow r$ and is $\neq p \wedge (q \rightarrow r)$

(iv) $p \rightarrow q \leftrightarrow r \quad = (p \rightarrow q) \leftrightarrow r$ and is $\neq p \rightarrow (q \leftrightarrow r)$

Ex. Verify the distributive laws :

(i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Sol.: (i)

p	**q**	**r**	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	
T	T	T	$T \wedge T = T$	$T \vee T = T$	
T	T	F	$T \wedge T = T$	$T \vee F = T$	As the columns (i) and (ii) representing L.H.S. and R.H.S. of (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ are identical, so the equivalence (i) holds true.
T	F	T	$T \wedge T = T$	$F \vee T = T$	
T	F	F	$T \wedge F = F$ (i)	$F \vee F = F$ (ii)	
F	T	T	$F \wedge T = F$	$F \vee F = F$	
F	T	F	$F \wedge T = F$	$F \vee F = F$	
F	F	T	$F \wedge T = F$	$F \vee F = F$	
F	F	F	$F \wedge F = F$	$F \vee F = F$	

(ii)

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	
T	T	T	$T \vee T = T$	$T \wedge T = T$	As the columns (i) and (ii) representing L.H.S. and R.H.S. of $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ are identically same, so the equivalence (ii) holds.
T	T	F	$T \vee F = T$	$T \wedge T = T$	
T	F	T	$T \vee F = T$	$T \wedge T = T$	
T	F	F	$T \vee F = T$ (i)	$T \wedge T = T$ (ii)	
F	T	T	$F \vee T = T$	$T \wedge T = T$	
F	T	F	$F \vee \cancel{T} = F$	$T \wedge F = F$	
F	F	T	$F \vee F = F$	$F \wedge T = F$	
F	F	F	$F \vee F = F$	$F \wedge F = F$	

Ex. Show that $\sim(p \vee (\sim p \wedge q))$ is (identically equivalent to) $\equiv \sim p \wedge \sim q$, without using truth table.

Sol.: L.H.S. $= \sim (p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim (\sim p \wedge q)$ by De Morgan's laws

$\equiv \sim p \wedge (p \vee \sim q)$, again by De Morgan's law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$, by distributive law

$\equiv F \vee (\sim p \wedge \sim q)$, by identity law

$\equiv \sim p \wedge \sim q$.

Ex. Prove that the following implications are tautologies:

(i) $p \wedge q \rightarrow p \vee q$, **(ii)** $\sim p \rightarrow (p \rightarrow q)$

Sol.: (i) $(p \wedge q) \rightarrow (p \vee q)$ By definition of conditional

$$\equiv \sim (p \wedge q) \vee (p \vee q)$$

$$\equiv (\sim p \vee \sim q) \vee p \vee q$$

using associative and commutative laws for disjunction

$$\equiv (\sim p \vee p) \vee (\sim q \vee q)$$

$$\equiv (\sim p \vee p) \vee (\sim q \vee q)$$

$$\equiv T \vee T = T$$ **Hence proved**

(ii) $\sim p \rightarrow (p \rightarrow q)$

$$\equiv \sim (\sim p) \vee (p \rightarrow q)$$

$$= p \vee (\sim p \vee q) = (p \text{ Ú} \sim p) \vee q = T \vee q = T$$

Ex. Prove the following equivalences:

(i) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(ii) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee r) \rightarrow r$

Sol.: (i) L.H.S. $= (p \rightarrow q) \wedge (p \rightarrow r)$

$\equiv (\sim p \vee q) \wedge (\sim p \vee r)$ as $\sim p$ is common so

$\equiv \sim p \vee (q \wedge r) \equiv p \rightarrow (q \wedge r)$

(ii) L.H.S. $= (p \rightarrow r) \wedge (q \rightarrow r)$

$\equiv (\sim p \vee r) \wedge (\sim q \vee r)$ as r is common so

By right distributive law

$\equiv (\sim p \wedge \sim q) \vee r$ using DeMorgan's law

$\equiv \sim (p \vee q) \vee r$ By definition of implication

$\equiv p \vee q \rightarrow r$

Ex. If p and q be the propositions

p : I purchased a lottery ticket last week

q : This week I won the lottery opened on Monday this week.

Given the english versions of the followign:

		English version
(i)	$\sim p$	I did not purchase the lottery ticket last week
(ii)	$p \wedge q$	I purchased a lottery ticket last week **and** has also won the lottery
(iii)	$q \rightarrow p$	I won the lottery on Monday **implies** I must have purchased a lottery ticket last week.
(iv)	$\sim p \rightarrow \sim q$	As I did not purchase a lottery ticket last week so I could not won the prize.
(v)	$\sim p \wedge \sim q$	I did not purchased the lottery ticket last week and did not won any prize.
(vi)	$\sim p \vee (p \wedge q)$	Either I did not purchase the lottery ticket last week or purchased and won.

Ex.16. Let p : I will study Discrete Mathematics

q : I will go to a movie

r : I will be in a good mood.

translate the following in English:

(i) $(\sim p \wedge q) \rightarrow r$ (ii) $r \rightarrow p \vee r$

(iii) $\sim r \rightarrow \sim q \vee p$ (iv) $q \wedge (\sim p) \leftrightarrow r$

Ans.: (i) If I do not study DMS and go to a movie I will be in good mood.

(ii) If I be in good mood then I will read DMS or will go to a movie.

(iii) If I am not in a good mood then I will not go to a movie or I will study DMS.

(iv) If I am in a good mood it implies and is implied by that I do not study DMS and go to a movie.

❖ **Predicate:** The statement

“x is a positive integer”

The statement cannot has a truth value unless the value of the variable x is specified. The first part, i.e., the variable ‘x’ is called the subject of the statement, while the second part, i.e., “is a positive integer”- refers to a property that the subject of the statement can have, is called the predicate. We can express the above statement by P(x), where P denotes the predicate “is a positive integer” and ‘x’ is the variable. The statement P(x) is also called a propositional function because once a value has been assigned to ‘x’, It's becomes a proposition and has it truth value.

We will use the notation $P(x)$ to denote the expression ‘x has the property P’. As some value is assigned to x, then $P(x)$ has a truth value T or F and it becomes a proposition.

Let D be a given set. A propositional function or predicate is a sentence $P(x)$ for which value of x in D is a proposition i.e. $P(x)$ is true or false for each $x \in D$. D is called the *domain or universe of discourse* and x is called a free variable.

❖**First order Logic :** The logic based upon the analysis of predicates in any statement is called predicate logic or first order logic.

Ex:- Consider the propositional functions

$P(x)$: x is a multiple of 3.

$Q(x,y)$: $x = y + 3$.

where the domain is the set of integers Z. What are the truth values of P(4), P(9), Q(7,4) and Q(2,2) ?

Solution:

$P(4)$: 4 is multiple of 3, is false.

$P(9)$: 9 is multiple of 3, is true.

$Q(7, 4)$: $7 = 4 + 3$ is true.

$Q(2, 2)$: $2 = 2 + 3$ is false.

❖Quantifier:

- **Universal Quantifier** **denoted by** $\forall$
- **Existential Quantifier** **denoted by** $\exists$

(*a*) **The Universal quantifier:** Suppose that P(x) is a propositional function with domain D. The universal quantification of P(x) is the proposition that asserts that P(x) is true for all values of x in the universe of discourse D i.e. 'P(x) is true for all values of x in D' written '$\forall\ x$ P(x)' and read 'for all x P(x)' or 'for every x P(x)'. The symbol $\forall$ is read as 'for all' or 'for every'.

Example 1.23 *What is the truth value of* $\forall\ x\ (x^2 \geq x)$ *if the universe of discourse consists of all real numbers and what is its truth value if the universe of discourse consists of all integers.*

Solution: $x^2 \geq x$

$\Rightarrow \quad x\ (x-1) \geq 0$

$\Rightarrow \quad x \leq 0$ or $x \geq 1$

$\therefore\ \forall\ x,\ (x^2 \geq x)$ is false if domain consists of all real numbers since the inequality is false for x with $0 < x < 1$. However, if the universe of discourse consists of integers, $\forall\ x(x^2 \geq x)$ is true since there are no integers x with $0 < x < 1$.

Thus, we can write

$$\forall\ x\ P(x) \text{ is false, } x \in R,$$

$$\forall\ x\ P(x) \text{ is true, } x \in Z.$$

(b) The Existential Quantifier: With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value is the universe of discourse.

The existential quantification of $P(x)$ is the proposition. 'There exists an element x in domain D such that $P(x)$ in true' denoted by '$\exists\, xP(x)$' read as 'There exists x such that $P(x)$' or 'for some x $P(x)$' or 'There is at least one x such that $P(x)$.' The symbol $\exists$ is read as 'there exists'.

Example 1.25 *Let $P(x)$: $x > 3$. What is the truth value of the quantification $\exists x\, P(x)$ where the domain is the set of real numbers.*

Solution: $\because$ $x > 3$ is true for instance when $x = 3.5$. Thus, there exists x, such that $x > 3$ is true.

$\therefore$ $\exists\, xP(x)$ is true.

Table 1.30 Table of Quantifiers for one variable

Statement	When True ?	When False ?
$\forall x P(x)$	$P(x)$ is true for every x.	There is at least one x for which $P(x)$ is false.
$\exists x P(x)$	There is atleast one x for which $P(x)$ is true.	$P(x)$ is false for every x.

❖Properties Quantifier:

The negation of a quantified statement changes the quantifier and also negates the given statement as mentioned below :

(i) $\sim(\forall x P(x)) \equiv \exists x \sim P(x)$ (De Morgan's Law)

(ii) $\sim(\exists x P(x)) \equiv \forall x \sim P(x)$ (De Morgan's Law)

(iii) $\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$

(iv) $\exists x P(x) \rightarrow \forall x Q(x) \equiv \forall x(P(x) \rightarrow Q(x))$

(v) $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

(vi) $\sim(\exists x \sim P(x)) \equiv \forall x P(x)$

❖Duality Law :

Two formulae A and A* involving logical connectives $\vee$, $\wedge$ and ~ are said to be duals of each other if either one can be obtained from the other by replacing

(i) '$\wedge$, conjunction in one by $\vee$, disjunction in other' and

(ii) '$\vee$ disjunction in one by $\wedge$, conjunction in other' and

(iii) T by F and (iv) F by T.

So these two connectives $\wedge$ and $\vee$ are called duals of each other.

Ex: Write the duals of the following

(i) $(p \vee q) \wedge r$

(ii) $\sim(p \vee q) \wedge (r \vee s)$

Sol:

(i) $(p \wedge q) \vee r$

(ii) $\sim(p \wedge q) \vee (r \wedge s)$

❖NORMAL FORMS:

(A) Disjunctive Normal Form (DNF) or the sum of the products form

(B) Conjunctive Normal Form (CNF) or the products of Sum form

Minterm: A *minterm* is conjunction of propositional variables where each variable appears only once either as variable or its negation. Each minterm has truth value **T** corresponding to exactly one row in its truth table.

There are 2^n minterms for n variables as there are two choices (p and $\sim p$) for each variable p. e.g. For propositions p and q, there are $2^2 = 4$ minterms namely $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$.

Maxterm: A *maxterm* is disjunction of propositional variables where each variable appears only once either as variable or its negation. Each maxterm has truth value F corresponding to exactly one row in the truth table.

There are 2^n maxterms for n variables.

e.g. for propositions p and q, there are $2^2 = 4$ maxterms namely $p \vee q$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$.

The compliment of a minterm is respective maxterm.

e.g. $\sim(\sim a \wedge b) \equiv a \vee \sim b$ [De Morgan's law]

There is a correspondence between the truth table and minterms and maxterms.

Minterms and maxterms for a 3 variable function $P(x, y, z)$.

Row	x	y	z	P	Minterm	Maxterm
0	F	F	F	P(F,F,F)	$\sim x \wedge \sim y \wedge \sim z$	$x \vee y \vee z$
1	F	F	T	P(F,F,T)	$\sim x \wedge \sim y \wedge z$	$x \vee y \vee \sim z$
2	F	T	F	P(F,T,F)	$\sim x \wedge y \wedge \sim z$	$x \vee \sim y \vee z$
3	F	T	T	P(F,T,T)	$\sim x \wedge y \wedge z$	$x \vee \sim y \vee \sim z$
4	T	F	F	P(T,F,F)	$x \wedge \sim y \wedge \sim z$	$\sim x \vee y \vee z$
5	T	F	T	P(T,F,T)	$x \wedge \sim y \wedge z$	$\sim x \vee y \vee \sim z$
6	T	T	F	P(T,T,F)	$x \wedge y \wedge \sim z$	$\sim x \vee \sim y \vee z$
7	T	T	T	P(T,T,T)	$x \wedge y \wedge z$	$\sim x \vee \sim y \vee \sim z$

(A) Disjunctive Normal Form (DNF): A proposition form which consists of disjunction (sum) of fundamental conjunctions (products) is called a *disjunctive normal form* (DNF). It is similar to canonical sum of products used in the circuit theory.

e.g. (i) $(p \wedge q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$

(ii) $(p \wedge \sim q) \vee (p \wedge r)$

(iii) $(p \wedge q \wedge r) \vee \sim r$

(iv) $(p \wedge q) \vee \sim r$

❖ **Principle Disjunctive Normal Form (PDNF) :** If disjunctive normal form consists of disjunction of minterms only , it is called Principle Disjunctive Normal Form (PDNF)

e.g. (i) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(ii) $(p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$

(iii) $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

Methods to obtain DNF:

(i) **Using Logical Equivalence:** In order to obtain DNF of a given proposition, first replace the conditionals and biconditionals by their equivalent formulae containing only the connectives, $\wedge$, $\vee$ and $\sim$. Next apply the laws of algebra of equivalence to obtain DNF.

Example 1.49 *Obtain the DNF of the proposition* $(p \rightarrow q) \wedge (\sim p \wedge q)$.

Solution:

$$(p \rightarrow q) \wedge (\sim p \wedge q) \equiv (\sim p \vee q) \wedge (\sim p \wedge q)$$

[$p \rightarrow q \equiv \sim p \vee q$ [Elimination of conditional]]

$$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q) \quad \text{[Distributive law]}$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge q) \quad \text{[Idempotent law]}$$

$$\equiv (\sim p \wedge q)$$

(ii) **Using Truth Table:** Let P be a statement containing n variables $p_1, p_2 \dots p_n$. Its DNF is obtained from the truth table as follows: For each row in which P assumes value T, form the conjunction $p_1 \wedge p_2 \wedge \dots \wedge \sim p_j \wedge \dots \sim p_n$, where we write p_j if it has truth table T in that row and $\sim p_j$ if it has value F in that row. Now write the disjunction of all the above obtained conjunctions (minterms). This method always gives you PDNF.

Example 1.50 *Find the PDNF of $(\sim p \to r) \wedge (p \leftrightarrow q)$ using truth table.*

Solution: Truth table for $(\sim p \to r) \wedge (p \leftrightarrow q)$ is

p	q	r	$\sim p$	$\sim p \to r$	$p \leftrightarrow q$	$(\sim p \to r) \wedge (p \leftrightarrow q)$
$\boxed{T}$	$\boxed{T}$	$\boxed{T}$	F	T	T	$\boxed{T}$
$\boxed{T}$	$\boxed{T}$	$\boxed{F}$	F	T	T	$\boxed{T}$
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
$\boxed{F}$	$\boxed{F}$	$\boxed{T}$	T	T	T	$\boxed{T}$
F	F	F	T	F	T	F

Here, T appears in last column in I, II and VII rows.

In I row, p, q, r all have value T. Hence the minterm corresponding to this row is $(p \wedge q \wedge r)$.

In II row, p and q take value T while r takes value F. Hence, the corresponding minterm is $(p \wedge q \wedge \sim r)$.

In VII row, r takes value T and p and q take the value of F. Therefore, the corresponding minterm is $(\sim p \wedge \sim q \wedge r)$.

Thus, the required PDNF is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r).$$

(b) Conjunctive Normal Form (CNF) (This is De Morgan dual of DNF):

A propositional form which consists of conjunction (product) of fundamental disjunctions (sum) is called a *Conjunctive normal form* **(CNF)**. It is similar to canonical product of sums used in circuit theory and Boolean algebra.

e.g. $(p \vee q) \wedge \sim q$, $(p \vee q \vee r) \wedge \sim r$, $(p \vee \sim q) \wedge (p \vee r)$,
$(p \vee q \vee r) \wedge (p \vee r) \wedge (q \vee r)$ etc.

❖ **Principle Conjunctive Normal Form (PCNF) :** If disjunctive normal form consists of conjunction of maxterms only , it is called Principle Conjunctive Normal Form (PCNF)

e.g. $(p \vee \sim q) \wedge (\sim p \vee \sim q)$, $(p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r)$,
$(p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$ etc.

Methods to obtain CNF:

(i) **Using Logical Equivalences:** The method to obtain CNF using logical equivalences is similar to the method of obtaining DNF. First eliminate the conditionals and biconditionals by their equivalent formulae containing only the connective $\wedge$, $\vee$ and $\sim$. Then apply the laws of algebra of equivalence to obtain CNF.

Example 1.52 *Obtain CNF of* $p \wedge (p \rightarrow q)$.

Solution: $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$ $[\because p \rightarrow q \equiv \sim p \vee q]$

$\equiv (p \vee (q \wedge \sim q)) \wedge (\sim p \vee q)$ $[\because q \wedge \sim q \equiv F]$

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$ Distributive law

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$

which is the required CNF.

(ii) **Using Truth Table:** Let P be a statement containing variables $p_1, p_2, \ldots p_n$. To find its CNF, for each row in which P assumes truth value F in truth table, form the disjunction $p_1 \vee p_2 \vee \ldots \vee \sim p_j \wedge \ldots \wedge p_n$ where we write $\sim p_j$ if it has truth value T and p_j if it has truth value F in that row. Now write the conjunction of all the above obtained disjunctions (maxterms). This method always gives you PCNF.

Example 1.53 *Find the PCNF of* $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$ *using truth table.*

Solution:

p	q	r	$\sim p$	$\sim p \rightarrow r$	$p \leftrightarrow q$	$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
$\boxed{\text{T}}$	$\boxed{\text{F}}$	$\boxed{\text{T}}$	F	T	F	$\boxed{\text{F}}$
$\boxed{\text{T}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$	F	T	F	$\boxed{\text{F}}$
$\boxed{\text{F}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$	T	T	F	$\boxed{\text{F}}$
$\boxed{\text{F}}$	$\boxed{\text{T}}$	$\boxed{\text{F}}$	T	F	F	$\boxed{\text{F}}$
F	F	T	T	T	T	T
$\boxed{\text{F}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$	T	F	T	$\boxed{\text{F}}$

Here, F appears in last column in III, IV, V, VI [illegible] is

$(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r)$.

1.7.1 To Obtain PCNF from PDNF and vice-versa

If PDNF (or PCNF) is known for a given proposition P containing n variables then PDNF (or PCNF) of ~P will consist of the minterms (or maxterms) which are not present in PDNF (or PCNF) of P. Since $\sim(\sim p) \equiv p$, PCNF (or PDNF) of P can be obtained by applying De-Morgan's laws to PDNF (or PCNF) of ~P.

Example 1.54 *Find PCNF of a statement* $(p \to q) \wedge (\sim p \wedge q)$ *whose PDNF is* $(\sim p \wedge q)$.

Solution: Let P : $(p \to q) \wedge (\sim p \wedge q) \equiv (\sim p \wedge q)$

Then PDNF of ~P is

$$\sim P \equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$$

[$\because$ For variables, p and q, there are $2^2 = 4$ minterms, namely $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$].

Thus, the required PCNF can be obtained by applying double negation law and De-Morgan's law.

$$\begin{aligned} P &\equiv \sim(\sim P) \equiv \sim((p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)) \\ &\equiv \sim(p \wedge q) \wedge \sim(p \wedge \sim q) \wedge \sim(\sim p \wedge \sim q) \\ &\equiv (\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q) \end{aligned}$$

2.2.2 Proof by Contradiction (Proving p)

In a proof by contradiction, we assume $\sim p$ i.e. the conclusion p is not true and then arrive at a contradiction. A famous example of a proof by contradiction is given below:

Example 2.7 *Show that $\sqrt{2}$ is irrational.*

Solution: Let p: $\sqrt{2}$ is irrational. Suppose that ~p is true i.e. $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some integer a and b having no common factor other than 1 and $b \neq 0$.

$\Rightarrow \quad a = b\sqrt{2}$

$\Rightarrow \quad a^2 = 2b^2$

$\Rightarrow \quad a^2$ is an even integer. ....(i)

As a^2 is an even integer, we must have a is an even integer.

$\Rightarrow \quad a = 2k$ for some integer k.

Substituting the value of a in (i), we get

$4k^2 = 2b^2$

$\Rightarrow \quad b^2 = 2k^2$

$\Rightarrow \quad b^2$ is even and hence b is an even integer.

Thus, we have proved that both a and b are even and have 2 as a common factor which is contradiction to our assumption that a and b have no common factors. Hence $\sqrt{2}$ is irrational.