Chapter 9

NUMERICAL DIFFERENTIATION AND INTEGRATION

9.1 INTRODUCTION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of data (x_i, y_i). To solve $\frac{dy}{dx}$, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = g(x)$ and then differentiate the latter as many times as we require. The interpolation formula to be used, will depend on the assigned value of x at which $\frac{dy}{dx}$ is desired.

If the values of x are equi-intervated and $\frac{dy}{dx}$ is required near the beginning of the table, we employ Newton's forward interpolation formula. If it is required near the end of the table, we employ Newton's Backward Interpolation formula and for values near the middle of the table we apply Stirling's formula.

9.2 NUMERICAL DIFFERENTIATION USING INTERPOLATION FORMULAE WITH EQUAL INTERVALS

Corresponding to each of the interpolation formulae, we can derive a formula for finding the derivatives,

(*i*) **Newton's Forward Difference Derivative Formula :**

Newton's forward interpolation formula is written as

$$P_n(x_0 + hu) = Y_n = y_0 + u\Delta y_0 + \frac{u(u-1)}{\underline{|2}} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\underline{|3}} \Delta^3 y_0 + \ldots\ldots \quad (9.1)$$

Differentiating it with respect to u, we get

$$\frac{dY_n}{du} = \Delta y_0 + \frac{2u-1}{\lfloor 2} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{\lfloor 3} \Delta^3 y_0 + \ldots\ldots \quad \ldots(9.2)$$

since $\quad u = \dfrac{x - x_0}{h}$, therefore $\dfrac{du}{dx} = \dfrac{1}{h}$

So,

$$\frac{dY_n}{dx} = \frac{dY_n}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{\lfloor 2} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{\lfloor 3} \Delta^3 y_0 + \ldots\ldots\right] \quad \ldots(9.3)$$

and at $x = x_0$, we get $u = 0$.

Hence, by putting $u = 0$, in equation (9.3), we get

$$\left(\frac{dY_n}{dx}\right)_{x = x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right] \quad \ldots(9.4)$$

Again, differentiating equation (9.3) w.r.t. x, we get

$$\frac{d^2 Y_n}{dx^2} = \frac{1}{h}\frac{d}{du}\left(\frac{dY_n}{du}\right) \cdot \frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\frac{2}{\lfloor 2}\Delta^2 y_0 + \frac{6u-6}{\lfloor 3}\Delta^3 y_0 + \frac{12u^2 - 36u + 22}{\lfloor 4} . \Delta^4 y_0 + \ldots\ldots\right]$$

or

$$\frac{d^2 Y_n}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \frac{6u^2 - 18u + 11}{12}\Delta^4 y_0 + \ldots\ldots\right] \quad \ldots(9.5)$$

and at $x = x_0$, by putting $u = 0$ in equation (9.5), we get

$$\left(\frac{d^2 Y_n}{dx^2}\right)_{x = x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\ldots\right] \quad \ldots(9.6)$$

Similarly,

$$\left(\frac{d^3 Y_n}{dx^3}\right)_{x = x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\right] \quad \ldots(9.7)$$

(*ii*) Newton's Backward Difference Derivative Formula :

Newton's backward interpolation formula is written as

$$P_n(x_n + hu) = Y_n = y_n + u\nabla y_n + \frac{u(u+1)}{\lfloor 2}\nabla^2 y_n + \frac{u(u+1)(u+2)}{\lfloor 3}\nabla^3 y_n + \ldots\ldots \quad (9.8)$$

Differentiating it with respect to u, we get

$$\frac{dY_n}{du} = \nabla y_n + \frac{2u+1}{\lfloor 2}\nabla^2 y_n + \frac{3u^2 + 6u + 2}{\lfloor 3}\nabla^3 y_n + \ldots\ldots \quad \ldots(9.9)$$

since $u = \frac{x - x_n}{h}$, therefore $\frac{du}{dx} = \frac{1}{h}$

So, $\frac{dY_n}{dx} = \frac{dY_n}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{\underline{|2}} \nabla^2 y_n + \frac{3u^2+6u+2}{\underline{|3}} \nabla^3 y_n + \ldots\ldots\right] \qquad \ldots(9.10)$$

and at $x = x_n$, we get $u = 0$.

Hence, by putting $u = 0$ in equation (9.10), we get

$$\left(\frac{dY_n}{dx}\right)_{x = x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\ldots\right] \qquad \ldots(9.11)$$

Again, differentiating equation (9.10) with respect to x, we get

$$\frac{d^2Y_n}{dx^2} = \frac{1}{h}\frac{d}{du}\left(\frac{dY_n}{du}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12} . \nabla^4 y_n + \ldots\ldots\right]$$

$$\ldots(9.12)$$

and at $x = x_n$, by putting $u = 0$ in equation (9.12), we get

$$\left[\frac{d^2Y_n}{dx^2}\right]_{x = x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \ldots\ldots\right] \qquad \ldots(9.13)$$

Similarly,

$$\left(\frac{d^3Y_n}{dx^3}\right)_{x = x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\ldots\right] \qquad \ldots(9.14)$$

***(iii)* Stirling's Central Difference Derivative Formula :**

Stirling's central interpolation formula is written as

$$P_n(x_0 + hu) = Y_n = y_0 + u\,(\mu\delta y_0) + \frac{u^2}{\underline{|2}}(\delta^2 y_0)$$

$$+ \frac{u(u^2-1)}{\underline{|3}}(\mu\delta^3 y_0) + \frac{u^2(u^2-1)}{\underline{|4}}(\delta^4 y_0) + \ldots\ldots \qquad \ldots(9.15)$$

differentiating it with respect to u, we get

$$\frac{dY_n}{du} = (\mu\delta y_0) + \frac{2u}{\underline{|2}}(\delta^2 y_0) + \frac{3u^2-1}{\underline{|3}}(\mu\delta^3 y_0)$$

$$+ \frac{4u^3-2u}{\underline{|4}}(\delta^4 y_0) + \ldots\ldots \qquad (9.16)$$

since $u = \frac{x - x_0}{h}$, therefore $\frac{du}{dx} = \frac{1}{h}$

So, $$\frac{dY_n}{dx} = \frac{dY_n}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[(\mu\delta y_0) + u\,(\delta^2 y_0) + \frac{(3u^2 - 1)}{3!}(\mu\delta^3 y_0) + \frac{4u^3 - 2u}{4!}(\delta^4 y_0) + \ldots\ldots\right] \quad \ldots(9.17)$$

and at $x = x_0$, we get $u = 0$.

Hence, by putting $u = 0$ in equation (9.17), we get

$$\left(\frac{dY_n}{dx}\right)_{x = x_0} = \frac{1}{h}\left[(\mu\delta y_0) - \frac{1}{6}(\mu\delta^3 y_0) + \frac{1}{30}(\mu\delta^5 y_0) - \ldots\ldots\right] \quad \ldots(9.18)$$

Again, differentiating equation (9.17) with respect to x, we get

$$\frac{d^2 Y_n}{dx^2} = \frac{1}{h}\frac{d}{du}\left(\frac{dY_n}{du}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\delta^2 y_0 + u\,(\mu\delta^3 y_0) + \frac{6u^2 - 1}{12}\delta^4 y_0 + \ldots\ldots\right] \quad \ldots(9.19)$$

and at $x = x_0$, by putting $u = 0$ in equation (9.19), we get

$$\left(\frac{d^2 Y_n}{dx^2}\right)_{x = x_0} = \frac{1}{h^2}\left[\delta^2 y_0 - \frac{1}{12}\delta^4 y_0 + \frac{1}{90}\delta^6 y_0 - \ldots\ldots\right] \quad \ldots(9.20)$$

Remark : While evaluating derivatives at tabular points, use equations (9.4), (9.6), (9.7), (9.11), (9.13), (9.14), (9.18), (9.20) etc.

9.3 NUMERICAL DIFFERENTIATION USING LAGRANGE'S INTERPOLATION FORMULA WITH UNEQUAL INTERVALS

Using given data with unequal intervals in Lagrange's interpolation formula, find the polynomial function representing that data in form of $y = f(x)$ and hence $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ etc. can be obtained at any value of x.

ILLUSTRATIVE EXAMPLES

1. *(i) Use following table to find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $x = 1.1$

x :	1.0	1.2	1.4	1.6	1.8	2.0
y :	0.000	0.1280	0.5440	1.2960	2.4320	4.0000

(Raj. 2006)

(ii) Use following table to find first and second derivatives of $f(x)$ *at* $x = 3.0$

x :	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	− 14.000	− 10.032	− 5.296	0.256	6.672	14.000

(iii) *A rod is rotating in a plane about one of its end points. If the following table give the angle θ radians through which the rod has turned for different values of time t seconds, then find its angular velocity and angular acceleration at t = 0.7 seconds*

t (in seconds) :	0.0	0.2	0.4	0.6	0.8	1.0
θ (in radians) :	0.00	0.12	0.48	1.10	2.00	3.20

(iv) *Use following table to find* $\frac{dy}{dx}$ *at x = 1.5*

x :	0.0	0.5	1.0	1.5	2.0
y :	0.3989	0.3521	0.2420	0.1295	0.0540

(Raj. 2002, 03)

(v) *Use following table to find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at x = 1.35*

x :	1.1	1.2	1.3	1.4	1.5	1.6
y :	– 1.62628	0.15584	2.45256	5.39168	9.12500	13.83072

(vi) *A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod, is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration at t = 0.3 seconds.*

t :	0.0	0.1	0.2	0.3	0.4	0.5	0.6
x :	30.13	31.62	32.87	33.64	33.95	33.81	33.24

[Raj. 1998, 2001]

(vii) *Use following table to find f '(6)*

x :	0	1	3	4	5	7	9
f(x) :	150	108	0	– 54	– 100	– 144	– 84

(Raj. 2007)

(viii) *Use following table to find value of x for which y is maximum and also find this value of y.*

x :	1.2	1.3	1.4	1.5	1.6
y :	0.9320	0.9636	0.9855	0.9975	0.9996

Solution. (*i*) Since x = 1.1 lies towards the starting of the table and variable x is equidistant, therefore we use Newton's forward difference derivative formula :

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.0000					
		0.1280				
1.2	0.1280		0.2880			
		0.4160		0.0480		
1.4	0.5440		0.3360		0	
		0.7520		0.0480		0
1.6	1.2960		0.3840		0	
		1.1360		0.0480		
1.8	2.4320		0.4320			
		1.5680				
2.0	4.0000					

Here, $u = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.2} = \frac{1}{2}$

We know that

$$Y_n = y_0 + u\Delta y_0 + \frac{u(u-1)}{\underline{|2}} \Delta^2 y_0 + \ldots\ldots \quad \ldots(1)$$

differentiating (1) w.r.t. x, we get

$$\frac{dY_n}{dx} = \frac{dY_n}{du} \frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \ldots\ldots\right] \quad \ldots(2)$$

and $$\frac{d^2 Y_n}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \ldots\ldots\right] \quad \ldots(3)$$

Replacing $h = 0.2$, $u = \frac{1}{2}$ in equations (2) and (3) and using difference table, we get

$$\left(\frac{dY_n}{dx}\right) = \left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.1280 + 0 + \frac{1}{6}\left(3 * \frac{1}{4} - 6 * \frac{1}{2} + 2\right) \times .0480\right]$$

$$= 0.63$$

and $$\left(\frac{d^2 Y_n}{dx^2}\right) = \left(\frac{d^2 y}{dx^2}\right)_{x=1.1} = \frac{1}{(0.2)^2}\left[0.2880 + \left(\frac{1}{2} - 1\right)(0.0480)\right]$$

$$= 6.6$$

(*ii*) Since $x = 3.0$ lies at the starting of the table and variable x is equidistant, therefore we use Newton's forward difference derivative formula.

Forward Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3.0	– 14.0000				
		3.968			
3.2	– 10.032		0.768		
		4.736		0.048	
3.4	– 5.296		0.816		0
		5.552		0.048	
3.6	0.256		0.864		0
		6.416		0.048	
3.8	6.672		0.912		
		7.328			
4.0	14.000				

Here, $u = \frac{x - x_0}{h} = \frac{3-3}{0.2} = 0$

So, $[f'(x)]_{x=3} = \left(\frac{dy}{dx}\right)_{x=3} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$

$$= \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(0.048)\right]$$

$$= 18$$

and $[f''(x)]_{x=3} = \left(\frac{d^2y}{dx^2}\right)_{x=3} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\ldots\right]$

$$= \frac{1}{.04}[0.768 - 0.048]$$

$$= 18$$

(*iii*) Since $t = 0.7$ lies towards the end of the table and variable t is equidistant, therefore we use Newton's backward difference derivative formula.

Backward Difference Table

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
0.0	0.00				
		0.12			
0.2	0.12		0.24		
		0.36		0.02	
0.4	0.48		0.26		0
		0.62		0.02	
0.6	1.10		0.28		0
		0.90		0.02	
0.8	2.00		0.30		
		1.20			
1.0	3.20				

Here, $u = \frac{t - t_n}{h} = \frac{0.7 - 1}{0.2} = -1.5$

So, angular velocity at $t = 0.7$ seconds

$$\left(\frac{d\theta}{dt}\right)_{t=0.7} = \frac{1}{h}\left[\nabla\theta_n + \frac{2u+1}{2}\nabla^2\theta_n + \frac{3u^2+6u+2}{6}\nabla^3\theta_n + \ldots\ldots\right]$$

$$= \frac{1}{0.2}\left[1.20 + \frac{2(-1.5)+1}{2}(0.30) + \frac{3(-1.5)^2 + 6(-1.5) + 2}{6}(0.02)\right]$$

$$= 5\,[1.20 - 0.30 - 0.0008] = 4.496 \text{ rad/sec.}$$

and angular acceleration at $t = 0.7$ seconds

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.7} = \frac{1}{h^2}\left[\nabla^2\theta_n + (u+1)\,\nabla^3\theta_n + \ldots\ldots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.30 - 0.5 \times 0.02\right]$$

$$= 7.25 \text{ rad/sec}^2.$$

(*iv*) Since $x = 1.5$ lies towards the end of the table and variable x is equidistant, therefore we use Newton's backward difference derivative formula.

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	0.3989				
		− 0.0468			
0.5	0.3521		− 0.0633		
		− 0.1101		0.0609	
1.0	0.2420		− 0.0024		− 0.0215
		− 0.1125		0.0394	
1.5	0.1295		0.0370		
		− 0.0755			
2.0	0.0540				

Here, $$u = \frac{x - x_n}{h} = \frac{1.5 - 1.5}{0.5} = 0$$

So,

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\ldots\right]$$

$$= \frac{1}{0.5}\left[-0.1125 + \frac{1}{2}(-0.0024) + \frac{1}{3}(0.0609)\right]$$

$$= -0.1868.$$

(*v*) Since $x = 1.35$ lies towards the middle of the table and variable x is equidistant, therefore we use Stirling's central difference derivative formula.

Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
1.1	– 1.62628					
		1.78212				
1.2	0.15584		0.5146			
		2.29672		0.1278		
1.3	2.45256		0.6424		0.0240	
		2.93912		0.1518		0.0024
1.4	5.39168		0.7942		0.0264	
		3.73332		0.1782		
1.5	9.12500		0.9724			
		4.70572				
1.6	13.83072					

Here, $$u = \frac{x - x_0}{h} = \frac{1.35 - 1.3}{0.1} = 0.5$$

So, $$\left(\frac{dy}{dx}\right)_{x = 1.35} = \frac{1}{h}\left[\mu\delta y_0 + u(\delta^2 y_0) + \frac{(3u^2 - 1)}{\lfloor 3}(\mu\delta^3 y_0) + \frac{4u^3 - 2u}{\lfloor 4}(\delta^4 y_0) + \ldots\ldots\right]$$

$$= \frac{1}{0.1}\left[2.61792 + (0.5)(0.6424) + \frac{1}{6}(0.75 - 1)(0.1398) + \frac{1}{24}(-0.5)(0.0240)\right]$$

$$= 29.32795$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x = 1.35} = \frac{1}{h^2}\left[\delta^2 y_0 + u(\mu\delta^3 y_0) + \frac{6u^2 - 1}{12}\delta^4 y_0 + \ldots\ldots\right]$$

$$= \frac{1}{(0.1)^2}\left[0.6424 + (0.5)(0.1398) + \frac{1}{12}(0.5)(0.0240)\right]$$

$$= 71.33$$

(*vi*) Since the velocity and acceleration are required at the midde time $t = 0.3$ with equidistant values of t, therefore we use Stirling's central difference derivative formula.

Central Difference Table

t	x	δx	$\delta^2 x$	$\delta^3 x$	$\delta^4 x$	$\delta^5 x$	$\delta^6 x$
0.0	30.13						
		1.49					
0.1	31.62		– 0.24				
		1.25		– 0.24			
0.2	32.87		– 0.48		0.26		
		0.77		0.02		– 0.27	
0.3	33.64		– 0.46		– 0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		– 0.45		0.01		
		– 0.14		0.02			
0.5	33.81		– 0.43				
		– 0.57					
0.6	33.24						

Here, $u \frac{x - x_0}{h} = \frac{0.3 - 0.3}{0.1} = 0$

So, velocity at x = 0.3 seconds

$$\left(\frac{dt}{dx}\right)_{x=0.3} = \frac{1}{h}\left[\mu\delta x_0 - \frac{1}{6}(\mu\delta^3 x_0) + \frac{1}{30}(\mu\delta^5 x_0) - ......\right]$$

$$= \frac{1}{0.1}\left[0.54 - \frac{1}{6}(0.015) + \frac{1}{30}(-0.125)\right]$$

$$= 5.33 \text{ cm/sec}$$

and acceleration at x = 0.3 seconds

$$\left(\frac{d^2t}{dx^2}\right)_{x=0.3} = \frac{1}{h^2}\left[\delta^2 x_0 - \frac{1}{12}(\delta^4 x_0) + \frac{1}{90}(\delta^6 x_0) - ......\right]$$

$$= \frac{1}{(0.1)^2}\left[(-0.46) - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29)\right]$$

$$= -45.594 \text{ cm/sec}^2.$$

(*vii*) Since variable x is given with unequal intervals, therefore we use Lagrange's interpolation formula first to find polynomial represented by the given data as

$$y = f(x) = \frac{(x-1)(x-3)(x-4)(x-5)(x-7)(x-9)}{(0-1)(0-3)(0-4)(0-5)(0-7)(0-9)}(150)$$

$$+ \frac{(x-0)(x-3)(x-4)(x-5)(x-7)(x-9)}{(1-0)(1-3)(1-4)(1-5)(1-7)(1-9)}(108)$$

$$+ 0 + \frac{(x-0)(x-1)(x-3)(x-5)(x-7)(x-9)}{(4-0)(4-1)(4-3)(4-5)(4-7)(4-9)}(-54)$$

$$+ \frac{(x-0)(x-1)(x-3)(x-4)(x-7)(x-9)}{(5-0)(5-1)(5-3)(5-4)(5-7)(5-9)}(-100)$$

$$+ \frac{(x-0)(x-1)(x-3)(x-4)(x-5)(x-9)}{(7-0)(7-1)(7-3)(7-4)(7-5)(7-9)}(-144)$$

$$+ \frac{(x-0)(x-1)(x-3)(x-4)(x-5)(x-7)}{(9-0)(9-1)(9-3)(9-4)(9-5)(9-7)}(-84)$$

$$\Rightarrow \quad y = f(x) = \frac{150}{3780}(x^3 - 8x^2 + 19x - 12)(x^3 - 21x^2 + 143x - 315)$$

$$- \frac{108}{1152}(x^3 - 7x^2 + 12x)(x^3 - 21x^2 + 143x - 315)$$

$$+ \frac{54}{180}(x^3 - 4x^2 + 3x)(x^3 - 21x^2 + 143x - 315)$$

$$- \frac{100}{320}(x^3 - 4x^2 + 3x)(x^3 - 20x^2 + 127x - 252)$$

$$+ \frac{144}{2016}(x^3 - 4x^2 + 3x)(x^3 - 18x^2 + 101x - 180)$$

$$- \frac{84}{17280}(x^3 - 4x^2 + 3x)(x^3 - 16x^2 + 83x - 140)$$

$$\Rightarrow \quad f'(x) = \frac{5}{126}[(x^3 - 8x^2 + 19x - 12)(3x^2 - 42x + 143)$$
$$+ (3x^2 - 16x + 19)(x^3 - 21x^2 + 143x - 315)]$$

$$- \frac{3}{32}[(x^3 - 7x^2 + 12x)(3x^2 - 42x + 143)$$
$$+ (3x^2 - 14x + 12)(x^3 - 21x^2 + 143x - 315)]$$

$$+ \frac{3}{10}[(x^3 - 4x^2 + 3x)(3x^2 - 42x + 143)$$
$$+ (3x^2 - 8x + 3)(x^3 - 21x^2 + 143x - 315)]$$

$$- \frac{5}{16}[(x^3 - 4x^2 + 3x)(3x^2 - 40x + 127)$$
$$+ (3x^2 - 8x + 3)(x^3 - 20x^2 + 127x - 252)]$$

$$+ \frac{1}{28}[(x^3 - 4x^2 + 3x)(3x^2 - 36x + 101)$$
$$+ (3x^2 - 8x + 3)(x^3 - 18x^2 + 101x - 180)]$$

$$- \frac{7}{1440}[(x^3 - 4x^2 + 3x)(3x^2 - 32x + 83)$$
$$+ (3x^2 - 8x + 3)(x^3 - 16x^2 + 83x - 140)]$$

$$\Rightarrow \quad f'(6) = \frac{5}{126}[(30)(-1) + (31)(3)]$$

$$- \frac{3}{32}[(36)(-1) + (36)(3)] + \frac{3}{10}[(90)(-1) + (63)(3)]$$

$$-\frac{5}{16}[(90)(-5)+(63)(6)]+\frac{1}{28}[(90)(-7)+(63)(-6)]$$
$$-\frac{7}{1440}[(90)(-1)+(63)(-2)]$$

$$\Rightarrow \quad f'(6)=\frac{5}{126}(63)-\frac{3}{32}(72)+\frac{3}{10}(99)-\frac{5}{16}(-72)$$
$$+\frac{1}{28}(-1008)-\frac{7}{1440}(-216)$$

$$\Rightarrow \quad f'(6)=\frac{5}{2}-\frac{27}{4}+\frac{297}{10}+\frac{45}{2}-36+\frac{21}{20}$$
$$=\frac{1}{20}(50-135+594+450-720+21)$$

$$\Rightarrow \quad f'(6)=\frac{1}{20}(260)=13$$
$$=-42-44+780-1360.8+695.772-51.84$$
$$=-22.868 \text{ (approx.)}$$

(*viii*)

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320			
		0.0316		
1.3	0.9636		− 0.0097	
		0.0219		− 0.0002
1.4	0.9855		− 0.0099	
		0.0120		− 0.0000
1.5	0.9975		− 0.0099	
		0.0021		
1.6	0.9996			

Assuming $x_0 = 1.2$ for maximum value of y, we have $\left(\frac{dy}{dx}\right)_{x=1.2} = 0$ and by using Newton's Forward difference derivative formula, we get

$$\left(\frac{dy}{dx}\right)_{x=1.2}=\frac{1}{h}\left[\Delta y_0+\frac{2u-1}{2}\Delta^2 y_0+\ldots\ldots\right]=0$$

$$\Rightarrow \quad \frac{1}{0.1}\left[0.0316+\frac{2u-1}{2}(-0.0097)\right]=0$$

$$\Rightarrow \quad u=3.8$$

So,
$$x=x_0+uh=1.2+3.8(0.1)$$
$$\Rightarrow \quad x=1.58$$

which is closer to $x = 1.6$.

Hence by applying backward interpolation formula, we get

$$Y_n = y_n + u\nabla y_n + \frac{u(u+1)}{\lfloor 2}\nabla^2 y_n + \ldots\ldots,$$

where
$$u = \frac{1.6 - 1.58}{0.1} = -0.2$$

$$\Rightarrow \quad y(1.58) = 0.9996 + (-0.2)(0.0021) + \frac{(-0.2)(-0.2+1)}{2!}(-0.0099)$$

$$= 1.0$$

EXERCISE 9.1

1. Use following table to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$

x :	1	2	3	4	5
y :	0	1	5	6	8

2. Use following table to find first and second derivatives of $f(x)$ at $x = 0$

x :	0	1	2	3	4	5
$f(x)$:	4	8	15	7	6	2

[Raj. 2001]

3. Use following table to find the annual rate of cloth sales of year 1935

Year :	1910	1920	1930	1940
Sale of : Cloth (in lakhs of metres)	250	285	328	444

4. Use following table to find $\frac{dy}{dx}$ at $x = 0.4$

x :	0.1	0.2	0.3	0.4
y :	1.10517	1.22140	1.34986	1.49182

5. Use following table to find $\frac{dy}{dx}$ at $x = 2.73$

x :	2.5	2.6	2.7	2.8	2.9	3.0
y :	0.4938	0.4953	0.4965	0.4974	0.4981	0.4987

6. If the following table gives the angle θ radians at different intervals of time t (in seconds), then find the angular velocity at $t = 0.06$ seconds.

t (in seconds) :	0.00	0.02	0.04	0.06	0.08	0.10	0.12
θ (in radians) :	0.052	0.105	0.168	0.242	0.327	0.480	0.489

7. Use following table to find $f'(2)$

x :	0	2	3
$f(x)$:	2	– 2	– 1

8. Use following table to find value of x for which y is minimum and also find this value of y.

x :	3	4	5	6	7	8
y :	0.205	0.240	0.259	0.262	0.250	0.224

ANSWERS

1. –0.82, 14.16
2. – 27.9, 117.67
3. 11.32917
4. 1.491
5. – 0.0282
6. 3.514 rad/sec
7. 0
8. $x = 5.6875$, $y = 0.263$

9.4 NUMERICAL INTEGRATION

The process of calculating a definite integral from a set of tabulated values of the integrand $f(x)$ is known as numerical integration. This process is known as quadrature, when applied to a function of one variable.

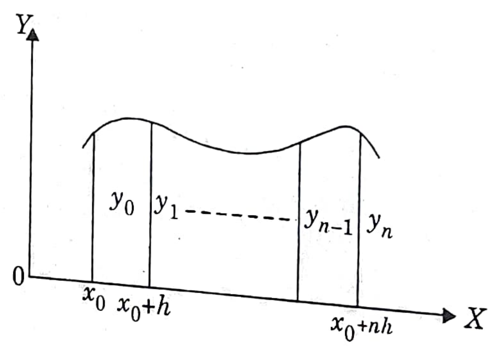

FIG. 9.1.

The problem of numerical integration, like that of numerical differentiation is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way we can formulate the quadrature formulae for approximate integration of a function defined by set of numerical values only.

9.5 GENERAL QUADRATURE FORMULAE FOR EQUIDISTANT VALUES OF INDEPENDENT VARIABLE

Here, we have to evaluate the definite integral $\int_a^b f(x)\, ds$, where $f(x)$ takes the values $y_0, y_1, y_2 \ldots\ldots y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$ as equal sub-intervals so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots\ldots, x_n = x_0 + nh = b$.

Thus,
$$I = \int_a^b f(x)\, dx = \int_{x_0}^{x_0+nh} f(x)\, dx = h \int_0^n f(x_0 + uh)\, du$$

where $x = x_0 + uh, \ dx = h\ du$

We know that

$$f(x_0 + uh) = y_0 + u\ \Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 y_0 + \ldots\ldots$$

(By Newton's forward interpolation formula)

$$\text{So, } I = h \int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 y_0 + \ldots\ldots \right) du$$

Integrating term by term, we get

$$I = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right) \frac{\Delta^4 y_0}{\lfloor 4} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n \right) \frac{\Delta^5 y_0}{\lfloor 5} + \ldots\ldots \right] \quad \ldots(9.21)$$

This is known as General Quadrature formula, form this general formula we deduce the following important quadrature rules by taking $n = 1, 2, 3$.

(*i*) **Trapezoidal Rule.** Putting $n = 1$ in equation (9.21) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.* a polynomial of first order so that differences of higher order than first order becomes zero, we get

$$\int_{x_0}^{x_0+h} f(x)\ dx = h \left(y_0 + \frac{1}{2} \Delta y_0 \right) = \frac{h}{2} (y_0 + y_1)$$

similarly

$$\int_{x_0+h}^{x_0+2h} f(x)\ dx = h \left(y_1 + \frac{1}{2} \Delta y_1 \right) = \frac{h}{2} (y_1 + y_2)$$

....

$$\int_{x_0+\overline{n-1}h}^{x_0+nh} f(x)\ dx = \frac{h}{2} [y_{n-1} + y_n]$$

Adding these n integrals, we obtain

$$I = \int_{x_0}^{x_0+nh} f(x)\ dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})] \quad \ldots(9.22)$$

This is known as the trapezoidal rule.

(*ii*) **Simpson's One Third Rule.** Putting $n = 2$ in equation (9.21) and taking the curve through (x_0, y_0) (x_1, y_1) and (x_2, y_2) as a parabola *i.e.* a polynomial of second order so that differences of higher order than second order vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

$$\int_{x_0+n-2h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots)$$

$$+ 2(y_2 + y_4 + \ldots\ldots)] \quad \ldots(9.23)$$

This is known as the simpson's one third rule and is most commonly used.

(iii) **Simpson's Three Eight Rule.** Putting $n = 3$ in equation (9.21) and taking the curve through (x_i, y_i) ; $i = 0, 1, 2, 3$ as a polynomial of third order so that differences of higher order than third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

similarly

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-2} + y_{n-1})$$

$$+ 2(y_3 + y_6 + \ldots\ldots\, y_{n-3})] \quad \ldots(9.24)$$

This is known as the Simpson's three eight rule.

ILLUSTRATIVE EXAMPLES

2. (i) *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

(a) *Trapezoidal rule*

(b) *Simpson's one third rule*

(c) *Simpson's three eight rule and hence obtain approximate value of* π *in each case.* (Raj. 2001, 03, 06, RTU 09)

(ii) *Evaluate* $\int_0^{\pi/2} \sin x \, dx$ *using*

(a) *Trapezoidal rule*

(b) *Simpson's rule (take 11 ordinates)*

(iii) *Find the value of* $\log_e 2$ *by evaluating* $\int_0^1 \frac{x^2}{1+x^3} dx$ *using Simpson's one third rule (take 4 equal intervals).* (Raj. 2001)

(iv) *A curve is drawn to pass through the points given by the following lable*

x :	0.0	0.5	1.0	1.5	2.0
y :	0.3989	0.3521	0.2420	0.1295	0.0540

Estimate the area bounded by the curve, the x-axis and the lines $x = 0$ *and* $x = 2$.

(v) *A solid of revolution is formed by rotating an arc between lines* $x = 0$ *and* $x = 1$ *about the x-axis corresponding to following table*

x :	0.00	0.25	0.50	0.75	1.00
y :	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of solid formed. (Raj. 1992)

Solution. (*i*) Since we have to use Trapezoidal rule, Simpson's $\frac{1}{3}$ rule and $\frac{3}{8}$ rule, we should divide the range (0, 1) in such manner, that it is divided by 2, 3, 6. Therefore we divide the range (0, 1) into six equal intervals with $h = \left(\frac{1}{6}\right)$. Thus, we get

	x	$y = f(x) = \frac{1}{1+x^2}$
(x_0)	0	1.000 (y_0)
(x_1)	$\frac{1}{6}$	0.97297 (y_1)

(x_2) $\frac{2}{6}$	0.90000 (y_2)
(x_3) $\frac{3}{6}$	0.80000 (y_3)
(x_4) $\frac{4}{6}$	0.69231 (y_4)
(x_5) $\frac{5}{6}$	0.59016 (y_5)
(x_6) $\frac{6}{6}$	0.5000 (y_6)

Also, $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$

(a) Trapezoidal rule

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + \ldots\ldots + y_5)]$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{1}{12}[(1 + 0.5000) + 2(0.97297 + 0.90000 + .80000 + .69231 + 0.59016)]$$

$$= 0.78424 \text{ approx.}$$

Hence, $\pi = 0.78424 \times 4 = 3.13696$

(b) Simpson's $\frac{1}{3}$ rule

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{1}{18}[(1 + 0.5) + 4(0.97297 + 0.8 + 0.59016) + 2(0.900 + 0.69231)]$$

$$= 0.78540 \text{ approx.}$$

Hence, $\pi = 0.78540 \times 4 = 3.14160$

(c) Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{3}{48}[(1 + 0.5) + 3(0.97297 + 0.9 + .69231 + .59016) + 2(0.8)]$$

$$= 0.78539 \text{ approx.}$$

Hence, $\pi = 0.78539 \times 4 = 3.14156$

(ii) Dividing the range $\left(0, \frac{\pi}{2}\right)$ into 11 ordinates *i.e.* 10 equal intervals with

$$h = \frac{b-a}{\text{no:of intervals}} = \frac{\frac{\pi}{2}-0}{10} = \frac{\pi}{20}, \text{ we get}$$

x :	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$
$y = f(x) = \sin x$:	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9510

x :	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
$y = f(x) = \sin x$:	0.9877	1

(a) Trapezoidal rule

$$\int_{x_0}^{x_{10}} f(x)\, dx = \frac{h}{2}\, [(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + \ldots\ldots y_9)]$$

$$\Rightarrow \int_0^{\pi/2} \sin x dx = \frac{\pi}{40}\, [(0 + 1) + 2(0.1564 + 0.309 + 0.4540$$

$$+ 0.5878 + 0.7071 + 0.809 + 0.891 + 0.951 + 0.9877]$$

$$= 0.9979 \text{ (approx.)}$$

(b) Simpson's rule

$$\int_{x_0}^{x_{10}} f(x)\, dx = \frac{h}{3}\, [(y_0 + y_{10}) + 4(y_1 + y_3 + \ldots\ldots + y_9)$$

$$+ 2(y_2 + y_4 + \ldots\ldots + y_8)$$

$$\Rightarrow \int_0^{\pi/2} \sin x dx = \frac{\pi}{60}\, [(0 + 1) + 4(0.1564 + 0.4540 + 0.7071$$

$$+ 0.891 + 0.9877) + 2(0.309 + 0.5878 + 0.809 + 0.9510)$$

$$= 0.9999 \text{ (approx).}$$

(iii) For taking 4 equal intervals, we choose $h = \frac{1-0}{4} = 0.25$

Thus, we get

x :	0.00	0.25	0.50	0.75	1.00
$y = f(x) = \frac{x^2}{1+x^3}$:	0.0000	0.0615	0.2222	0.3956	0.5000

Now, using Simpson's $\frac{1}{3}$ rule

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$\Rightarrow \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{12}\left[(0 + .5) + 4(.06248 + .3956) + 2(0.2222)\right]$$

$$= 0.23139 \text{ approx}$$

Also, $$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\log_e 2$$

$$\Rightarrow \quad \log_e 2 = 3 \times 0.23139 = 0.69324$$

(*iv*) Using simpson's $\frac{1}{3}$ rule, we have

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$\Rightarrow$ Required area

$$= \int_0^2 y\,dx = \frac{0.5}{3}\left[(0.3989 + 0.0540) + 4(0.3521 + 0.1295) + 2(0.2420)\right] \text{ (using given data)}$$

$$= 0.4772 \text{ (approx.)}$$

(*v*) Using simpson's $\frac{1}{3}$ rule, we have

$$\int_{x_0}^{x_4} [f(x)]^2\,dx = \frac{h}{3}\left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2\right]$$

$\Rightarrow$ Required volume

$$= \pi\int_0^1 y^2\,dx = \frac{0.25\pi}{3}\{(1^2 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.9589)^2]$$

$$= \frac{0.25\pi}{3}\left[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)\right]$$

$$= 2.8178 \text{ (approx.)}$$

EXERCISE 9.2

1. Evaluate $\int_0^1 \frac{dx}{1+x}$ using Simpson's rule [Raj. 2003]

2. Evaluate $\int_1^2 \frac{dx}{x}$ using Simpson's rule

3. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using

(i) Trapezoidal rule

(ii) Simpson's $\frac{1}{3}$ rule and

(iii) Simpson's $\frac{3}{8}$ rule and compare the results with true value. [Raj. 2004]

4. Evaluate $\int_0^6 y dx$ using an appropriate formula by virtue of following table :

x :	0	1	2	3	4	5	6
y :	0.146	0.161	0.176	0.190	0.204	0.217	0.230

5. Evaluate $\int_1^2 \sqrt{x - \frac{1}{x}}\, dx$ using Simpson's rule (take five ordinates) [MREC (Auto) 2001]

6. Evaluate $\int_0^1 \frac{\log_e (1+x^2)}{(1+x^2)} dx$ using (i) Trapezoidal rule (ii) Simpson's one third rule (take 10 equal intervals) (Raj. 2006)

7. Evaluate $\int_{0.2}^{1.4} e^x\, dx$ using Trapezoidal rule and compare the result with true value (take $h = 0.1$) (MNIT 2006)

8. A curve is drawn to pass through the points given by the following table

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Estimete the area bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$.

ANSWERS

1. 0.69315
2. 0.6932
3. (i) 1.4108 (ii) 1.3662 (iii) 1.3571 and true value = 1.4056
4. 1.1363
5. 0.8397
6. (i) 1.1730 (ii) 1.1728
7. 2.8376 and exact value = 2.8399
8. 7.78

MORE PROBLEMS ON NUMERICAL DIFFERENTIATION AND INTEGRATION

1. Use following table to find $f'(4)$

x :	1	2	4	8	10
$f(x)$:	0	1	5	21	27

2. A rod is rotating in a plane. The following table provides the angle θ (in rad.) through which the rod has turned for various values of time t (in seconds).

t :	0	0.2	0.4	0.6	0.8	1.0	1.2
θ :	0.00	0.12	0.49	1.12	2.02	3.20	4.67

Evaluate the angular velocity and the angular acceleration of the rod when $t = 0.6$ seconds.

3. If $y = f(x)$ and $y_n = f(x_0 + nh)$, then prove that

$$f'(x_0) = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$

where powers of h above h^6 are neglected. [MREC (Auto) 2001]

4. Use following table to find value of x for which y is minimum/maximum and also find this value of y

x :	0	1	2	3	4	5
y :	0.00	0.25	0.00	2.25	16.00	56.25

5. Using central difference operator, prove the following

(i) $y'_n = \frac{1}{2h}(y_{n+1} - y_{n-1})$ (ii) $y''_n = \frac{1}{h^2}(y_{n+1} - 2y_n + y_{n-1})$

6. Evaluate $\int_{-1.6}^{-1} e^x \, dx$ using Simpson's rule (take 6 equal intervals)

7. Using Simpson's one third rule to find the approximate value of the area of the cross-section of a 80 meter wide river, whose depth y (in meters) at a distance x from one bank being given by the following table [MREC (Auto) 2001]

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

ANSWERS

1. 2.833

2. angular velocity = 3.82 radian/sec. and angular acceleration = 6.75 radian/sec^2.

4. y is minimum at $x = 0$, 2 and $y_{min.} = 0$

6. 0.166

7. 710 square meter