ADVANCED ENGINEERING MATHEMATICS

IT (III SEM)

RANDOM VARIABLES

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INTRODUCTION

In many experiments we are interested not in knowing which of the outcomes has occurred,but in the numbers associated with them.

For example, when four coins are tossed, one may be interested in knowing the number of heads obtained. When a pair of dice are tossed, one may seek information about the sum of points. Thus, we associate a real number with each outcome of an experiment.

In other words, "A variable whose value is determined by the outcome of a random experiment is called a random variable." A random variable is also known as a chance variable or stochastic variable.

TYPES OF RANDOM VARIABLE

A random variable may be discrete or continuous. So, there are two types of random variables.

(I) Discrete Random Variable

(II) Continuous Random Variable

(I) **Discrete Random Variable :** If the random variable takes on the integer values (Countable number of values) such as 0, 1, 2,... then it is called a discrete random variable (DRV).

e.g. (*i*) The number of printing mistakes in each page of a book.

(*ii*) Number of students in a class.

(II) **Continuous Random Variable :** If the random variable takes all values, within a certain interval, then the random variable is called a Continuous Random Variable (CRV).

e.g. (*i*) The height of a person.

(*ii*) The amount of rainfall on a rainy day or in a rainy reason.

(*iii*) $X = \{x \in R : o < x < 1\}$

DISTRIBUTION FUNCTION

Let X be a random variable then a function $F(x)$ defined for all real x as follows.

$$F_X(x) = F(x) = P(X \leq x) \text{ Where } -\infty < x < \infty$$

is called distribution function.

The domain of distribution function is $(-\infty, \infty)$ and its range is $[0, 1]$.

PROPERTIES OF DISTRIBUTION FUNCTION

(I) Let X be a Random variable and $F(x)$ be the distribution function of X and if $a < b$ then

$$P(a < X \leq b) = F(b) - F(a)$$

Proof : The events $a < X \leq b$ and $X \leq a$ are two disjoint events, and their union is the event $X \leq b$,

Hence by addition theorem of probability

$$P(a < X \leq b) + P(X \leq a) = P(X \leq b)$$

$$\Rightarrow \quad P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$\Rightarrow \quad P(a < X \leq b) = F(b) - F(a) \qquad \text{(by def.)}$$

(II) If $F(x)$ is a distribution function then

(i) $0 \le F(x) \le 1$

(ii) $F(x) \le F(y)$ if $x < y$

In other words, all distribution functions are monotonically non -decreasing and lie between 0 and 1.

Cor. 1 : $$P(a \le X \le b) = P\{(X = a) \cup (a < X \le b)\}$$
$$= P(X = a) + P(a < X \le b)$$
$$= P(X = a) + F(b) - F(a) \qquad \text{(by def.)}$$

Similarly, We get

Cor. 2 : $$P(a < X < b) = P(a < X \le b) - P(X = b)$$
$$= F(b) - F(a) - P(X = b)$$

Cor. 3 : $$P(a \le X < b) = P(a < X < b) + P(X = a)$$
$$= F(b) - F(a) - P(X = b) + P(X = a)$$

(III) If F is $d.f.$ of one-dimensional $r.v.$ x, then

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0$$

and $$F(\infty) = \lim_{x \to \infty} F(x) = 1$$

ONE DIMENSIONAL RANDOM VARIABLE

DISCRETE RANDOM VARIABLE

PROBABILITY MASS FUNCTION (PMF)

The function which describe the probabilistic behaviour of discrete *r.v.* is called probability mass function.

Mathematically, let X be a discrete random variable with different values $x_1, x_2, \ldots, x_n$ then the function defined as follows :

$$f_x(x) = p(x) = \begin{cases} P(X = x_i) = p_i & \forall \quad x = x_i \\ 0 & \text{if} \quad x \neq x_i, \quad i = 1, 2, \ldots \end{cases}$$

is called *p.m.f.* of discrete *r.v.X*.

PROPERTIES OF PMF

The numbers $p(x_i)$; $i = 1, 2, \ldots$ must satisfy the following conditions :

(*i*) $p(x_i) \geq 0 \; \forall \, i$

(*ii*) $\sum_{i=1}^{\infty} p(x_i) = 1$

CONTINUOUS RANDOM VARIABLE

PROBABILITY DENSITY FUNCTION (PDF)

The function which describe the probabilistic behaviour of continuous random variables is called probability density function.

Mathematically, let X be a continuous random variable and $f(x)$ be the *p.d.f.* of X.

PROPERTIES OF PDF

The *p.d.f.* satisfies the following conditions :

(*i*) $f(x) \geq 0$ for all $x \in I_X$

(*ii*) $\int_{I_X} f(x)dx = 1$

Q.1. *A random variable has the following probability distribution values of X :*

x	:	0	1	2	3	4	5	6	7
$p(x)$	:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(a) *Find k*

(b) *Evaluate* $P(3 < X \leq 6)$

(c) *Evaluate* $P\left(\dfrac{1.5 < X < 4.5}{X > 2}\right)$

Sol.

x	:	0	1	2	3	4	5	6	7
$p(x)$	:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(a) Since $\sum_{x=0}^{7} p(x) = 1$,

we have, $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$

or $10k^2 + 9k - 1 = 0$

or $(10k - 1)(k + 1) = 0$

or $k = -1, \dfrac{1}{10}$

But, $k = -1$ is not possible as $p(x) \geq 0$.

Hence, $k = \dfrac{1}{10}$.

(b) $P\ (3 < X \le 6) = p\ (4) + p\ (5) + p\ (6)$

$$= 3k + k^2 + 2k^2$$

$$= 3k^2 + 3k$$

$$= \frac{3}{100} + \frac{3}{10} = \frac{33}{100}$$

(c) $P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{P(1.5 < X < 4.5) \cap P(X > 2)}{P(X > 2)}$

$$\left[\because P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}\right]$$

$$= \frac{P(2 < X < 4.5)}{P(X > 2)} = \frac{P(X = 3) + p(X = 4)}{P(X > 2)}$$

$$= \frac{P(X = 3) + p(X = 4)}{1 - [P(X = 0) + P(X = 1) + P(X = 2)]}$$

$$= \frac{2k + 3k}{1 - [0 + k + 2k]} = \frac{5k}{1 - 3k} = \frac{5/10}{1 - \frac{3}{10}} = \frac{5}{7}$$

Q.2. *A continuous random variable X that can assume any value between $x = 1$ and $x = 4$ and zero otherwise, has the density function is given by $f(x) = k\,(1 + x)$, find $P(X \le 3)$.*

Sol. Given $p.d.f$ is
$$f(x) = \begin{cases} k(1+x) & ; \quad 1 < x < 4 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

We have, $\int_{-\infty}^{\infty} f(x)dx = 1$

or $$\int_{-\infty}^{1} f(x)dx + \int_{1}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx = 1$$

or $$0 + \int_{1}^{4} f(x)dx + 0 = 1$$

or $$k\int_{1}^{1} (1+x)dx = 1$$

or $$k\left[x + \frac{x^2}{2}\right]_1^4 = 1$$

or $$k\left[\{4+8\} - \left\{1 + \frac{1}{2}\right\}\right] = 1$$

or $$k\left(\frac{21}{2}\right) = 1$$

or $$k = \frac{2}{21}$$

Now, $$P(X \leq 3) = \int_1^3 f(x)dx$$

$$= k \int_1^3 (1 + x)dx$$

$$= \frac{2}{21}\left(x + \frac{x^2}{2}\right)_1^3$$

$$= \frac{2}{21}\left[\left\{3 + \frac{9}{2}\right\} - \left\{1 + \frac{1}{2}\right\}\right]$$

$$= \frac{2}{21}\left(\frac{12}{2}\right)$$

$$= \frac{12}{21}.$$

or $$P(X \leq 3) = \frac{4}{7}$$

Q.3. *The distribution function for a random variable X is :*

$$F(x) = \begin{cases} 1 - e^{-2x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

Find : (a) The density function

(b) The probability that $-3 < X \leq 4$

Sol. Given, $$F(x) = \begin{cases} 1 - e^{-2x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

(a) The *p.d.f.* of X is given by

$$f(x) = \frac{d}{dx}\{F(x)\} = \begin{cases} 2e^{-2x} & ; \quad x \geq 0 \\ o & ; \quad x < 0 \end{cases}$$

(b) $$P(-3 < X < 4) = \int_{-3}^{4} f(x)dx = \int_{-3}^{0} f(x)\,dx + \int_{0}^{4} f(x)dx$$

$$= \int_{-3}^{0} 0.dx + \int_{0}^{4} 2e^{-2x}dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_0^4 = -(e^{-8} - e^{0}) = 1 - e^{-8}$$

TWO DIMENSIONAL RANDOM VARIABLE

TWO DIMENSIONAL RANDOM VARIABLE OR BIVARIATE RANDOM VARIABLE

In the previous section, we have considered one dimensional random variable X on a sample space, that is, the outcome of a random experiment has only one characteristic and assumes a single real value. But, in many times, the outcome of an experiment may have two or more characteristics.

For example, we have three colour balls, red, blue and green respectively. We may be interested in choosing two different coloured balls out of three each time. To describe such experiments mathematically we introduce the study of two dimensional random variables.

Discrete Random Variable

If the values assigned for (X, Y) are integers, then (X, Y) is called "two dimensional discrete random variable."

Continuous Random Variable

If the random variable (X, Y) can assume all possible values in a region R in the xy-plane, then (X, Y) is called "two dimensional continuous random variable"

DISCRETE RANDOM VARIABLE

TWO DIMENSIONAL P.M.F. OR JOINT P.M.F. OF (X,Y)

Let X and Y be two discrete random variables s.t.

$$X(s) = \{x_1, x_2, \ldots, x_n\} \; ; \; Y(s) = \{y_1, y_2, \ldots, y_m\}$$

and If (X, Y) is a bivariate random variable, then the joint probability mass function of (X, Y) denoted by P_{XY} is defined as

$$P_{XY} = P_{XY}(x_i, y_i) = \begin{cases} P(X = x_i, Y = y_j) & ; \quad \forall (x_i, y_j) \in (X, Y) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Remark : $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\left(X = x_i, Y = y_j\right) = 1$

MARGINAL P.M.F. OF X, Y

Using joint **P.M.F.** of X, Y i.e. P_{XY}, we can find marginal **P.M.F.** of **X** and **Y** i.e. $P(X = x_i)$ and $P(Y = y_j)$ respectively in tabular form, which is shown in one problem in further section of this chapter.

CONDITIONAL DISTRIBUTION FOR DISCRETE RANDOM VARIABLE

If X and Y are two discrete $r.v.$ and $P(X = x_i, Y = y_j)$ be their joint probability mass function and $P(Y = y_j)$ is the value of the marginal distribution of Y at y, then the function defined as

$$P\left(\frac{X = x_i}{Y = y_j}\right) = \frac{P[X = x_i \cap Y = y_j]}{P(Y = y_j)} = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

is called the conditional probability distribution of $X = x_i$, given $Y = y_j$.

similarly, we can define conditional probability distribution of $Y = y_j$, given $X = x_i$.

INDEPENDENT RANDOM VARIABLES

For discrete random variable : The two random variables X and Y are said to be independent if

$$P\,(X = x_i, Y = y_j) = P\,(X = x_i).\,P\,(Y = y_j) \;\; \forall\, i, j$$

CONTINUOUS RANDOM VARIABLE

TWO DIMENSIONAL P.D.F. OR JOINT P.D.F. OF (X, Y)

Let X and Y be two continuous random variables and If (X, Y) is a bivariate random variable, then the joint probability density function of (X, Y) is denoted as $f_{XY}(x,y)$ or simply as $f(x,y)$.

Remark : $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

MARGINAL PDF OF X, Y

If X and Y be two continuous $r.v.$ and $f_{XY}(x, y)$ be their joint $p.d.f.$, then marginal $p.d.f.$ of X is given as

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \qquad \text{(treating } x \text{ as a constant)}$$

Similarly, the marginal $p.d.f.$ of Y is given as

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \qquad \text{(treating } y \text{ as a constant)}$$

CONDITIONAL DISTRIBUTION FOR CONTINUOUS RANDOM VARIABLE

If X and Y are two continuous $r.v.$ and $f(x,y)$ be their joint probability density function and $f(y)$ is the value of the marginal $p.d.f.$ of Y, then the function defined as

$$P\left(\frac{X=x}{Y=y}\right) = \frac{P[X=x \cap Y=y]}{P(Y=y)} = \frac{f(x,y)}{f(y)}$$

is called the conditional probability distribution of $X=x$, given $Y=y$.

similarly, we can define conditional probability distribution of $Y=y$, given $X=x$.

INDEPENDENT RANDOM VARIABLES

For Continuous Random Variable: The two random variables X and Y are said to be independent if

$$f(x,y) = f(x).f(y)$$

Q.1. *The joint probability function of two discrete random variables X and Y is given by $f(x, y) = k(2x + y)$, where $x = 0, 1, 2$; $y = 0, 1, 2, 3$ and $f(x, y) = 0$, otherwise. Find :*

(a) *The value of constant k*

(b) $P(X = 2, Y = 1)$

(c) $P(X \geq 1, Y \leq 2)$

(d) *The marginal probability function of X and Y*

Sol. The joint probability mass function is given by

$$P(X = x, Y = y) = k(2x + y)$$

The joint probability distribution of (X, Y) can be represented in the tabular form for the values of $X = 0, 1, 2$ and $Y = 0, 1, 2, 3$.

Y \ X	0	1	2	Total
0	0	$2k$	$4k$	$6k$
1	k	$3k$	$5k$	$9k$
2	$2k$	$4k$	$6k$	$12k$
3	$3k$	$5k$	$7k$	$15k$
Total	$6k$	$14k$	$22k$	$42k$

(*a*) We know that

$$\sum_x \sum_y P(X = x, Y = y) = 1$$

$$\Rightarrow \qquad 42\,k = 1 \qquad \text{[From table]}$$

$$\Rightarrow \qquad k = \frac{1}{42}$$

(*b*) $$P(X = 2, Y = 1) = 5\,k = \frac{5}{42} \qquad \text{[From table]}$$

(*c*) $$P(X \geq 1, Y \leq 2) = (2\,k + 3\,k + 4\,k) + (4\,k + 5\,k + 6\,k)$$

$$= 24\,k \qquad \text{(From table)}$$

$$= \frac{24}{42} = \frac{4}{7}$$

(*d*) The marginal probability of X is given in tabular form as shown below :

$X = x$	$P(X = x)$
0	$6k = \frac{6}{42} = \frac{1}{7}$
1	$14k = \frac{14}{42} = \frac{1}{3}$
2	$22k = \frac{22}{42} = \frac{11}{21}$
Total	1

The marginal probability of Y is given in tabular form as shown below :

$Y = y$	$P(Y = y)$
0	$6k = 1/7$
1	$9k = 3/14$
2	$12k = 2/7$
3	$15k = 5/14$
Total	1

Q.2. **The joint probability distribution of random variable (X, Y) is given by**

$$f_{XY}(x, y) = \begin{cases} \frac{1}{27}(x + 2y) & ; \quad x = y = 0,1,2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find : (a) The marginal distribution of X and Y
(b) The conditional distribution of Y for $X = 1$
(c) Are X and Y independent?

Sol. **The joint probability mass function is given by**

$$P(X = x, Y = y) = \frac{1}{27}(x + 2y)$$

The joint probability distribution of (X,Y) can be represented in the tabular form for the values of $X = 0, 1, 2$ and $Y = 0, 1, 2$

Y \ X	0	1	2	Total
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
Total	6/27	9/27	12/27	1

(a) The marginal distribution of X is given in tabular form as shown below :

$X = x$	$P(X = x)$
0	6/27
1	9/27
2	12/27
Total	1

The marginal distribution of Y is given in tabular form as shown below :

$Y = y$	$P(Y = y)$
0	3/27
1	9/27
2	15/27
Total	1

(b) The conditional distribution of Y for X = 1 is given as:

$$P\left(\frac{Y=y}{X=1}\right)=\frac{P(Y=y, X=1)}{P(X=1)},$$

which is given in tabular form as shown below :

$Y=y$	$P\left(\frac{Y=y}{X=1}\right)$
0	$\frac{1/27}{9/27}=\frac{1}{9}$
1	$\frac{3/27}{9/27}=\frac{3}{9}$
2	$\frac{5/27}{9/27}=\frac{5}{9}$
Total	**1**

(c) For checking whether *X* and *Y* are independent or not, we have to check

$$P(X=x,Y=y)=P(X=x)P(Y=y)\ \forall \text{ values of } X,Y \text{ is true or not}$$

For $x=0, y=0$, we have $P(X=0,Y=0)=0$

$$\text{and } P(X=0)P(Y=0)=\frac{6}{27}\cdot\frac{3}{27}$$

so, $P(X=0,Y=0)\neq P(X=0)P(Y=0)$

Thus, X and Y are not independent.

Q.3. *The joint density function of two continuous random variables X and Y is*

$$f(x,y) = \begin{cases} kxy & ; \quad 0 < x < 4, 1 < y < 5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find :

(a) *The value of the constant k.*

(b) $P(1 < X < 2, 2 < Y < 3)$

sol. The joint density function of two continuous *r.v.'s* X and Y is given by

$$f(x,y) = \begin{cases} kxy & ; \quad 0 < x < 4 \;;\; 1 < y < 5 \\ 0 & ; \quad \text{ohterwise} \end{cases}$$

(a) Constant 'k' is determined from the consideration that total probability is unity

i.e. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ i.e. $\displaystyle\int_{x=0}^{4}\int_{y=1}^{5} kxy\,dx\,dy = 1$

$\Rightarrow$ $\displaystyle k\int_0^4 \left\{\int_1^5 xy\,dy\right\} dx = 1$ $\Rightarrow$ $\displaystyle k\int_0^4 \left\{\frac{xy^2}{2}\right\}_1^5 dx = 1$

$\Rightarrow$ $\displaystyle \frac{k}{2}\int_0^4 x(25-1)dx = 1$ $\Rightarrow$ $\displaystyle \frac{k}{2}\int_0^4 24x\,dx = 1$

$\Rightarrow$ $\displaystyle 12k\int_0^4 x\,dx = 1$ $\Rightarrow$ $\displaystyle 12\,k\left(\frac{x^2}{2}\right)_0^4 = 1$

$\Rightarrow$ $6\,k\,(16) = 1$ $\Rightarrow$ $k = \dfrac{1}{96}$

(b) $P(1 < X < 2, 2 < Y < 3) = \int_{x=1}^{2} \int_{y=2}^{3} kxy \, dx \, dy$

$$= \frac{1}{96} \int_1^2 \left\{ \int_2^3 xy \, dy \right\} dx \qquad = \frac{1}{96} \int_1^2 \left\{ \frac{xy^2}{2} \right\}_2^3 dx$$

$$= \frac{1}{192} \int_1^2 x(9-4) \, dx \qquad = \frac{5}{192} \int_1^2 x \, dx$$

$$= \frac{5}{192} \left(\frac{x^2}{2} \right)_1^2 \qquad = \frac{5}{192 \times 2}(4-1)$$

$$= \frac{3 \times 5}{2 \times 192} \qquad = \frac{5}{128}$$

Q.4. *The joint density function of two continuous random variables X and Y is*

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & ; \quad 0<x<1, 0<y<1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find : (a) *Marginal density of X and Y*

(b) *Conditional density of X given $Y = y$ and use it to evaluate*

$$P = \left(\frac{X \le 1/2}{Y = 1/2}\right)$$

(c) *Show that X and Y are dependent.*

Sol. (*a*) By the definition of marginal distribution function of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_{-\infty}^{0} f(x, y)\,dy + \int_{0}^{1} f(x, y)\,dy + \int_{1}^{\infty} f(x, y)\,dy$$

$$= \int_{0}^{1} f(x, y)\,dy \quad \left\{\because \int_{-\infty}^{0} f(x, y)\,dy = \int_{1}^{\infty} f(x, y)\,dy = 0\right\}$$

$$= \frac{2}{3}\int_{0}^{1} (x + 2y)\,dy = \frac{2}{3}\left\{xy + \frac{2y^2}{2}\right\}_0^1 = \frac{2}{3}\left(xy + y^2\right)_0^1$$

$$= \frac{2}{3}(x + 1)$$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{2}{3}(x+1) & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Again the definition of marginal distribution function of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \frac{2}{3}\int_0^1 (x+2y)\,dx = \frac{2}{3}\left\{\frac{x^2}{2} + 2xy\right\}_0^1$$

$$= \frac{2}{3}\left\{\frac{1}{2} + 2y\right\} = \frac{1}{3}(1+4y)$$

$$\Rightarrow \quad f(y) = \begin{cases} \frac{1}{3}(1+4y) & ; \quad 0 < y < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(b) By the definition the conditional density of X for $Y = y$ is

$$f_{\frac{X}{Y}}\left(\frac{x}{y}\right) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = \frac{2(x+2y)}{(1+4y)}$$

So, the conditional density of X given $Y = y$ is

$$f_{\frac{X}{Y}}\left(\frac{x}{y}\right) = \begin{cases} \dfrac{2(x+2y)}{1+4y} & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Now, we have to find the value of

$$P\left(\frac{X \le 1/2}{Y = 1/2}\right)$$

$$\text{So, } P\left(\frac{X \le 1/2}{Y = 1/2}\right) = \int_0^{1/2} f_{X/Y}\left(\frac{x}{y}\right)dx$$

$$= \int_0^{1/2} f\left(\frac{X}{Y}\right)\left(\frac{x}{y = 1/2}\right)dx$$

$$= \int_0^{1/2} \frac{2(x+1)}{1+2}dx \quad \text{[Using first part of (b)]}$$

$$= \frac{1}{3}\int_0^{1/2} 2(x+1)dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^{1/2} = \frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right]$$

$$= \frac{2}{3}\left[\frac{5}{8}\right] = \frac{5}{12}$$

(c) It is clear from the given function that $f(x,y) \neq f(x) \cdot f(y)$

Hence, X and Y are dependent.

Exercise

1. Find the probability mass function of the total number of heads obtained in four tosses of a balanced coin.
2. A random variable X has the following probability distribution:

x:	0	1	2	3	4	5	6	7
$p(x)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$

   Find (a) Value of k (b) $P(X \geq 3)$, $P(X < 3)$, $P(0 < X < 5)$
3. If X is a continuous random variable with the following probability density function:

$$f(x) = \begin{cases} \lambda\left(2x - x^2\right) & ; \ 0 < x < 2 \\ 0 & ; \ \text{otherwise} \end{cases}$$

   Find (a) Value of λ (b) $P(X > 1)$
4. A continuous random variable X has the following distribution function:

$$F(x) = \begin{cases} 0 & ; \ x < 1 \\ \lambda(x-1)^4 & ; \ 1 \leq x \leq 3 \\ 1 & ; \ x > 3 \end{cases}$$

   Find (a) The density function (b) Value of λ

5. The joint probability function of two discrete random variables X and Y is given by

$f(x,y)=\lambda(2x+3y)$, where $x=0,1,2$; $y=1,2,3$

Find (a) Value of λ

(b) Marginal probability distribution of X and Y

(c) Conditional distribution of Y, given $X=2$ i.e. $P\left(\dfrac{Y=y}{X=2}\right)$

6. The joint probability function of two discrete random variables X and Y is given by

$f(x,y)=\dfrac{xy}{27}$, where $x=1,2$; $y=2,3,4$

Prove that X and Y are independent.

7. The joint density function of two continuous random variables X and Y is given by

$$f(x,y)=\begin{cases} 8xy & ;\ 0\le x\le 1,\ 0\le y\le 1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Find (a) Marginal density of X and Y

(b) Conditional density of X and Y

8. The joint density function of two continuous random variables X and Y is given by

$$f(x,y)=\begin{cases} \lambda e^{-(ax+by)} & ;\ x,y>0 \text{ and } a,b>0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Find (a) Value of λ

(b) Are X and Y independent?

Answers

1. $P(X=x)=\dfrac{{}^4C_x}{16}$; $x=0,1,2,3,4$

2. $(a)\ k=\dfrac{1}{64}\quad (b)\ \dfrac{55}{64},\dfrac{9}{64},\dfrac{3}{8}$

3. $(a)\ \lambda=\dfrac{3}{4}\quad (b)\ \dfrac{1}{2}$

4. $(a)\quad f(x)=\begin{cases}\dfrac{1}{4}(x-1)^3 & ;\ 1\le x\le 3\\ 0 & ;\ \text{otherwise}\end{cases}\quad (b)\ \lambda=\dfrac{1}{16}$

5. $(a)\ \lambda=\dfrac{1}{72}$

   (b) Marginal probability distribution of $X=\dfrac{18}{72},\dfrac{24}{72},\dfrac{30}{72}$;

   Marginal probability distribution of $Y=\dfrac{15}{72},\dfrac{24}{72},\dfrac{33}{72}$

   (c) Conditional distribution of Y, given $X=2=P\left(\dfrac{Y=y}{X=2}\right)=\dfrac{7}{30},\dfrac{1}{3},\dfrac{13}{30}$

7. (a) Marginal density of $X = \begin{cases} 4x & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

and Marginal density of Y = $\begin{cases} 4y & ; \ 0 \le y \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

(b) Conditional density of $X = \begin{cases} 2x & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

and Conditional density of Y = $\begin{cases} 2y & ; \ 0 \le y \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

8. (a) $\lambda = ab$ (b) Yes, X and Y are independent

THANKS