ADVANCED ENGINEERING MATHEMATICS

IT (III SEM)

DIFFERENT METHODS FOR SOLVING LPP

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SIMPLEX METHOD

The simplex algorithm consists of the following steps:

Step 1. Finding BFS of the given LPP.

Step 2. Testing whether the BFS obtained in step 1 is optimal or not.

Step 3. If it is not optimal, then imporving it by a set of rules.

Step 4. Repeating the steps 2 and 3 till the optimal solution is obtained.

Prerequisites

1. Restatement of LPP

We recall that the general LPP is:

Optimize $$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \;(\leq \text{ or } = \text{ or } \geq)\; b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \;(\leq \text{ or } = \text{ or } \geq)\; b_2 \\
\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \;(\leq \text{ or } = \text{ or } \geq)\; b_m
\end{aligned}$$

and $x_j \geq 0$, $j = 1, 2, \ldots, n$

2. Slack and surplus variables

The constraints of LPP may involve any of the three signs $\leq$, $=$, $\geq$. In case the constraints are inequalities, they can be changed to equations by adding or subtracting non-negative variables in the left hand side of each such constraint. These new variables, if they are added, are called *slack variables* and in case when they are substracted, are called *surplus variables*.

For example, if we have the constraints

$$4x_1 - 6x_2 \leq 14$$

$$10x_1 + x_2 \geq 10$$

then we can change them to equations by introducing two new variables x_3 and x_4 as follows;

$$4x_1 - 6x_2 + s_1 = 14$$

$$10x_1 + x_2 - s_2 = 10$$

Here, s_1 is a slack variable and s_2 is a surplus variable and they are non-negative i.e. $s_1 \geq 0$ and $s_2 \geq 0$. Also, these slack and surplus variables are introduced in the objective function and the prices corresponding to them are taken zero.

3 Standard from of LPP

The standard form of LPP possesses the following characteristics

(i) The objective function is to be of maximization type.

(ii) All the constraints are to be in the form of equations, except the non-negativity restrictions.

(iii) The components of the requirement vector (i.e. the right hand side elements of all the constraints) are to be positive.

(iv) All the variables involved are to be non-negative.

(v) If there are n constraints in the given LPP, then we need an identity matrix of order n x n in the all equations of constraints.

EXAMPLE 1.

Solve by Simplex method :

Maximize $z = 4x_1 + 3x_2$

subject to $2x_1 + x_2 \le 30,$

$x_1 + 2x_2 \le 24,$

and $x_1, x_2 \ge 0.$

Solution.

The given linear programming problem reduces to :

Maximize $z = 4x_1 + 3x_2 + 0S_1 + 0S_2$

subject to $2x_1 + x_2 + S_1 + 0S_2 = 30$

$x_1 + 2x_2 + 0S_1 + S_2 = 24$

and $x_1, x_2, S_1, S_2 \ge 0$

$$\text{Maximize } z = 4x_1 + 3x_2 + 0S_1 + 0S_2$$

$$\text{subject to} \quad 2x_1 + x_2 + S_1 + 0S_2 = 30$$

$$x_1 + 2x_2 + 0S_1 + S_2 = 24$$

$$\text{and} \quad x_1, x_2, S_1, S_2 \geq 0$$

TABLE 1

c_B (contribution per unit)	Basic variables B	Solution or b	$c_j \rightarrow 4$	3	0 Identity	0 Matrix	Replacement Ratio
			x_1	x_2	S_1	S_2	
0	S_1	30	2	1	1	0	$\frac{30}{2} = 15$
0	S_2	24	1	2	0	1	$\frac{24}{1} = 24$
Contribution loss per unit z_j			0	0	0	0	
Net contribution loss per unit $(z_j - c_j)$			−4	−3	0	0	

Key row Key column Entering variable Key element

TABLE 1

c_B (contribution per unit)	Basic variables B	Solution or b	$c_j \to 4$	3	0 Identity	0 Matrix	Replacement Ratio
			x_1	x_2	S_1	S_2	
0	S_1	30	2	1	1	0	$\frac{30}{2} = 15$
0	S_2	24	1	2	0	1	$\frac{24}{1} = 24$
Contribution loss per unit z_j			0	0	0	0	
Net contribution loss per unit $(z_j - c_j)$			−4	−3	0	0	

Key row Key column Entering variable Key element

TABLE 2

c_B (contribution per unit)	Basic variables B	Solution or b	$c_j \to 4$	3	0	0	Replacement Ratio
			x_1	x_2	S_1	S_2	
4	x_1	$\frac{30}{2} = 15$	$\frac{2}{2} = 1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{0}{2} = 0$	$\frac{15}{1/2} = 30$
0	S_2	$24 - 15 \times 1 = 9$	$1 - 1 \times 1 = 0$	$2 - \frac{1}{2} \times 1 = \frac{3}{2}$	$0 - \frac{1}{2} \times 1 = \frac{-1}{2}$	$1 - 0 \times 1 = 1$	$\frac{9}{3/2} = 6$
$z_j \to$			4	2	2	0	
$z_j - c_j \to$			0	−1	2	0	

Key Row Key Column $z_2 - c_2$ Key Element

This is not an optimal solution as $z_2 - c_2$ is negative. So, further improve-ment is needed.

Here x_2 is an entering variable.

From the replacement ratio column, minimum ratio occurs for the variable S_2, hence S_2 will be a departing or outgoing variable.

TABLE 2

c_B (contribution per unit)	Basic variables B	Solution or b	$c_j \to 4$	3	0	0	Replacement Ratio
			x_1	x_2	S_1	S_2	
4	x_1	$\frac{30}{2} = 15$	$\frac{2}{2} = 1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{0}{2} = 0$	$\frac{15}{1/2} = 30$
0	S_2	$24 - 15 \times 1 = 9$	$1 - 1 \times 1 = 0$	$2 - \frac{1}{2} \times 1 = \frac{3}{2}$	$0 - \frac{1}{2} \times 1 = \frac{-1}{2}$	$1 - 0 \times 1 = 1$	$\frac{9}{3/2} = 6$
$z_j \to$			4	2	2	0	
$z_j - c_j \to$			0	-1	2	0	

Key Row   Key Column $z_2 - c_2$ Key Element

This is not an optimal solution as $z_2 - c_2$ is negative. So, further improve-ment is needed.
Here x_2 is an entering variable.
From the replacement ratio column, minimum ratio occurs for the variable S_2, hence S_2 will be a departing or outgoing variable. $\frac{3}{2}$ is the key element.

TABLE 3

c_B	Basic variables	Solution or b	$c_j \to 4$	3	0	0
			x_1	x_2	S_1	S_2
4	x_1	$15 - 6 \times \frac{1}{2} = 12$	$1 - 0 \times \frac{1}{2} = 1$	$\frac{1}{2} - 1 \times \frac{1}{2} = 0$	$\frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}$	$0 - \frac{2}{3} \times \frac{1}{2} = -\frac{1}{3}$
3	x_2	$\frac{9}{\frac{3}{2}} = 6$	$\frac{0}{\frac{3}{2}} = 0$	$\frac{\frac{3}{2}}{\frac{3}{2}} = 1$	$\frac{\frac{-1}{2}}{\frac{3}{2}} = \frac{-1}{3}$	$\frac{1}{\frac{3}{2}} = \frac{2}{3}$
z_j			4	3	$\frac{5}{3}$	$\frac{2}{3}$
$z_j - z_j$			0	0	$\frac{5}{3}$	$\frac{2}{3}$

In above table 3, all the values of $z_j - c_j$ are either positive or zero which means that further improvement in the programme is not possible. Hence, this is an optimal solution.

Optimal solution is $x_1 = 12, x_2 = 6$ and

$$\text{Maximize} = 4 \times 12 + 3 \times 6 = 66.$$

EXAMPLE 2.

Solve the following LPP by simplex method :

$$\text{Min. } z = x_1 - 3x_2 + 2x_3$$

s.t. $3x_1 - x_2 + 3x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

and $x_1, x_2, x_3 \ge 0.$

Solution.

Adding slack variables S_1, S_2 and S_3 in constraints and objective function and arranging it in standard form, we get

$$\text{Max. } z' = (-z) = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$

s.t. $3x_1 - x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 7$

$-2x_1 + 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12$

$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$

and $x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

$$\text{Max. } z' = (-z) = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$

$$\text{s.t. } 3x_1 - x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 7$$

$$-2x_1 + 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$$

and $$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

TABLE 1

c_B	Basic variables B	Solution or b	$c_j \to -1$	3	-2	0	0	0	Replacement Ratio
			x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	7	3	-1	3	1	0	0	—
0	S_2	12	-2	4	0	0	1	0	$\frac{12}{4} = 3$
0	S_3	10	-4	3	8	0	0	1	$\frac{10}{3}$
$z_j - c_j$			1	-3	2	0	0	0	

This is not an optimal solution as some of the $(z_j - c_j)$ enteries are negative. Hence $z_2 - c_2$ is most negative. Therefore x_2 is an entering variable. From the replacement ratio column, it is obvious to say that minimum ratio occurs for the variable S_2. S_2 is an outgoing variable 4 is the key element.

TABLE 1

c_B	Basic variables B	Solution or b	$c_j \rightarrow 1$ x_1	3 x_2	−2 x_3	0 S_1	0 S_2	0 S_3	Replacement Ratio
0	S_1	7	3	−1	3	1	0	0	—
0	S_2	12	−2	4	0	0	1	0	$\frac{12}{4} = 3$
0	S_3	10	−4	3	8	0	0	1	$\frac{10}{3}$
$z_j - c_j$			1	−3	2	0	0	0	

This is not an optimal solution as some of the $(z_j - c_j)$ enteries are negative. Hence $z_2 - c_2$ is most negative. Therefore x_2 is an entering variable. From the replacement ratio column, it is obvious to say that minimum ratio occurs for the variable S_2. S_2 is an outgoing variable 4 is the key element.

TABLE 2

c_B	Basic variables B	Solution or b	$c_j \rightarrow -1$ x_1	3 x_2	−2 x_3	0 S_1	0 S_2	0 S_3	Replacement Ratio
0	S_1	$7 - 3(-1)$ $= 10$	$3 - \frac{-1}{2}(-1)$ $= \frac{5}{2}$	$-1 - 1(-1)$ $= 0$	$3 - 0(-1) = 3$ $= 3$	$1 - 0(-1) = 1$ $= 1$	$0 - \frac{1}{4}(-1)$ $= \frac{1}{4}$	$0 - 0(-1)$ $= 0$	$\frac{10}{5/2} = 4$
3	x_2	$\frac{12}{4} = 3$	$\frac{-2}{4} = \frac{-1}{2}$	$\frac{4}{4} = 1$	$\frac{0}{4} = 0$	$\frac{0}{4} = 0$	$\frac{1}{4}$	$\frac{0}{4} = 0$	—
0	S_3	$10 - 3 \times 3 = 1$	$-4 - \left(\frac{-1}{2}\right) \times 3$ $= \frac{-5}{2}$	$3 - 1 \times 3 = 0$	$8 - 0 \times 3 = 8$	$0 - 0 \times 3 = 0$	$0 - \frac{1}{4} \times 3$ $= \frac{-3}{4}$	$1 - 0 \times 3 = 1$	—
$z_j - c_j$			$\frac{-1}{2}$	0	0	0	$\frac{3}{4}$	0	

This is not an optimal solution as $z_1 - c_1$ is most negative. Therefore x_1 is an entering variable. From the replacement ratio column, it is obvious to say that minimum ratio occurs for the variable S_1. S_1 is an outgoing variable $\frac{5}{2}$ is the key element.

TABLE 2

c_B	Basic variables B	Solution or b	$c_j \rightarrow -1$ x_1	3 x_2	−2 x_3	0 S_1	0 S_2	0 S_3	Replacement Ratio
0	S_1	$7 - 3(-1)$ $= 10$	$3 - \frac{-1}{2}(-1)$ $= \frac{5}{2}$	$-1 - 1(-1)$ $= 0$	$3 - 0(-1) = 3$ $= 3$	$1 - 0(-1) = 1$ $= 1$	$0 - \frac{1}{4}(-1)$ $= \frac{1}{4}$	$0 - 0(-1)$ $= 0$	$\frac{10}{5/2} = 4$
3	x_2	$\frac{12}{4} = 3$	$\frac{-2}{4} = \frac{-1}{2}$	$\frac{4}{4} = 1$	$\frac{0}{4} = 0$	$\frac{0}{4} = 0$	$\frac{1}{4}$	$\frac{0}{4} = 0$	—
0	S_3	$10 - 3 \times 3 = 1$	$-4 - \left(\frac{-1}{2}\right) \times 3$ $= \frac{-5}{2}$	$3 - 1 \times 3 = 0$	$8 - 0 \times 3 = 8$	$0 - 0 \times 3 = 0$	$0 - \frac{1}{4} \times 3$ $= \frac{-3}{4}$	$1 - 0 \times 3 = 1$	—
$z_j - c_j$			$\frac{-1}{2}$	0	0	0	$\frac{3}{4}$	0	

TABLE 3

c_B	Basic variables B	Solution or b	$c_j \rightarrow -1$ x_1	3 x_2	−2 x_3	0 S_1	0 S_2	0 S_3
−1	x_1	$\frac{10}{5/2} = 4$	$\frac{5/2}{5/2} = 1$	0	$\frac{3}{5/2} = \frac{6}{5}$	$\frac{1}{5/2} = \frac{2}{5}$	$\frac{1/4}{5/2} = \frac{1}{10}$	0
3	x_2	$3 - 4 \times \frac{-1}{2} = 5$	$\frac{-1}{2} - 1 \times \frac{-1}{2} = 0$	$1 - 0 \times \frac{-1}{2} = 1$	$0 - \frac{6}{5} \times \frac{-1}{2} = \frac{3}{5}$	$0 - \frac{2}{5} \times \frac{-1}{2} = \frac{1}{5}$	$\frac{1}{4} - \frac{1}{10} \times \frac{-1}{2} = \frac{3}{10}$	$0 - 0 \times \frac{-1}{2} = 0$
0	S_3	$1 - 4 \times \frac{-5}{2} = 11$	$\frac{-5}{2} - 1 \times \frac{-5}{2} = 0$	$0 - 0 \times \frac{-5}{2} = 0$	$8 - \frac{6}{5} \times \frac{-5}{2} = 11$	$0 - \frac{2}{5} \times \frac{-5}{2} = 11$	$\frac{-3}{4} - \frac{1}{10} \times \frac{-5}{2} = \frac{-16}{20}$	$1 - 0 \times \frac{-5}{2} = 1$
		$z_j - c_j$	0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

This is an optimal solution as all the values of $(z_j - c_j)$ are either zero or positive.

Hence, an optimal solution is as follows :

$$x_1 = 4, \quad x_2 = 5, \quad x_3 = 0$$

Min. $$z = 4 - 3 \times 5 + 2 \times 0 = -11$$

SIMPLEX METHOD
(BIG M METHOD)

Example 3.

Solve the following LPP :

Minimize $z = 4x_1 + x_2$

subject to $3x_1 + 4x_2 \geq 20$

$-x_1 - 5x_2 \leq -15$

and $x_1, x_2 \geq 0.$

Solution.

In this problem, the right hand side component of second inequality is negative which shall be converted into positive by multiplying this inequality by -1. After multiplication, it can be expressed as $x_1 + 5x_2 \geq 15$. After introducing surplus variables S_1 and S_2, the above LPP reduces to :

Maximize $z' = -4x_1 - x_2 + 0\,S_1 + 0\,S_2$

subject to $3x_1 + 4x_2 - S_1 + 0\,S_2 = 20$

$x_1 + 5x_2 + 0\,S_1 - S_2 = 15$

and $x_1, x_2, S_1, S_2 \geq 0.$

Since, we are not getting identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ from the constraint equations; therefore in order to get identity matrix *two* artificial variables are needed.

Maximize $z' = -4x_1 - x_2 + 0\,S_1 + 0\,S_2 - MA_1 - MA_2$

subject to $3x_1 + 4x_2 - S_1 + 0\,S_2 + A_1 + 0\,A_2 = 20$

$x_1 + 5x_2 + 0\,S_1 - S_2 + 0\,A_1 + A_2 = 15$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where M is very large number.

TABLE 1

c_B	Basic variables B	Solution or b	$c_j \to -4$ x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Replacement Ratio
$-M$	A_1	20	3	4	-1	0	1	0	$\frac{20}{4} = 5$
$-M$	A_2	15	1	5	0	-1	0	1	$\frac{15}{5} = 3$
z_j			$-4M$	$-9M$	M	M	$-M$	$-M$	
$z_j - c_j$			$-4M+4$	$-9M+1$	M	M	0	0	

This is not an optimal solution as some of the entries of $(z_j - c_j)$ row are negative. Here, to select entering variable, we may assign $M = 100$ (arbitrary). Then most negative of $(z_j - c_j)$ row comes out to be $(-9M + 1) = -899$. Hence x_2 is an entering variable. From the replacement ratio column it is obvious to say that minimum ratio occurs for the variable A_2. A_2 is an outgoing variable and 5 is the key element.

TABLE 2

c_B	Basic variables B	Solution or b	$c_j \rightarrow -4$ x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	Replacement Ratio
$-M$	A_1	$20 - 3 \times 4$ $= 8$	$3 - \frac{1}{5} \times 4$ $= \frac{11}{5}$	$4 - 1 \times 4$ $= 0$	$-1 - 0 \times 4$ $= -1$	$0 + \frac{1}{5} \times 4$ $= \frac{4}{5}$	$1 - 0 \times 4$ $= 1$	$0 - \frac{1}{5} \times 4$ $= \frac{-4}{5}$	$\frac{8}{11/5} = \frac{40}{11}$
-1	x_2	$\frac{15}{5} = 3$	$\frac{1}{5}$	$\frac{5}{5} = 1$	$\frac{0}{5} = 0$	$\frac{-1}{5}$	$\frac{0}{5} = 0$	$\frac{1}{5}$	$\frac{3}{1/5} = 15$
z_j			$\frac{-11M-1}{5}$	-1	M	$\frac{-4M+1}{5}$	$-M$	$\frac{4M-1}{5}$	
$z_j - c_j$			$\frac{-11M+19}{5}$	0	M	$\frac{-4M+1}{5}$	0	$\frac{9M-1}{5}$	

This is not an optimal solution as some of the entries of $(z_j - c_j)$ row are negative. Here, to select entering variable we may assign $M = 100$ (arbitrary). Then most negative of $(z_j - c_j)$ row comes out to be $\left(\frac{-1100+19}{5}\right) = \frac{-1081}{5}$. Hence x_1 is an entering variable. From the replacement ratio column, it is obvious to say that minimum ratio occurs for the variable A_1. A_1 is an outgoing variable and $\frac{11}{5}$ is the key element.

TABLE 3

c_B	*Basic variables B*	*Solution or b*	$c_j \rightarrow -4$ x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	*Replacement Ratio*
-4	x_1	$\frac{8}{11/5} = \frac{40}{11}$	$\frac{11/5}{11/5} = 1$	$\frac{0}{11/5} = 0$	$\frac{-1}{11/5} = \frac{-5}{11}$	$\frac{4/5}{11/5} = \frac{4}{11}$	$\frac{1}{11/5} = \frac{5}{11}$	$\frac{-4/5}{11/5} = \frac{-4}{11}$	$\frac{40/11}{4/11} = 10$
-1	x_2	$3 - \frac{40}{11} \times \frac{1}{5} = \frac{25}{11}$	$\frac{1}{5} - 1 \times \frac{1}{5} = 0$	$1 - 0 \times \frac{1}{5} = 1$	$0 + \frac{5}{11} \times \frac{1}{5} = \frac{1}{11}$	$\frac{-1}{5} - \frac{4}{11} \times \frac{1}{5} = \frac{-3}{11}$	$0 - \frac{5}{11} \times \frac{1}{5} = \frac{-1}{11}$	$\frac{1}{5} + \frac{4}{11} \times \frac{1}{5} = \frac{3}{11}$	—
z_j			-4	-1	$\frac{19}{11}$	$\frac{-13}{11}$	$\frac{-19}{11}$	$\frac{13}{11}$	
$z_j - c_j$			0	0	$\frac{19}{11}$	$\frac{-13}{11}$	$M - \frac{19}{11}$	$M - \frac{13}{11}$	

This is not an optimal solution as $(z_4 - c_4)$ is still negative. Therefore S_2 is an entering variable. From the replacement ratio column, it is obvious that minimum ratio occurs for the variable x_1 is an outgoing variable and $\frac{4}{11}$ is the key element.

TABLE 4

c_B	Basic variables B	Solution or b	$c_j \to -4$ x_1	-1 x_2	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2
0	S_2	$\frac{40/11}{4/11} = 10$	$\frac{1}{4/11} = \frac{11}{4}$	$\frac{0}{4/11} = 0$	$\frac{-5/11}{4/11} = \frac{-5}{4}$	$\frac{4/11}{4/11} = 1$	$\frac{5/11}{4/11} = \frac{5}{4}$	$\frac{-4/11}{4/11} = -1$
−1	x_2	$\frac{25}{11} - 10 \times \frac{-3}{11}$ $= 5$	$0 - \frac{11}{4} \times \frac{-3}{11}$ $= \frac{3}{4}$	$1 - 0 \times \frac{-3}{11}$ $= 1$	$\frac{1}{11} + \frac{5}{4} \times \frac{-3}{11}$ $= \frac{-1}{4}$	$\frac{-3}{11} - 1 \times \frac{-3}{11}$ $= 0$	$\frac{-1}{11} - \frac{5}{4} \times \frac{-3}{11}$ $= \frac{1}{4}$	$\frac{3}{11} + 1 \times \frac{-3}{11}$ $= 0$
z_j			$\frac{-3}{4}$	-1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0
$z_j - c_j$			$\frac{13}{4}$	0	$\frac{1}{4}$	0	$M - \frac{1}{4}$	M

This is an optimal solution as all the values of $(z_j - c_j)$ row are either zero or positive. Hence optimal solution is as follows :

$$x_1 = 0, x_2 = 5$$

and Minimize $z = 4 \times 0 + 5 = 5.$

SPECIFIC PROBLEMS AND THEIR REMEDIES IN SIMPLEX METHOD

Case - 1: If their is a tie between most negative value of $z_j - c_j$

c_B	Basic variables B	Solution or b	$c_j \to 1$	1	0	0	0	Replacement Ratio
			x_1	x_2	S_1	S_2	S_3	
0	S_1	3	1	2	1	0	0	$\frac{3}{2}$
0	S_2	2	1	1	0	1	0	$\frac{2}{1}$
0	S_3	6	3	2	0	0	1	$\frac{6}{2}$
$z_j - c_j$			−1	−1	0	0	0	

This is not an optimal solution as some of the $(z_j - c_j)$ entries are negative. Here $z_1 - c_1$ and $z_2 - c_2$ both are equal to −1, so that we can choose any one of them, choosing x_2 as an entering variable.

Case - 2: If their is a tie between minimum value of R.R.

c_B	Basic variables B	Solution or b	$c_j \to 10$ x_1	5 x_2	0 S_1	0 S_2	Replacement Ratio
0	S_1	120	4	5	1	0	$\frac{120}{4} = 30$
0	S_2	60	2	2	0	1	$\frac{60}{2} = 30$
z_j			0	0	0	0	
$z_j - c_j$			−10	−5	0	0	

At this stage there are two nonnegative minimum values in replacement ratio column for I and II row. Thus, there is a problem of degeneracy. Therefore, we divide the I and II row by the elements of key column *i.e.*, row I by 4 and row II by 2 and observe the ratios for S_1, S_2, x_1 and x_2.

	S_1	S_2	x_1	x_2
Row I	$\frac{1}{4}$	0	1	$\frac{5}{4}$
Row II	0	$\frac{1}{2}$	1	1

Hence, at this stage comparing step by step from left to right, we observe that the ratio corresponding to column S_1 in row II is least possible. Therefore, we take row II as the key row and S_2 becomes the departing variable and x_1 is the entering variable. 2 is the key element.

Case - 3: If it is not possible to find any value of R.R.

c_B	Basic	Solution	$c_j \to 107$	1	2	0	0	0
	variables B	or b	x_1	x_2	x_3	x_4	S_1	S_2
0	x_4	$\frac{7}{3} - 0 \times \frac{14}{3}$ $= \frac{7}{3}$	$\frac{14}{3} - 1 \times \frac{14}{3}$ $= 0$	$\frac{1}{3} + \frac{1}{3} \times \frac{14}{3}$ $= \frac{17}{9}$	$-2 + \frac{1}{3} \times \frac{14}{3}$ $= \frac{-4}{9}$	$1 - 0 \times \frac{14}{3}$ $= 1$	$0 - 0 \times \frac{14}{3}$ $= 0$	$0 - \frac{1}{3} \times \frac{14}{3}$ $= \frac{-14}{9}$
0	S_1	$5 - 0 \times 16$ $= 5$	$16 - 1 \times 16$ $= 0$	$\frac{1}{2} + \frac{1}{3} \times 16$ $= \frac{35}{6}$	$-6 + \frac{1}{3} \times 16$ $= \frac{-2}{3}$	$0 - 0 \times 16$ $= 0$	$1 - 0 \times 16$ $= 1$	$0 - \frac{1}{3} \times 16$ $= \frac{-16}{3}$
107	x_1	$\frac{0}{3} = 0$	$\frac{3}{3} = 1$	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{0}{3} = 0$	$\frac{0}{3} = 0$	$\frac{1}{3}$
z_j			107	$\frac{-107}{3}$	$\frac{-107}{3}$	0	0	$\frac{107}{3}$
$z_j - c_j$			0	$\frac{-110}{3}$	$\frac{-113}{3}$	0	0	$\frac{107}{3}$

This is not an optimal solution as some of the elements of $(z_j - c_j)$ row are negative. The most negative element of $(z_j - c_j)$ row is $\frac{-113}{3}$. x_3 is an incoming variable. Since there is no positive element in x_3 column. These exist an unbounded solution to the given linear programming problem.

Case - 4: If an artificial variable exist with non-zero value when all $z_j - c_j \geq 0$

c_B	Basic variables B	Solution or b	$c_j \rightarrow 6$	4	0	0	$-M$
			x_1	x_2	S_1	S_2	A_1
4	x_2	5	1	1	1	0	0
$-M$	A_1	$8-5\times1=3$	$0-1\times1=-1$	$1-1\times1=0$	$0-1\times1=-1$	$-1-0\times1=-1$	$1-0\times1=1$
z_j			$4+M$	4	$4+M$	M	$-M$
z_j-c_j			$M-2$	0	$4+M$	M	0

Since all $z_j - c_j \geq 0$, the solution is optimal. But this solution is not feasible for the given problem since it has $x_1 = 0$ and $x_2 = 5$ whereas $x_2 \geq 8$ (second constraint). The fact that artificial variable $A_1 = 3$ is in the solution also indicate that the final solution violates the second constraint by 3 units.

TWO PHASE METHOD

Two Phase Method : In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints (known as auxilliary linear programming problem). Second phase then optimizes the original objective function, starting with the basic feasible solution obtained at the end of phase I.

EXAMPLE 4. *Solve the following linear programming problem by two phase method :*

Minimize $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0.$

SOLUTION. Convert the objective function into maximization form :

Maximize $z' = -x_1 - x_2$

Introducing surplus and artificial variables the constraints can be rewritten as : $2x_1 + x_2 - S_1 + 0\,S_2 + A_1 + 0\,A_2 = 4$

$$x_1 + 7x_2 + 0\,S_1 - S_2 + 0\,A_1 + A_2 = 7$$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0.$

Phase I :

Maximize $z = 0\,x_1 + 0\,x_2 + 0\,S_1 + 0\,S_2 - A_1 - A_2$

subject to $2x_1 + x_2 - S_1 + 0\,S_2 + A_1 + 0\,A_2 = 4$

$$x_1 + 7x_2 + 0\,S_1 - S_2 + 0\,A_1 + A_2 = 7$$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Phase I : Maximize $z = 0\,x_1 + 0\,x_2 + 0\,S_1 + 0\,S_2 - A_1 - A_2$

subject to $2x_1 + x_2 - S_1 + 0\,S_2 + A_1 + 0\,A_2 = 4$

$$x_1 + 7x_2 + 0\,S_1 - S_2 + 0\,A_1 + A_2 = 7$$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

TABLE 1

c_B	*Basic variables B*	*Solution or b*	$c_j \rightarrow 0$ x_1	0 x_2	0 S_1	0 S_2	−1 A_1	−1 A_2	*Replacement Ratio*
−1	A_1	4	2	1	−1	0	1	0	$\frac{4}{1} = 4$
−1	A_2	7	1	7	0	−1	0	1	$\frac{7}{7} = 1$
z_j			−3	−8	1	1	−1	−1	
$z_j - c_j$			−3	−8	1	1	0	0	

This is not an optimal solution as some of the elements of $(z_j - c_j)$ row are negative. The most negative element of $(z_j - c_j)$ row is -8 ; *i.e.*, x_2 is an entering variable. From replacement ratio column, it is obvious to say that minimum ratio occurs for the variable A_2. A_2 is a departing variable and 7 is the key element.

TABLE 2

c_B	Basic variables B	Solution or b	$c_j \to$ 0 x_1	0 x_2	0 S_1	0 S_2	-1 A_1	Replacement Ratio
-1	A_1	$4-1\times1=3$	$2-\frac{1}{7}\times1=\frac{13}{7}$	$1-1\times1=0$	$-1-0\times1=-1$	$0+\frac{1}{7}\times1=\frac{1}{7}$	$1-0\times1=1$	$\frac{3}{13/7}=\frac{21}{13}$
0	x_2	$\frac{7}{7}=1$	$\frac{1}{7}$	$\frac{7}{7}=1$	$\frac{0}{7}=0$	$\frac{-1}{7}$	$\frac{0}{7}=0$	$\frac{1}{1/7}=7$
z_j			$\frac{-13}{7}$	0	1	$\frac{-1}{7}$	-1	
z_j-c_j			$\frac{-13}{7}$	0	1	$\frac{-1}{7}$	0	

This is not an optimal solution as some of the elements of (z_j-c_j) row are negative. The most negative element of (z_j-c_j) row is $\frac{-13}{7}$ *i.e.*, x_1 is an entering variable. From the replacement ratio column, it is clear that minimum ratio occurs for the variable A_1. A_1 is an outgoing variable and $\frac{13}{7}$ is the key element.

TABLE 3

c_B	Basic variables B	Solution or b	$c_j \to 0$ x_1	0 x_2	0 S_1	0 S_2
0	x_1	$\frac{3}{13/7} = \frac{21}{13}$	$\frac{13/7}{13/7} = 1$	$\frac{0}{13/7} = 0$	$\frac{-1}{13/7} = \frac{-7}{13}$	$\frac{1/7}{13/7} = \frac{1}{13}$
0	x_2	$1 - \frac{21}{13} \times \frac{1}{7} = \frac{10}{13}$	$\frac{1}{7} - 1 \times \frac{1}{7} = 0$	$1 - 0 \times \frac{1}{7} = 1$	$0 + \frac{7}{13} \times \frac{1}{7} = \frac{1}{13}$	$\frac{-1}{7} - \frac{1}{13} \times \frac{1}{7} = \frac{-2}{13}$
z_j			0	0	0	0
$z_j - c_j$			0	0	0	0

There exist no artificial variable in the basis and $Z = 0$.

Phase I ends. We proceed to phase II.

For phase II the objective function will be :

$$\text{Max. } z = -x_1 - x_2 + 0\,S_1 + 0\,S_2.$$

TABLE 4

c_B	Basic variables B	Solution or b	$c_j \to -1$	-1	0	0
			x_1	x_2	S_1	S_2
-1	x_1	$\frac{21}{13}$	1	0	$\frac{-7}{13}$	$\frac{1}{13}$
-1	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$\frac{-2}{13}$
z_j			-1	-1	$\frac{6}{13}$	$\frac{1}{13}$
$z_j - c_j$			0	0	$\frac{6}{13}$	$\frac{1}{13}$

Since all the entries of $(z_j - c_j)$ row are either zero or positive, the above table gives the optimal solution.

Tha optimal solution is as follows : $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$

Minimize $z = \frac{21}{13} + \frac{10}{13} = \frac{31}{13}$.

DUALITY OF LPP

FORMATION OF DUAL PROBLEM

From the given Primal Problem

Consider the standard form of an LPP

Maximize $z_p = c_1x_1 + c_2x_2 + ... + c_nx_n$

s.t.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m$$
$$x_1, x_2 ....., x_n \ge 0$$

Before finding and its dual, all the constraints should be transformed to less than or equal to type for maximization problem and to greater than or equal to type for minimization problem. It can be done by multiplying with –1 to both the sides of the constraints, so that sign of inequality gets reversed.

Dual of the above problem can be found as follows :

Minimize $z_D = b_1W_1 + b_2W_2 + ... + b_mW_m$

s.t.

$$a_{11}W_1 + a_{21}W_2 + ... + a_{m1}W_m \ge C_1$$
$$a_{12}W_1 + a_{22}W_2 + ... + a_{m2}W_m \ge C_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{1n}W_1 + a_{2n}W_2 + ... + a_{mn}W_m \ge C_n$$

and $\qquad W_1, W_2, ..., W_m \ge 0.$

EXAMPLE 5. *Give the dual of the following linear programming problem:*

$$\begin{aligned}
&\text{Minimize } z = x_1 + 2x_2 - x_3\\
&\text{subject to} \quad 2x_1 - 3x_2 + 4x_3 \geq 5\\
&\qquad 2x_1 - 2x_2 \geq 6\\
&\qquad 3x_1 - x_3 \leq 4 \quad \text{and} \quad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Solution.

After putting all the constraints into ($\geq$) type, above problem reduces to

$$\begin{aligned}
&\text{Minimize } z_p = x_1 + 2x_2 - x_3\\
&\text{subject to} \quad 2x_1 - 3x_2 + 4x_3 \geq 5\\
&\qquad 2x_1 - 2x_2 \geq 6\\
&\qquad -3x_1 + x_3 \geq -4\\
&\text{and} \qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Then dual of the above linear programming problem is

$$\begin{aligned}
&\text{Maximise } z_D = 5W_1 + 6W_2 - 4W_3\\
&\text{subject to} \quad 2W_1 + 2W_2 - 3W_3 \leq 1\\
&\qquad -3W_1 - 2W_2 \leq 2\\
&\qquad 4W_1 + W_3 \leq -1\\
&\text{and} \qquad W_1, W_2, W_3 \geq 0.
\end{aligned}$$

EXAMPLE 6. *Find the dual of the given linear programming problem :*

$$\text{Minimize } z_p = x_1 + x_2 + x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ 2x_1 - 2x_2 &\le 3 \\ 2x_2 - x_3 &\ge 5 \end{aligned}$$

and $x_1, x_2 \ge 0$, x_3 *is unrestricted in sign.*

Solution. Since x_3 is unrestricted in sign, so we can take x_3 as a difference of two non-negative variables x_3' and x_3''.

Above linear programming problem reduces to

$$\text{Minimize } z_p = x_1 + x_2 + x'_3 - x_3''$$

$$\text{subject to} \quad \begin{aligned} x_1 - 3x_2 + 4x_3' - 4x_3'' &\ge 5 \\ x_1 - 3x_2 + 4x_3' - 4x_3'' &\le 5 \\ 2x_1 - 2x_2 &\le 3 \\ 2x_2 - x_3' + x_3'' &\ge 5 \end{aligned}$$

and $x_1, x_2, x_3', x_3'' \ge 0$; where $x_3' - x_3'' = x_3$

After putting all the contraints into ($\geq$) type, above problem becomes

Minimize $z_p = x_1 + x_2 + x_3' - x_3''$

subject to

$$
\begin{aligned}
x_1 - 3x_2 + 4x_3' - 4x_3'' &\geq 5 \\
-x_1 + 3x_2 - 4x_3' + 4x_3'' &\geq -5 \\
-2x_1 + 2x_2 &\geq -3 \\
2x_2 - x_3' + x_3'' &\geq 5 \quad \text{and} \quad x_1, x_2, x_3', x_3'' \geq 0
\end{aligned}
$$

Then the dual of the above linear programming problem is

Minimize $z_D = 5W_1' - 5W_1'' - 3W_2 + 5W_3$

subject to

$$
\begin{aligned}
W_1' - W_1'' - 2W_2 &\leq 1 \\
-3W_1' + 3W_1'' + 2W_2 + 2W_3 &\leq 1 \\
4W'_1 - 4W_1'' - W_3 &\leq 1 \\
-4W'_1 + 4W_1'' + W_3 &\leq -1 \quad \text{and} \quad W_1', W_1'', W_2, W_3 \geq 0
\end{aligned}
$$

Let $W_1' - W_1'' = W_1$ the above dual problem becomes

Maximize $z_D = 5W_1 - 3W_2 + 5W_3$

subject to

$$
\begin{aligned}
W_1 - 2W_2 &\leq 1 \\
-3W_1 + 2W_2 + 2W_3 &\leq 1 \\
4W_1 - W_3 &\leq 1 \\
-4W_1 + W_3 &\leq -1 \quad \text{and } W_2, W_3 \geq 0,
\end{aligned}
$$

W_1 is unrestricted in sign.

RELATION BETWEEN PRIMAL AND DUAL PROBLEM

The primal and the dual problems are related to each other in the following manner :

1. If one is a maximization problem then the other is a minimization problem.
2. If one of them has an optimum solution, the other also has an optimum solution. The solution of the other problem can be read from the $z_j - c_j$ row below the columns of slack variables. Both will have the same optimum value

   i.e., Max. z = Min. z^*
3. If the primal problem has an unbounded solution, then the dual has either no solution or an unbounded solution.
4. If dual has no feasible solution, then the primal also admits no feasible solution.

EXAMPLE 7. *Solve the following linear programming problem by solving its dual :*

$$\text{Minimize } z_p = 4x_1 + x_2$$

$$\text{subject to} \quad 3x_1 + 4x_2 \geq 20$$

$$-x_1 - 5x_2 \leq -15$$

$$\text{and} \quad x_1, x_2 \geq 0.$$

Solution. The given primal problem can be rewritten as

$$\text{Minimize } z_p = 4x_1 + x_2$$

$$\text{subject to} \quad 3x_1 + 4x_2 \geq 20$$

$$x_1 + 5x_2 \geq 15$$

$$\text{and} \quad x_1, x_2 \geq 0$$

The dual of the above lpp is as follows :

$$\text{Maximize } z_D = 20W_1 + 15W_2$$

$$\text{subject to} \quad 3W_1 + W_2 \leq 4$$

$$4W_1 + 5W_2 \leq 1$$

$$\text{and} \quad W_1, W_2 \geq 0$$

After introducing slack variables S_1 and S_2, the given problem can be rewritten as

$$\text{Maximize } z_D = 20W_1 + 15\,W_2 + 0S_1 + 0S_2$$

$$\text{subject to} \quad 3W_1 + W_2 + S_1 + 0S_2 = 4$$

$$4W_1 + 5W_2 + 0S_1 + S_2 = 1$$

$$\text{and} \quad W_1, W_2, S_1, S_2 \geq 0$$

TABLE 1

c_B	Basic variables B	Solution or b	$c_j \to 20$	15	0	0	Replacement Ratio
			W_1	W_2	S_1	S_2	
0	S_1	4	3	1	1	0	$\frac{4}{3}$
0	S_2	1	4	5	0	1	$\frac{1}{4}$
z_j			0	0	0	0	
$z_j - c_j$			−20	−15	0	0	

This is not an optimal solution Here, most negative element of $(z_j - c_j)$ row is −20 , *i.e.*, W_1 is an incoming variable. From the replacement ratio column, S_2 is an outgoing variable 4 is the key element

TABLE 2

c_B	Basic variables B	Solution or b	$c_j \rightarrow 20$	15	0	0
			W_1	W_2	S_1	S_2
0	S_1	$4-\frac{1}{4}\times 3 = \frac{13}{4}$	$3-1\times 3 = 0$	$1-\frac{5}{4}\times 3 = \frac{-11}{4}$	$1-0\times 3 = 1$	$0-\frac{1}{4}\times 3 = \frac{-3}{4}$
20	W_1	$\frac{1}{4}$	$\frac{4}{4}=1$	$\frac{5}{4}$	$\frac{0}{4}=0$	$\frac{1}{4}$
z_j			20	25	0	5
z_j-c_j			0	10	0	5

Since all the entries of $(z_j - c_j)$ row either zero or positive. Hence the above programme is an optimal The optimal solution of dual problem is $W_1 = \frac{1}{4}$, $W_2 = 0$ and Max. $z = 5$.

The optimal value of the primal objective function $(z_p = 5)$ is same as optimal value of dual objective function $(z_D = 5)$. The values of x_1 and x_2 in the primal programme are 0 and 5 respectively. These values are identical with the entries of $(z_j - c_j)$ row of the final simplex table of dual under the columns labled S_1 and S_2. Thus, the solution of primal problem is $x_1 = 0$, $x_2 = 5$ and $z_p = 5$.

Exercise

1. Solve the following LPP:
   Maximize $z = 5x_1 + 3x_2$
   subject to $3x_1 + 5x_2 \le 15$
   $5x_1 + 2x_2 \le 10$
   and $x_1, x_2 \ge 0$

2. Solve the following LPP:
   Minimize $z = -x_1 - x_2$
   subject to $x_1 + 2x_2 \le 3$
   $x_1 + x_2 \le 2$
   $3x_1 + 2x_2 \le 6$
   and $x_1, x_2 \ge 0$

3. Use big - M method to solve the following LPP:
   Minimize $z = 4x_1 + 3x_2$
   subject to $x_1 + 2x_2 \ge 8$
   $3x_1 + 2x_2 \ge 12$
   and $x_1, x_2 \ge 0$

4. Use big - M method to solve the following LPP:

Maximize $z = 5x_1 + 3x_2$

subject to $2x_1 + 4x_2 \le 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \ge 10$

and $x_1, x_2 \ge 0$

5. Solve the following LPP by two phase method:

Maximize $z = 4x_1 + 3x_2$

subject to $2x_1 + 3x_2 \le 6$

$4x_1 + 6x_2 \ge 24$

and $x_1, x_2 \ge 0$

6. Find the dual of the following LPP:

Minimize $z_p = 30x_1 + 20x_2$

subject to $-x_1 - x_2 \ge -8$

$-6x_1 - 4x_2 \le -12$

$5x_1 + 8x_2 = 20$

and $x_1, x_2 \ge 0$

7. Solve the following LPP by converting it into its dual:

Minimize $z_p = 20x_1 + 10x_2$

subject to $x_1 + x_2 \ge 10$

$3x_1 + 2x_2 \ge 24$

and $x_1, x_2 \ge 0$

Answers

1. $x_1 = 20/19, \quad x_2 = 45/19$ and max. $z = 235/19$

2. $x_1 = 1, \quad x_2 = 1$ and min. $z = -2$

3. $x_1 = 2, \quad x_2 = 3$ and min. $z = 17$

4. $x_1 = 4, \quad x_2 = 1$ and min. $z = 23$

5. No feasible solution

6. Minimize $z_D = 8W_1 - 12W_2 + 20W_3 - 20W_4$
   subject to $W_1 - 6W_2 + 5W_3 - 5W_4 \geq 30$
   $W_1 - 4W_2 + 8W_3 - 8W_4 \geq 20$
   and $W_1, W_2, W_3, W_4 \geq 0$

7. Optimal solution of dual problem is: $W_1 = 0, \quad W_2 = 5$ and max. $z_D = 120$

   and optimal solution of primal problem is: $x_1 = 0, \quad x_2 = 12$ and min. $z_p = 120$

THANKS