Oscillation, Chaos and Control of Nonlinear Dynamical Models of Musical Instruments with Sustained Sound

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Abstract

Basic models of musical instruments are studied as nonlinear dynamical systems with delay. The behavior of models, stability, oscillation, periodicity and chaos is determined. The number of periodic solutions is reduced. In chaotic-like signals, the proportion of sinusoidal versus non-sinusoidal components can be varied at will.

1. Introduction

This paper describes an approach to the functioning of musical instruments with sustained sound as nonlinear dynamical systems. We highlight the existence of delay terms in the equations of models and study a family of delay equations which retain the essence of the behavior of certain classes of instruments. We have determined the behavior of our models, in terms of stability, oscillation, periodicity and finally chaos. Moreover we have found analytically some conditions for these behaviors. Our models usually have several periodic solutions depending on initial conditions. This occurs in natural instruments but can be a serious inconvenience for an electronic instrument. We therefore show how to limit the number of solutions. We have realised real-time digital simulations of our models on a workstation. Our models also provide interesting chaotic-like signals where the proportion of sinusoidal components versus non-sinusoidal components can be varied at will.

2. Basic Models of Musical Instruments

Many sustained musical instruments [Fletcher & al 91], can be described by an autonomous system of integral and differential delay equations. Seen from the mouthpiece, the bore of a flute appears first as a delay line with a sign inversion reflection at the other extremity, and some low pass filtering. The corresponding delay T or some related period is responsible for the pitch played by the instrument. The basis of the oscillatory behavior is to be found in the coupling of this passive linear part with the nonlinearity at the reed. For string instruments the delay comes from transverse waves travelling back and forth along the string. The nonlinearity is due to the bow-string interaction and represented as a nonlinear velocity-force function.

2.1. Clarinet Basic Model

The bore of a clarinet is cylindrical and therefore does not induce much reflection until the bell where the sound wave is filtered through an impulse response h_b (including sign inversion), reflected and transmitted back to the mouthpiece. The delay T corresponds to the total transmission back and forth along the bore. The reed induces a nonlinear relation between the pressure and the flow velocity between the reed and the mouthpiece [McIntyre & al. 83]. One can assume that the reed has no mass, leading to a *instantaneous* nonlinearity G.

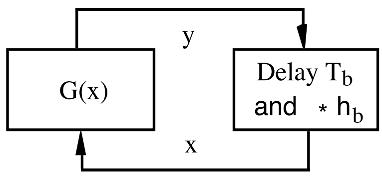


Fig. 1. Clarinet Basic Model

In [Rodet 93b] we showed that a basic model of a clarinet-like instrument (Fig. 1.) is given by the following equation where x, $y: R \rightarrow R$ represent pressure or flow velocity and * is the convolution operator:

$$x(t) = G(x(t-T_b)) * h_b \quad \text{or} \quad y(t) = G(y(t-T_b) * h_b) \quad (1)$$

2.2. Flute Basic Model

The bore of a flute is also a cylindrical pipe. At the mouthpiece, an air jet blown through a flue channel is directed towards a sharp edge. This jet is deviated alternatively on each side of the edge by the influence, at the flue exit, of sound waves induced in the pipe by jet oscillations [Verge 95]. The portion Q_{in} of the volume flux of the jet entering the pipe is a nonlinear function N of the transverse acoustic flow in the tube at the flue exit. The pipe is represented again by a delay T_b and an impulse response h_b. Another delay appears in the model. Jet deviations are essentially initiated at the flue exit and propagate along the jet, reaching the edge after a delay T_j. The jet deviation at the edge is noted d and is assumed to be a linear function of the flux Q in the mouthpiece, with impulse response h_j: $d = h_j * Q(t-T_j)$. Finally, we have studied [Rodet 95a] a basic model of the flute-like instruments in the form of the system in Fig. 2.

Like the basic clarinet model, this flute model is composed of a linear feedback loop $d\square=L\{Q_{in}\}$ closed on an instantaneous nonlinearity N, which complies with the form of equation 1. But L is composed here of one loop with delay (T_b, h_b) in series with a delayed element. This configuration provides a more complicated transfer function, in phase as well as in magnitude, than the one of the single delayed element of the clarinet case.

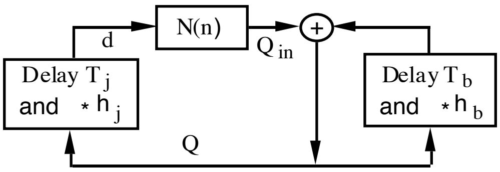


Fig. 2. Flute Basic Model

Another difference is that N is rather smooth while the nonlinearity G, in the clarinet case, even has a discontinuous derivative when the reed enters in contact with the mouthpiece.

2.3. Violin Basic Model

Following [McIntyre & al. 83] and [Smith 86], we are studying a basic model of the oscillation of violin-like instruments (Fig. 3). The force f exerted by the bow in the direction of its displacement (friction) is a function of its velocity v. Two waves propagate along the string in opposite directions from the point of contact and reflect at the bridge and at the nut. this propagation is represented here by the elements (T_1, h_1) and (T_2, h_2).

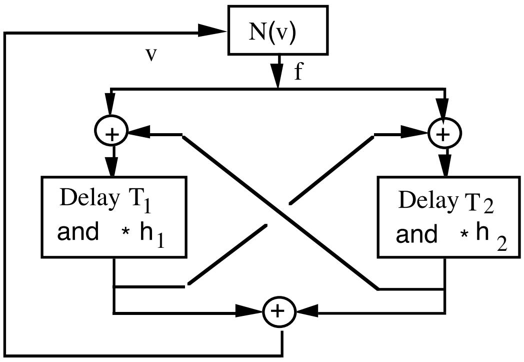


Fig. 3. Violin Basic Model

Again, this violin model is composed of a linear feedback loop v=L{f} closed on an instantaneous nonlinearity N and complies with the form of equation 1. L is here composed of two interleaved loops with delays, (T_1, h_1) and (T_2, h_2). This configuration also provides a rather complicated transfer function in phase and magnitude. The nonlinearity N corresponds to friction and should be a highly discontinuous function. However some effects such as the torsion of the string tend to attenuate the discontinuity [McIntyre & al. 83].

2.2. Trumpet Basic Model

In [Rodet 95b] we propose and study a basic model of the oscillation of trumpet like instruments. As for clarinet and flute, the bore is represented by an impulse response h_b and a delay T_b. This linear feedback loop is different from the ones of the previous bores since a trumpet first has a semi-hemispherical *cup*, then is partly cylindrical, partly conical and terminated by a large bell. But the most specific part is the second loop due to the mass. Contrary to the clarinet reed, the reed here, i.e. the lip, has now a non-negligible mass, the resonance frequency of which plays an essential role. This is represented by an impulse response h_l (Fig. 4.). But even more important is the fact that the non linearity N is now a function of two variables, the position x of the mass (lip) limiting more or less the air flow, and the pressure Ph of the incoming wave due to the feedback loop of the bore. The dimension 2 of the argument of N complicates stability and behavior analysis as we explain in section 3. We think that this system captures an important part of the behavior of trumpet-like instruments in the interaction of the modes of the tube loop with the resonance of the mass subsystem [Rodet 92].

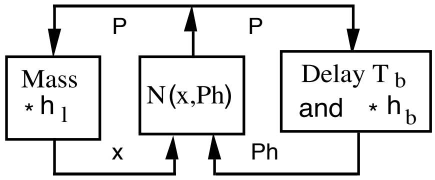


Fig. 4. Trumpet Basic Model

3. Stability and oscillation of basic models

We first consider equation 1 which, for x, y, h_b, $G:R \rightarrow R$, represents the first model proposed above. This equation defines the class of systems composed of an instantaneous nonlinearity G and a single linear feedback loop including an impulse response h_b and a delay T_b. In [Rodet 93b] we showed that we can easily determine the stability and some oscillation properties of models in an even larger class which encompasses the four models proposed above. The main result is given in terms of the Graphical Hopf Theorem and its algebraic version. Then under certain conditions on G and h_b, the system has a unique stable periodic solution. Therefore it provides the existence, uniqueness and stability test of the solution required for our application. Note that the only restriction on the linear element is that its impulse response be *stable*. In particular the continuous transfer function of the linear element does not need to be a rational function and thus can include delays. The application is straightforward in the 1 dimensional case, and we are applying the theorem also to the 2x1 dimensions case of the basic trumpet model.

Solutions of such a system are usually numerous and relatively difficult to find. Since they are the most important for the normal playing conditions of a musical instrument, let us consider the *2T-periodic solutions* of (1), i.e. those with period length approximately 2T. We have noticed in [Rodet 93b] that the period can differ from 2T because of additional delays added by the convolution by h_b. A period of 2T exactly can be obtained by choosing h_b even symmetric. We study here only exactly-2T periodic solutions. Let us rewrite (1) in the discrete signals domain, as used in our computer simulations for synthesis of musical sounds:

$$x_n = (h*G(x))_{n-T} = \sum_{m=-m_1}^{m_2} h_m G(x_{n-m-T}), \text{ or}$$

$$y_n = G\Big(\big(h*y \big)_{n-T} \Big) = G\left(\sum_{m=-m_1}^{m_2} h_m y_{n-m-T} \right) \quad (2)$$

Since we are dealing with physical systems, it is natural to suppose that $\{h_m\}$ is causal, $m_1 < T$, and finite and $m_2 < \infty$. The sample n can be computed when the past samples from $n-m_2-T$ to $n+m_1-T$ are known. Thus, the minimal set of samples necessary to continue the computation of samples forever after n is $[x_{n-m_2-T}, x_{n-1}]$, which is of dimension m_2+T. Therefore, solutions of equation (2) are also solutions of an iterated map $Q:R^{m_2+T} \rightarrow R^{m_2+T}$. Notice that the 2T-periodic solution of (2) are usually not fixed points of $Q \circ Q$. But from Q it is easy to derive a map $\Psi:R^{2T} \rightarrow R^{2T}$ with the same solutions, such that the 2T-periodic solutions of (2) are the fixed points of Ψ. The search for solutions of (2) is eased if G is a rational function $G(x) = B(x)/A(x)$ where A and B are polynomials. Then (2) can be rewritten into a polynomial equation in the variables y_n:

$$y_n \,.\, B\Big(\big(h*y \big)_{n-T} \Big) = A\Big(\big(h*y \big)_{n-T} \Big), \quad (3)$$

In consequence, the search for solutions of (2) can be replaced by the search of solutions of a system of polynomial equations. Interestingly enough, this is true also in the Fourier domain. Let $\{s_n\}$ be a 2T-periodic solution of (2). Let us call $S=\{S_k\}$ and $\Gamma(S)=\{\Gamma_k(S)\}$ respectively the Fourier series of $\{s_n\}$ and of $\{G(s_n)\}$, of length 2T. Then the solutions $\{s_n\}$ are such that:

$$S_k = (-1)^k H_k \Gamma_k(S), \; k=1, 2, ...2T \quad (4)$$

where $(-1)^k$ accounts for the time delay T, and $H=\{H_k\}$ is the Fourier series taken after time aliasing of $\{h_n\}$ on an interval of length 2T and overlap-adding. From equation (4) we define a map $M:B \rightarrow B$. The set $B \subset C^{2T}$ (where C is the complex plane) is the set of Fourier series of real periodic discrete signals of period length 2T. When $\Gamma_k(S)$ is the k^{th} Fourier coefficient of G(s) and s is the inverse Fourier series of S, the k^{th} component of M is:

$$M_k(S) = (-1)^k H_k \Gamma_k(S), \; k=1,2,...2T \quad (5)$$

In the time domain, we call $P:R^{2T} \rightarrow R^{2T}$ the map associated with M and defined by:

$$P(s) = \text{Fourier}^{-1}(M(S)) \quad (6)$$

where *Fourier*$^{-1}$ designates the inverse Fourier series, or equivalently:

$$P(s) = r * G(s_{n-T}) \quad (7)$$

where $r = \text{Fourier}^{-1}(H_k)$. Fixed points of the map P (or M) are the periodic solutions of (2). As said above, if G is a rational function, then the fixed points of P are the solutions of a system of polynomial equations. This is also true for M in the Fourier domain: Note first that the Fourier series of x^2, $x \in R^{2T}$, is the circular convolution $X \otimes X$ which is a polynomial in the variables $\{X_k$, $k=1,2,...2T\}$. Therefore the Fourier Transform of (3) is a polynomial equation, q.e.d.. The advantages of a rational function nonlinearity are explained in section 4. Once the 2T-periodic solutions of (2) have been found, there stability can easily be checked. Let $D\Psi_x$ be the derivative of Ψ at the point $s \in R^{2T}$. The stability of a 2T-periodic solution of (2), i.e. of a fixed point s of Ψ is given by the eigenvalues of $D\Psi_s$.

In [Rodet 94] we show how the 2T periodic solutions of (2) can be found by using a homotopy method: these solutions are trivial when $h_0 = 1$ and $h_m = 0$ for $m \neq 0$, i.e. there is no filtering in the feedback loop. From these trivial solutions, general solutions are followed when filtering by h is gradually introduced in the loop.

4. Rational function nonlinearity

A polynomial nonlinearity G is easy to implement, can be efficiently computed on a digital processor and provides easy access to its important characteristics: e.g., the characteristics of G which we are interested in for easy control of oscillations are the negative slope around a fixed point and a slope of smaller magnitude further from this point [Rodet 93a]. A serious inconvenience, for musical usage, of the choice of a polynomial G is that it is not possible to avoid the existence of other operating points than the one cited above, and that the system can be unbounded. A rational function nonlinearity avoids these difficulties. For instance, in [Rodet 94] we look at:

$$G(x) = (x^3 + a_1 x)/(1 + b_2 x^2) \quad (8)$$

Then (1) can have no spurious fixed points and be bounded. Solutions with periods 2T or multiples and chaotic solutions can easily be obtained and controlled.

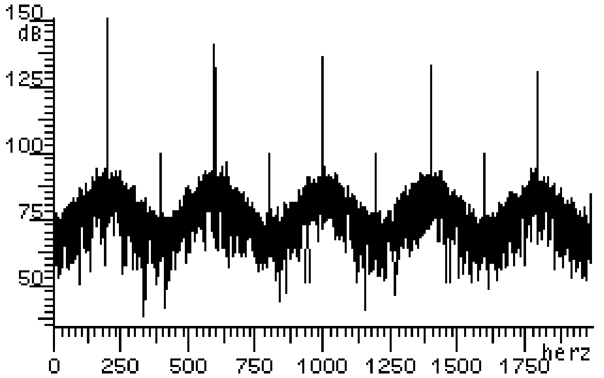
5. Number of periodic solutions

We have mentioned in the introduction our interest for the design of a system which would model the usual playing behavior of an instrument but could avoid the other behaviors if requested. For the feedback systems studied here we have shown in [Rodet 94] that a path toward such a goal could be based upon the low pass character of the linear element. As we gradually introduce low pass filtering in the feedback loop, we observe that the number of 2T-periodic solutions decreases and can be reduced to unity.

We have looked at all the musical instruments with sustained sounds (double reed instruments only have been omitted, but they can be described in the same way). It is remarkable that all the basic models have the same structure, with the delay feedback loop playing three essential roles. First it allows a precise and easy control of the period of the produced signal, which is responsible for the perceived pitch. Secondly it has a stabilisation effect such that 2T-periodic solutions are very often obtained, making the system *robust*. Finally it provides a large number of states leading to a large number of different periodic and chaotic solutions.

6. Chaotic signals and musical applications

We have developed real-time simulations of our systems on a workstation. Harmonic sounds and, in "chaotic-like" regions, sounds with non-sinusoidal components are obtained, as occurs for the majority of natural instruments. The non-sinusoidal components are heard as "noise". The noise and sinusoidal components coming from our system are correlated and *fuse* together. We have found the possibility to control precisely the proportion of noise components which the chaotic behavior introduces in the signal (Fig. 5). Chaotic signals, even when they are extremely "noisy", keep some of the harmonic structure derived from the fundamental frequency that corresponds to the delay T. The persistence of the harmonic structure in the chaotic signal is heard as a *pitch* even though the sound is noisy . Moreover, the value of the pitch and the amount of tonal (harmonic) sound perceived, as compared to "noise", can easily be controlled at will. The proportion of non-sinusoidal versus sinusoidal components can be changed continuously. In any case, the harmonic structure induced by the delay line persists in the chaotic signal.



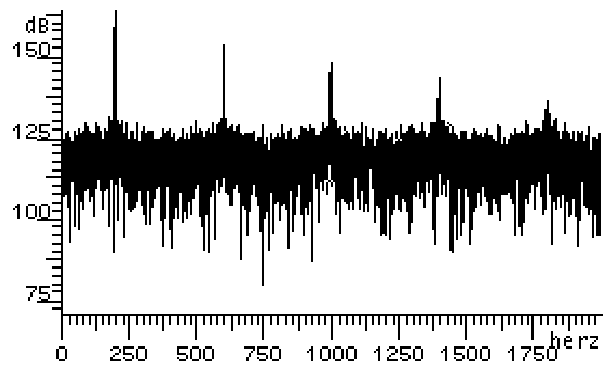


Fig. 5. Different proportions of sinusoidal/non-sinusoidal components in chaotic-like signals

7. Conclusion

We have studied here some problems stemming from physical models of musical instruments. We have proposed basic models of sustained musical instruments based on a linear element with delay, coupled with a nonlinear function. We have explained the different roles of the delay feedback loop. It has a stabilisation effect and provides a large number of states leading to a large number of different periodic and chaotic solutions. The number of solutions can be drastically reduced by the introduction of a low pass filter in the loop. We have been able to determine the main behaviour of such systems and the regions of the parameter space where this behaviour occurs. Bifurcation from stability usually leads to periodic oscillation of the instrument. Periodic oscillation is generally used in the normal playing conditions but the question of chaos is posed. We have found how the parameters of the models can be used to control the main characteristics of the sound that they produce, such as pitch, richness, transient onset velocity, and proportion of "noise" in the sound.

Acknowledgements

I. Steinecke, N. Misdariis, C. Vergez and A. Ward.

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