

# LINEAR ALGEBRA REVIEW (OVERVIEW)

Joseph Nelson, Data Science Immersive

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# AGENDA

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- ▶ Notation
- ▶ Linear Operations
- ▶ Advanced Topics

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## WHAT IS LINEAR ALGEBRA?

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- ▶ Linear algebra is the math of vectors and matrices.
- ▶ It is distills math problems down to their fundamental principles, which enables us to solve complex problems in a simpler way

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## WHAT IS LINEAR ALGEBRA?

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- ▶ Solve this:
- ▶  $4x_1 - 5x_2 = -13$
- ▶  $-2x_1 + 3x_2 = 9$

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## WHAT IS LINEAR ALGEBRA?

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- ▶ Solve this:
- ▶  $4x_1 - 5x_2 = -13$
- ▶  $-2x_1 + 3x_2 = 9$
  
- ▶  $X_2 = 5$
- ▶  $X_1 = 3$

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## WHAT IS LINEAR ALGEBRA?

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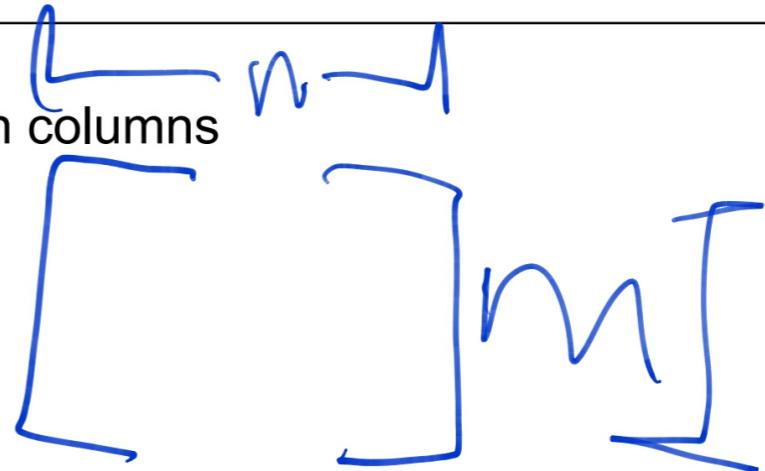
- Matrix algebra:

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

## BASIC NOTATION - MATRIX

- A matrix with real-valued entries, m rows, and n columns

$$A \in \mathbb{R}^{m \times n}$$



- Denote a single entry in this matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} A_{ij}$$

## BASIC NOTATION - VECTOR

- ▶ A (column) vector with n real-valued entries

$$x \in \mathbb{R}^n$$

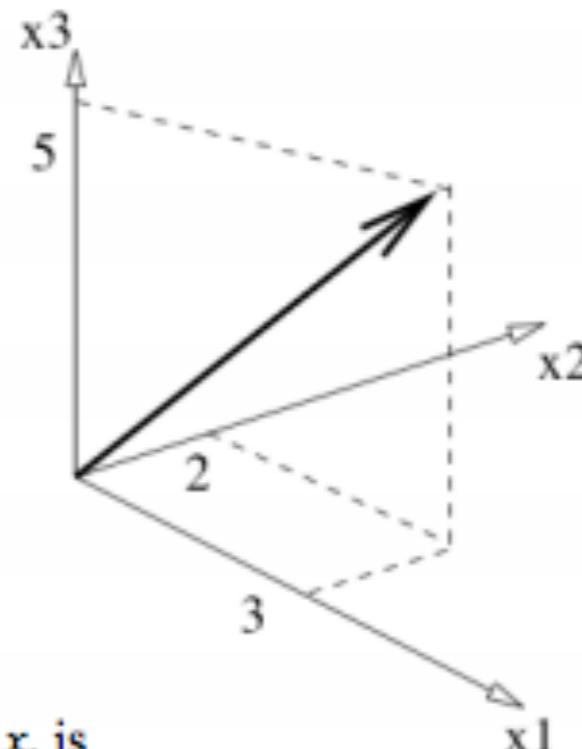
- ▶ Denote a single entry in this vector:

$x_i$  denotes the  $i^{th}$  entry

## VECTOR: SPATIAL REPRESENTATION

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

e.g.  $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$



The **length** of  $x$ , a.k.a. the **norm** or **2-norm** of  $x$ , is

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

e.g.,

$$\|\boldsymbol{x}\| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$$

## BASIC NOTATION - TRANSPOSE

- The transpose operator  $A^T$  switches rows and columns of a matrix

$$A_{ij} = (A^T)_{ji}$$

$A = \begin{bmatrix} n \\ m \end{bmatrix}$   
 $A^T = \begin{bmatrix} m \\ n \end{bmatrix}$

- Denote a single entry in this vector:

$$x = \begin{bmatrix} n \\ 1 \end{bmatrix}$$
$$x^T = \begin{bmatrix} 1 \end{bmatrix}$$

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## BASIC NOTATION – ELEMENTS OF A MATRIX

- We can write a matrix in terms of its columns

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

- Denote a single entry in this vector:

$a_i \in \mathbb{R}^m$   $(a_i)_j := A_{ji}$

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## LINEAR OPERATIONS - ADDITION

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- ▶ Vectors of equal dimensions may be added together:

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

Denote a single entry in this vector:

Represent a matrix by its rows:

$$A = \begin{bmatrix} -a^T \\ -a^T \\ -a^T \\ -a^T \end{bmatrix}$$

$$a_i \in \mathbb{R}^n$$

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## LINEAR OPERATIONS - ADDITION

► Vectors of equal dimensions may be added together:

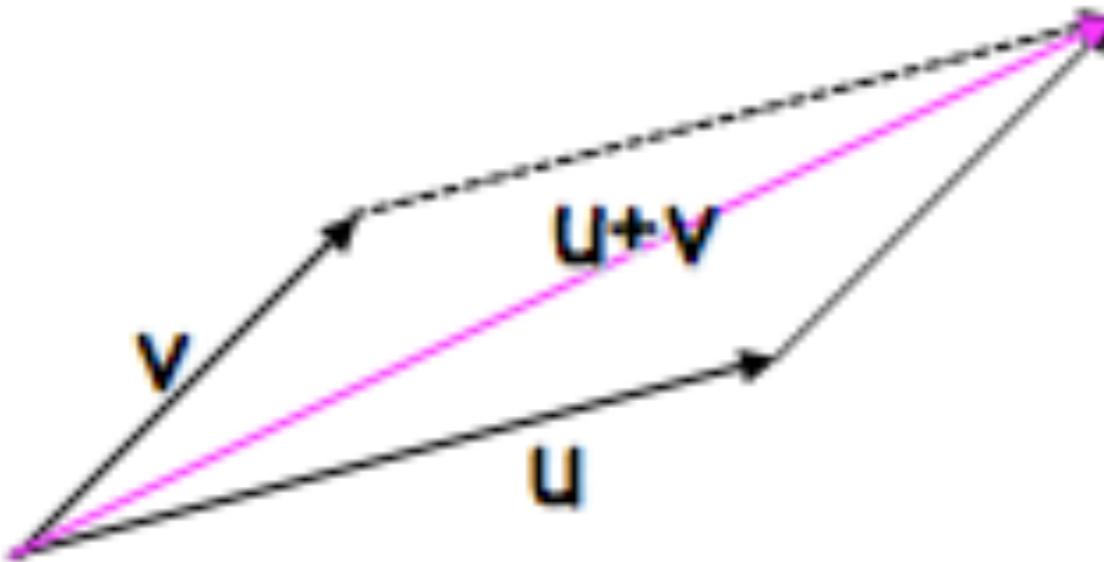
$$\begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} 12 \\ 16 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

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## LINEAR OPERATIONS - ADDITION

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- ▶ Vectors of equal dimensions may be added together:



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## LINEAR OPERATIONS - ADDITION

► Matrices of compatible dimensions may be added together:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

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## LINEAR OPERATIONS - SUBTRACTION

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- ▶ Vectors of equal dimensions may be subtracted:

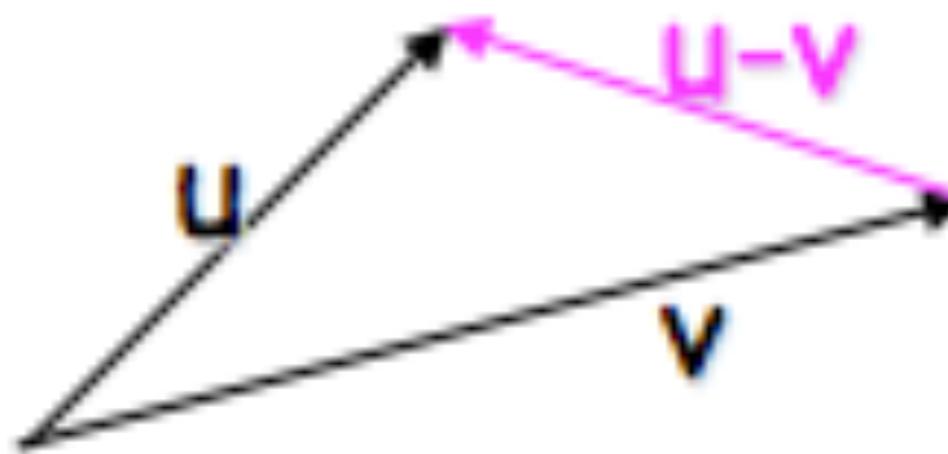
$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

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## LINEAR OPERATIONS - SUBTRACTION

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- ▶ Vectors of equal dimensions may be subtracted:



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## LINEAR OPERATIONS - SCALING

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- ▶ Scaling a vector: shrinking/growing a vector by some value

$$z = \alpha \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3\alpha \\ 2\alpha \\ 5\alpha \end{pmatrix}$$

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## LINEAR OPERATIONS - SCALING

► Graphically:

The diagram shows a 2D Cartesian coordinate system with a horizontal x-axis and a vertical y-axis. A vector  $x$  is drawn from the origin. A dashed vector  $10x$  is drawn parallel to  $x$  but longer. Another dashed vector  $\alpha x$  is drawn parallel to  $x$  but longer than  $10x$ . A third dashed vector  $\alpha = 3$  is shown, with an arrow pointing towards the vector  $\alpha x$ .

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## LINEAR OPERATIONS – DOT PRODUCT/INNER PRODUCT/SCALAR

- Multiply vectors of compatible dimensions to produce a scalar

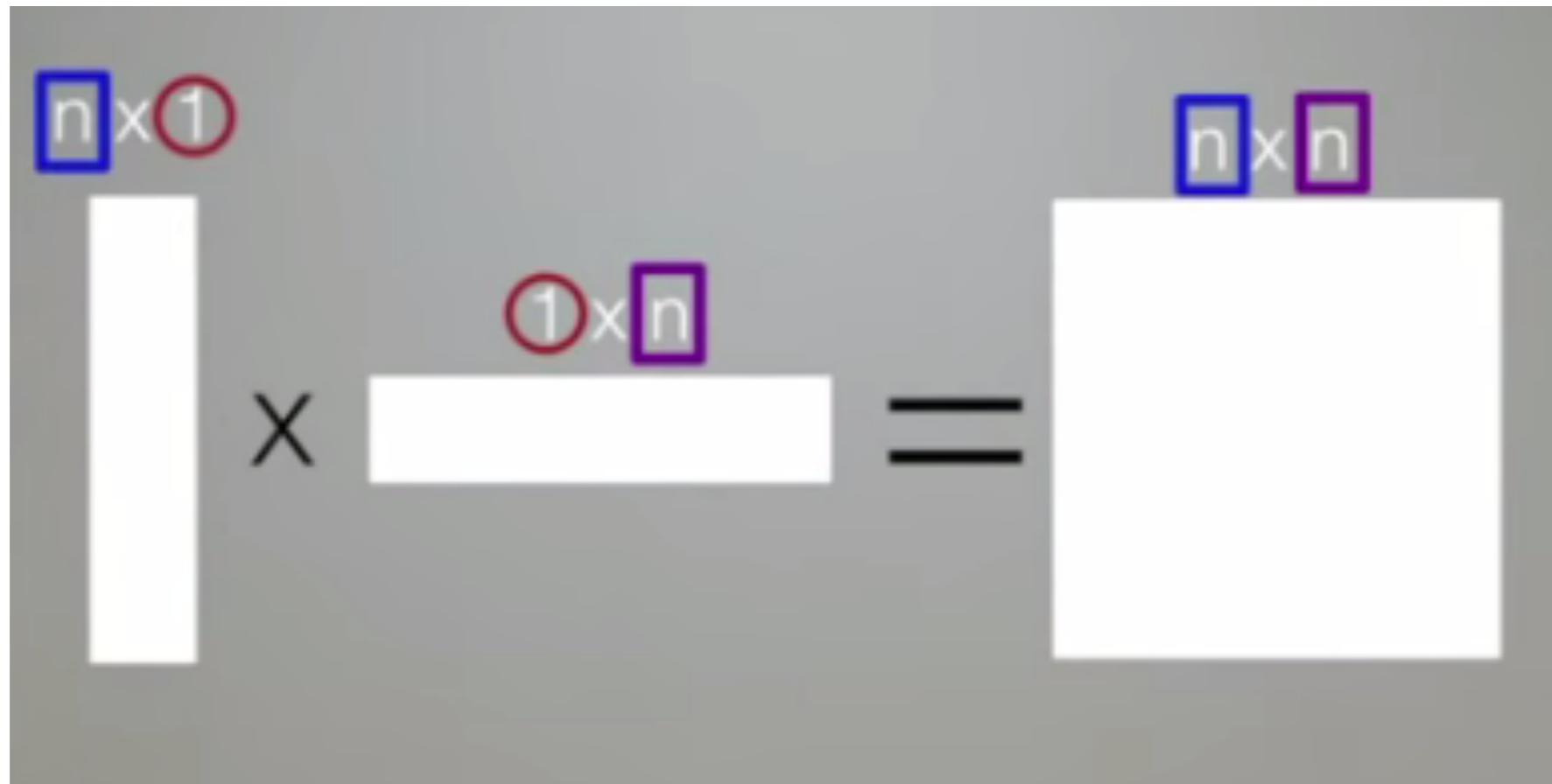
The diagram shows a multiplication operation between two matrices, A and B, resulting in matrix C. Matrix A is labeled  $[n \times n]$  and is represented by a square bracketed block of  $n^2$  white squares. Matrix B is labeled  $[n \times 1]$  and is represented by a vertical column of  $n$  white squares. The result, matrix C, is also labeled  $[n \times 1]$  and is represented by a vertical column of  $n$  white squares. Handwritten annotations in blue indicate the dimensions:  $1 \times h$  and  $l \times n$  are written above the first row of A;  $(kn, n \times 1)$  is written below the second row of A; and  $l \times n$  is written next to the third row of A. A handwritten arrow points from the first row of A to the first column of B. Handwritten labels A, B, and C are placed near their respective matrices.

If the dot product of two vectors is zero, they are orthogonal

$$A \cdot B \neq A \cdot B^\top$$

## LINEAR OPERATIONS – DOT PRODUCT/INNER PRODUCT/SCALAR

- ▶ The inverse (to produce a scalar) does not make sense



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## LINEAR OPERATIONS – MATRIX MULTIPLICATION

- We can multiple matrixes if the dimensions are compatible

The diagram shows two matrices, A and B, being multiplied. Matrix A is a 3x2 matrix with elements  $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ . Matrix B is a 2x2 matrix with elements  $b_{11}, b_{12}, b_{21}, b_{22}$ . A bracket on the left indicates the width of matrix A is 2, and a bracket on the top indicates the height of matrix B is 2. An arrow points from the bottom right of matrix A to the top left of matrix B, indicating the result of the multiplication.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

3 x 2      2 x 2

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## LINEAR OPERATIONS – MATRIX MULTIPLICATION

► We can multiple matrixes if the dimensions are compatible

The image shows a handwritten derivation of the formula for multiplying a 3x2 matrix A by a 2x2 matrix B. The result is a 3x2 matrix C. The first column of C is derived as follows:

$$\begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} \end{bmatrix}$$

The second column of C is derived as follows:

$$\begin{bmatrix} a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \\ a_{31} \cdot b_{12} + a_{32} \cdot b_{22} \end{bmatrix}$$

## LINEAR OPERATIONS – MATRIX MULTIPLICATION

- We can multiple matrixes if the dimensions are compatible

$$\begin{pmatrix} \textcolor{blue}{3 \times 2} \\ 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & \textcolor{red}{2 \times 4} \\ 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \end{pmatrix} = \begin{pmatrix} \textcolor{blue}{3 \times 4} \\ 29 & 32 & 35 & 38 \\ 65 & 72 & 79 & 86 \\ 101 & 112 & 123 & 134 \end{pmatrix}$$

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## LINEAR OPERATIONS – MATRIX MULTIPLICATION

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- ▶ Matrix multiplication is not commutative, but it is distributive and associative
- ▶  $AB \neq BA$
- ▶  $A(B+C) = A(B) + A(C)$

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## LINEAR OPERATIONS – IDENTITY MATRIX

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- ▶ Any matrix times an identity matrix equals the matrix itself

$$I \in \mathbb{R}^{n \times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- ▶  $AI = A$

