Lecture Notes on Quantum Mechanics & Quantum Computing Part - II

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1.1 Linear Algebra Basics

some useful method in our calculations

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \tag{1.1}$$

and

$$AB \neq BA \tag{1.2}$$

normally $AB \neq BA$ but if A and B are diagonal then and only then $AB = BA$
for a free particle if we measure the position momentum and energy.In general The energy matrix is diagonal on the other hand the position and momentum matrix are non diagonal.

One trick to find out hermitian operators, They will always give real eigenvalues.

$$A_{ij} = A^*_{ji} \tag{1.3}$$

Some properties of Hermitian matrix

- Main diagonal values are real.
- Symmetric
- Normal
- Diagonalizable
- Inverse is Hermitian

We use Harmitian operations to find the specific eigenvalue or stuffs like that. But Unitary operators are used when we change coordinates kind of like an transformation matrix. Since the physics is the same. It doesn't change anything. In simple terms unitary doesnt change states but hermitian operators do .

Eigen value properties

The eigenvectors x^i and eigenvalues λ_i of a matrix A are defined by $Ax^i = \lambda_i x^i$

$$\lambda_i = Tr(A)$$

$$\prod_i \lambda_i = |A|$$

- The eigenvectors of a normal matrix corresponding to different eigenvalues which are orthogonal.
- The eigenvalues of an Hermitian (or real orthogonal) matrix are real.
- The eigenvalues of a unitary matrix have unit modulus.
- Two normal matrices commute means they have a set of eigenvectors in common.
- A square matrix is singular means at least one of its eigenvalues is zero.

Eigen value calculations

Name	Effect on matrix product	Notes
Trace	$\mathrm{Tr}\,(\mathsf{AB}\ldots\mathsf{G}) = \mathrm{Tr}\,(\mathsf{B}\ldots\mathsf{GA})$	The product matrix $\mathsf{AB}\ldots\mathsf{G}$ must be square, though the individual matrices need not be. However, they must be compatible.
Determinant	$\lvert\mathsf{AB}\ldots\mathsf{G}\rvert = \lvert\mathsf{A}\rvert\lvert\mathsf{B}\rvert\ldots\lvert\mathsf{G}\rvert$	All matrices must be $N \times N$. Product is singular $\Leftrightarrow$ one or more of the individual matrices is singular.
Transpose	$(\mathsf{AB}\ldots\mathsf{G})^{\mathsf{T}} = \mathsf{G}^{\mathsf{T}}\ldots\mathsf{B}^{\mathsf{T}}\mathsf{A}^{\mathsf{T}}$	Matrices must be compatible but need not be square.
Complex conjugate	$(\mathsf{AB}\ldots\mathsf{G})^{*} = \mathsf{A}^{*}\mathsf{B}^{*}\ldots\mathsf{G}^{*}$	
Hermitian conjugate	$(\mathsf{AB}\ldots\mathsf{G})^{\dagger} = \mathsf{G}^{\dagger}\ldots\mathsf{B}^{\dagger}\mathsf{A}^{\dagger}$	Matrices must be compatible but need not be square.
Inverse	$(\mathsf{AB}\ldots\mathsf{G})^{-1} = \mathsf{G}^{-1}\ldots\mathsf{B}^{-1}\mathsf{A}^{-1}$	All matrices must be $N \times N$ and non-singular.

Table 1.1: Matrix operations

- Solve the characteristic equation $|A - \lambda I| = 0$ for N values of λ
- for each value i solve $Ax^i = \lambda_i x^i$ for x^i
- Construct the unitary matrix S whose columns are normalized eigenvectors $\hat{x}^i$ of A

1.2 Dirac notation

Bra notation basically means row vectors and kets are column vectors but in Hilbert space.

$$|\Psi\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\langle\Psi| = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

The effect a quantum gate has on a $|\psi\rangle$ is almost the same as the above figure of rotation of axis.
braket is basically the multiplication of bra matrix and ket matrix which we know as inner product.

$$\langle\psi|\phi\rangle = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Similarly there's outer product. defined as the inner product of ket and bra.

$$|\phi\rangle\langle\psi| = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$$

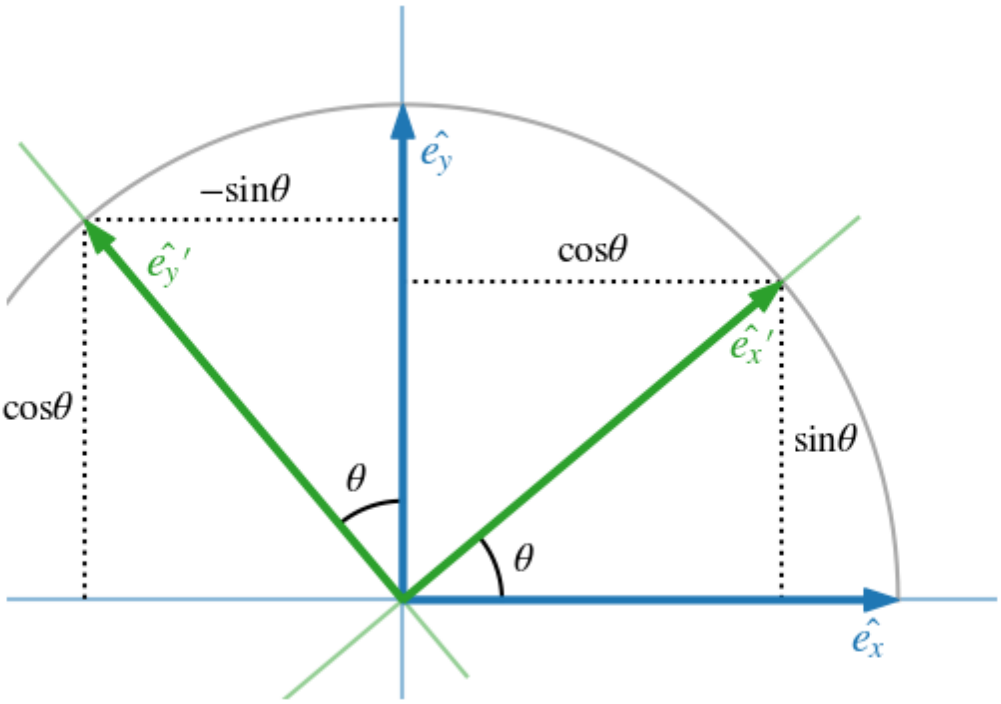
1.3 Rotation matrix using bra and ket operations

Define the following

$$\begin{aligned} \hat{R}\,|x\rangle &= |x'\rangle \\ \hat{R}\,|y\rangle &= |y'\rangle \end{aligned} \tag{1.4}$$

Name	Symbol	How obtained	Notes
Trace	$\mathrm{Tr}\,\mathsf{A}$	Sum the elements on the leading diagonal	Needs $M = N$
2×2 determinant		$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{12}a_{21}$	Definition[a]
Determinant	$\lvert\mathsf{A}\rvert$	Make a Laplace expansion (Section 1.9) to reduce to a sum of 2×2 determinants	Needs $M = N$
Rank	$R(\mathsf{A})$	The largest value of r for which A has an $r \times r$ submatrix with a non-zero determinant	$R \leq \min\{M, N\}$
Transpose	A^{T}	Interchange rows and columns: $(\mathsf{A}^{\mathsf{T}})_{ij} = A_{ji}$	A^{T} is $N \times M$
Complex conjugate	A^*	Take the complex conjugate of each element: $(\mathsf{A}^*)_{ij} = A^*_{ij}$	A^* is $M \times N$
Hermitian conjugate	$\mathsf{A}^\dagger$	Transpose the complex conjugate *or* complex conjugate the transpose: $(\mathsf{A}^\dagger)_{ij} = A^*_{ji}$	$\mathsf{A}^\dagger$ is $N \times M$
Minor	M_{ij}	Evaluate the determinant of the $(N-1) \times (N-1)$ matrix formed by deleting the ith row and the jth column	Needs $M = N$
Cofactor	C_{ij}	Multiply the minor M_{ij} by $(-1)^{i+j}$: $C_{ij} = (-1)^{i+j} M_{ij}$	Needs $M = N$
Inverse	A^{-1}	Divide each element of the transpose C^{T} of the matrix of cofactors C by the determinant of A; $(\mathsf{A}^{-1})_{ij} = C_{ji}/\lvert\mathsf{A}\rvert$	Needs $M = N$

Table 1.2: A table of useful quantities used in quantum mechanics and linear algebra



$$\boxed{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

Figure 1.1: Similarity in rotation and quantum gates

using this we can get the following

$$\begin{aligned} \hat{R}\,|x\rangle\,\langle x| &= |x'\rangle\,\langle x| \\ \hat{R}\,|y\rangle\,\langle y| &= |y'\rangle\,\langle y| \end{aligned} \tag{1.5}$$

adding these two up we get

$$\hat{R}(|x\rangle\,\langle x| + |y\rangle\,\langle y|) = |x'\rangle\,\langle x| + |y'\rangle\,\langle y|$$

which becomes

$$\hat{R}I = |x'\rangle\,\langle x| + |y'\rangle\,\langle y|$$

$$\hat{R} = |x'\rangle\,\langle x| + |y'\rangle\,\langle y|$$

$$\hat{R} = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

This is possible due to completeness theorem

2. Lecture 8

2.1 Another trick to calculate the rotation matrix(same basis)

Lets define once again that

$$|x\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$|y\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\hat{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$\hat{R}\,|x\rangle = \begin{bmatrix} a \\ c \end{bmatrix} = |x'\rangle = a\,|x\rangle + c\,|y\rangle$$

and

$$\hat{R}\,|y\rangle = \begin{bmatrix} b \\ d \end{bmatrix} = |y'\rangle = b\,|x\rangle + d\,|y\rangle$$

using the definition of inner product and a bit of algebra we get the following form of the rotation matrix.

$$\hat{R} = \begin{bmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{bmatrix} = \begin{bmatrix} \langle x|\hat{R}|x'\rangle & \langle x|\hat{R}|y'\rangle \\ \langle y|\hat{R}|x'\rangle & \langle y|\hat{R}|y'\rangle \end{bmatrix}$$

The elements are now like a expectation value. We can exploit this. Basically these elements are replaced with the inner product of input and output to build a quantum gate.

Perturbation can change the matrix we are expecting. Real life energy eigen matrix may contain non zero values at places where we were hoping to get value 0.

2.2 Change of basis

Fourier Transform is a type of change of basis, in quantum mechanics we assign no real value to Ψ so that we can take the projection of required information onto Ψ and get actual information of the state. This is done by taking the Fourier Transformation, in other words change of basis (which is known as inner product in Hilbert space) from quantum wave mechanics we know.

$$\Psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{ikx} dx$$

but in matrix mechanics , or inner product in Hilbert space , we can write the following

$$\begin{aligned} \Psi(k) = \langle k|\Psi\rangle &= \int_{-\infty}^{\infty} \langle x|\Psi\rangle \langle k|x\rangle \, dx \\ &= \int_{-\infty}^{\infty} \langle k|x\rangle \langle x|\Psi\rangle \, dx \\ &= \langle k|I\Psi\rangle \\ &= \langle k|\Psi\rangle \end{aligned}$$

Lets now consider the case of rotation or gate matrix where the basis has changed.

$$\hat{A}\begin{bmatrix} B_{x,o} \\ B_{y,o} \end{bmatrix} = \begin{bmatrix} B'_{x,n} \\ B'_{y,n} \end{bmatrix}$$

That is to say

$$|B\rangle = B_{x,o}|x\rangle + B_{y,o}|y\rangle = B_{x,n}|x'\rangle + B_{y,n}|y\rangle'$$
$$\rightarrow |B\rangle = \langle x'|B\rangle|x'\rangle + \langle y'|B\rangle|y'\rangle$$

back to our original problem writing the operator with its elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} B_{x,o} \\ B_{y,o} \end{bmatrix} = \begin{bmatrix} B'_{x,n} \\ B'_{y,n} \end{bmatrix} = \begin{bmatrix} \langle x'|B\rangle \\ \langle y'|B\rangle \end{bmatrix}$$

We get the following then

$$aB_{x,o} + bB_{y,o} = \langle x'|B\rangle = B_{x,o}\langle x'|x\rangle + B_{y,o}\langle x'|y\rangle$$
$$cB_{x,o} + dB_{y,o} = \langle y'|B\rangle = B_{x,o}\langle y'|x\rangle + B_{y,o}\langle y'|y\rangle$$

equating we get the following matrix

$$\hat{A} = \begin{bmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{bmatrix}$$

Which is interesting cause

$$(\hat{R}^T)^* = \hat{A}$$

These are harmitian operator. Meaning the rotation matrix and change of basis matrix are harmitians

$\|x\rangle$ and $\|y\rangle$ used in the rotation matrix are input outputs but in $\hat{A}$ they are input's and output's basis vectors.

2.3 A useful proof

$$\begin{aligned}
|\Psi\rangle &= C_1|x\rangle + C_2|y\rangle + C_3|z\rangle \\
&= \langle x|\Psi\rangle|x\rangle + \langle y|\Psi\rangle|y\rangle + \langle z|\Psi\rangle|z\rangle \\
&= |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle + |z\rangle\langle z|\Psi\rangle \\
&= (|x\rangle\langle x| + |y\rangle\langle y| + |z\rangle\langle z|)|\Psi\rangle \\
&= I|\Psi\rangle \\
&= |\Psi\rangle
\end{aligned}$$

Using this we can write Ψ as the following vector

$$|\Psi\rangle = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

and

$$\langle\Psi| = \begin{bmatrix} C_1^* & C_2^* & C_3^* \end{bmatrix}$$

then

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= \begin{bmatrix} C_1^* & C_2^* & C_3^* \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \\
&= C_1^2 + C_2^2 + C_3^2 \\
&= 1
\end{aligned}$$

2.4 Mathematical formulation based on hilbert space

1. **State of a system :** The state of any physical system is specified, at each time t, by a state vector $|\psi(t)\rangle$ in a Hilbert space $\mathscr{H}$; $|\psi(t)\rangle$ contains (and serves as the basis to extract) all the needed information about the system. Any superposition of state vectors is also a state vector.

2. **Observables and operators :** To every physically measurable quantity A, called an observable or dynamical variable, there corresponds a linear Hermitian operator $\hat{A}$ whose eigenvectors form a complete basis.

3. **Measurements and eigenvalues of operators:** The measurement of an observable A may be represented formally by the action of $\hat{A}$ on a state vector $|\psi(t)\rangle$. The only possible result of such a measurement is one of the eigenvalues a_n (which are real) of the operator $\hat{A}$. If the result of a measurement of A on a state $|\psi(t)\rangle$ is a_n , the state of the system immediately after the measurement changes to $|\psi_n\rangle$:

$$\hat{A}\,|\psi(t)\rangle = a_n\,|\psi_n\rangle$$

   where a_n is $\langle\psi_n|\psi(t)\rangle$. Note: a_n is the component of $|\psi(t)\rangle$ when projected onto the eigen- vector $|\psi_n\rangle$

4. **Outcome of measurement** When measuring an observable A of system in a state $|\psi\rangle$ the probability of obtaining one of the nondegenerate eignevalues a_n of the corresponding operator $\hat{A}$ is given by

$$P_n(a_n) = \frac{|\,\langle\psi_n|\psi\rangle\,|^2}{\langle\psi|\psi\rangle} = \frac{|a_n|^2}{\langle\psi|\psi\rangle}$$

   The act of measurement changes the state of the system from $|\psi\rangle$ to ψ. If the system is already in an eigenstate $|\psi_n\rangle$ of $\hat{A}$, a measurement of A yileds with certainty that corresponding eigenvalue a_n:

$$\hat{A}\,|\psi_n\rangle = a_n\,|\psi_n\rangle$$

5. **Time evolution of a system:** The time evolution of the state vector $|\psi(t)\rangle$ of a system is governed by the time-dependent Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle$$

   where $\hat{H}$ is the Hamiltonian operator corresponding to the total energy of the system.

3. Lecture 9

3.1 Matrix representation of polarization

Consider a transverse electromagnetic wave propagating in the z direction and oscillating in the xy plane.

$$\mathbf{E} = E_0 e^{i(\mathbf{kr}-\omega t)}$$

$$\mathbf{E} = (E_{0x}e^{i\alpha}, E_{0y}e^{i\beta}, 0)e^{i(kz-\omega t)}$$

changing the phase and setting

$$\alpha = 0$$

we get

$$\mathbf{E} = (E_{0x}, E_{0x}e^{i\delta}, 0)e^{i(kz-\omega t)}$$

For linearly polarised wave, we have $\delta = 0$ ignoring the $e^{i(kz-\omega t)}$ and normalizing we can write the following state vectors

$$|E_h\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

and

$$|E_v\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

for circularly polarised wave we have $\delta = \pm\frac{\pi}{2}$ in this case we dont have any polarising component in the z direction, so we use a complex number to represent this phenomena.

$$|E_r\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

$$|E_l\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

$\frac{1}{\sqrt{2}}$ is used to normalize the vectors.Note here that $|E_h\rangle$ and $|E_v\rangle$ are orthonormal basis. Similarly $|E_r\rangle$ and $|E_l\rangle$

$$\langle E_h|E_v\rangle = \langle E_r|E_l\rangle = 0$$

and

$$\langle E_r|E_r\rangle = \langle E_l|E_l\rangle = \langle E_h|E_h\rangle = \langle E_v|E_v\rangle = 1$$

But both denote the same space. So we have found two pairs of linearly independent basis. We can now write the horizontal and vertical vectors with the help of left and right vectors. The inverse is also possible.

$$|h\rangle = \frac{1}{\sqrt{2}}(|r\rangle + |l\rangle)$$
$$|v\rangle = \frac{1}{i\sqrt{2}}(|r\rangle - |l\rangle)$$

and

$$|r\rangle = \frac{1}{\sqrt{2}}(|h\rangle + i\,|v\rangle)$$
$$|l\rangle - \frac{1}{\sqrt{2}}(|h\rangle - i\,|v\rangle)$$

3.1.1 Mallus law

$$\hat{R}\,|h\rangle = |\hat{h}\rangle$$

$$\rightarrow \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$$

$$\rightarrow |\hat{h}\rangle = cos\theta\,|h\rangle + sin\theta\,|v\rangle$$

so for horizontally polarised waves the intensity is $cos^2\theta$ which is knows as mallus law.

3.1.2 Electric field amplitudes for alternating paths using state vectors

Let's consider the Experimental setup of figure 1. To model this problem we consider the wave after passing through the block as $|\Psi_{int}\rangle$ then

$$|\Psi_0\rangle = |v\rangle = |y\rangle$$

y' and x' are basically basis in the block which denotes the rotation of the system. then we can write

$$\begin{aligned} |\Psi_{int}\rangle &= C_1\,|x'\rangle + C_2\,|y'\rangle = |y\rangle \\ &=> \langle x'|y\rangle\,|x'\rangle + \langle y'|y\rangle\,|y'\rangle = |y\rangle \\ &=> cos\theta\,|y'\rangle - sin\theta\,|x'\rangle = |y\rangle \\ |\Psi_{int}\rangle &= cos\theta\,|y'\rangle - sin\theta\,|x'\rangle \\ \rightarrow C_1 &= -sin\theta, C_2 = cos\theta \end{aligned}$$

But after one of the orientations is blocked (eg :$|x'\rangle = 0$)
the input wave function is just $|y'\rangle$
and the intensity measured is($\theta = 30°$)

$$I_0 = C_2^2 = cos^2\theta = \frac{3}{4}$$

at the detector the orientation is once again in the basis $|h\rangle = |x\rangle\,, |v\rangle = |y\rangle$

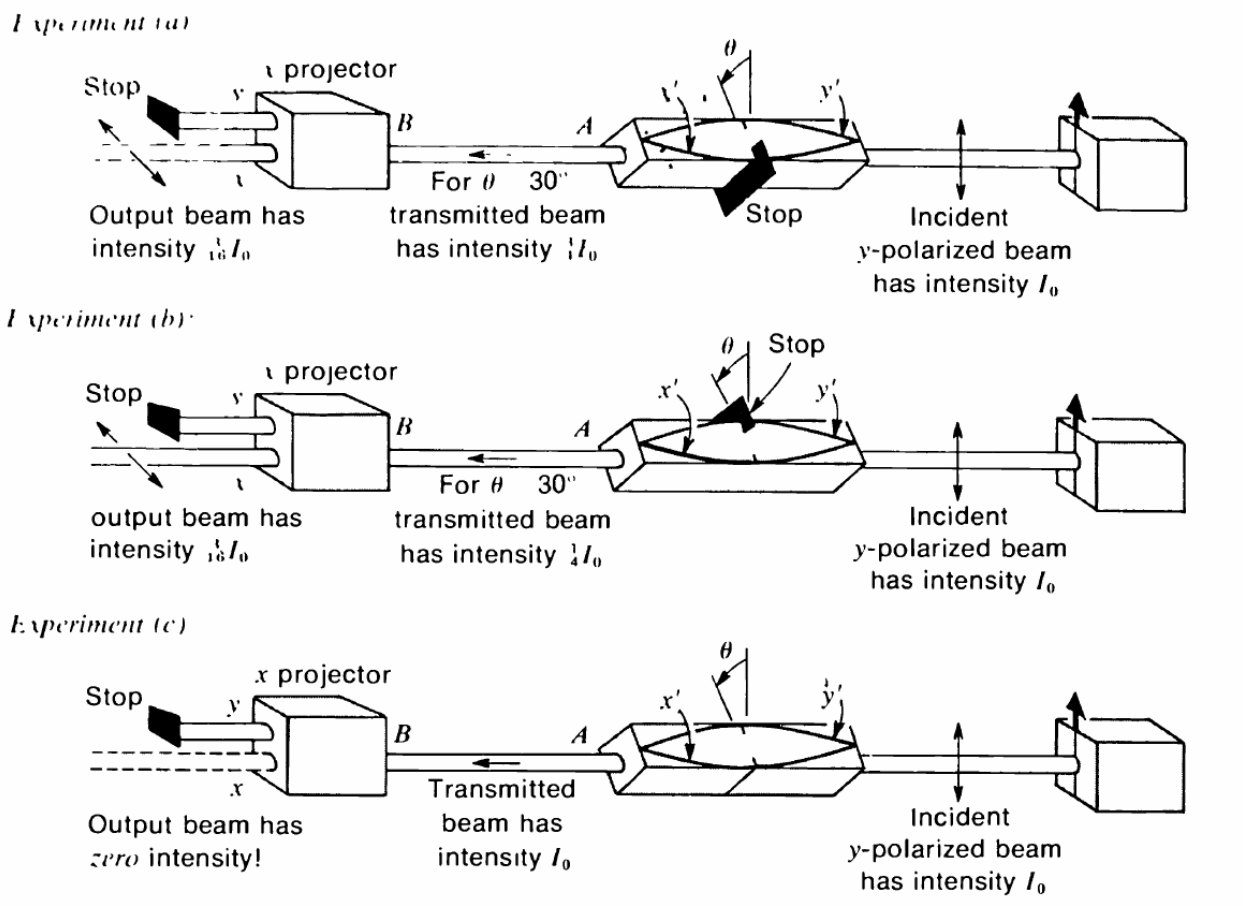


Figure 3.1: Experimental setup of Electric field amplitudes for alternating paths

$$
\begin{aligned}
|\Psi_{final}\rangle &= A_1 |x\rangle + A_2 |y\rangle = |y'\rangle \\
&=> \langle x|y'\rangle |x\rangle + \langle y|y'\rangle |y\rangle = |y'\rangle \\
&=> -sin\theta |x\rangle + cos\theta |y\rangle = |y'\rangle \\
|\Psi_{final}\rangle &= -sin\theta |x\rangle + cos\theta |y\rangle \\
\rightarrow A_1 &= -sin\theta, A_2 = cos\theta
\end{aligned}
$$

now if we let one of the outputs zero (eg : $|y\rangle = 0$ then the Intensity measured is

$$I_f = C_2^2 \times A_1^2 = \frac{3}{16}$$

for the inverse case ($|x'\rangle = 0, |y_f\rangle = 0$) the intensity would still be same . but the intermediate intensity will now be $\frac{1}{4}$

for the case of no obstacle at the middle, the input wave function is splitting to form two wave functions which than interfere at the final ouput. meaning

$$\Psi_1 0 = \langle x|y'\rangle \langle y'|y\rangle$$
$$\Psi_2 0 = \langle x|x'\rangle \langle x'|y\rangle$$

The final Intensity is thus

$$I_f = |\Psi_f|^2 = (cos\theta sin\theta - cos\theta sin\theta)^2 = 0$$

Adding probabilities is not a viable option here so if we measure the other component we will get 0 as output as seen in the experiment.

Here is a summary

- The probability amplitude for two successive probabilities is the product of amplitudes for the individual possibilities.

- The amplitude for a process that can take place in one of several indistinguishable ways is the sum of amplitudes for each of the individual ways.
- The total probability for the process to occur is the absolute value squared of the total amplitude calculated by 1 and 2.

4. Lecture 10

4.1 Maths from zettili 3.7,3.8,3.11

[**Problem: Zettili 3.7**] Consider a system whose Hamiltonian $\hat{H}$ and an operator $\hat{A}$ are given by the matrices

$$\hat{H} = \epsilon_0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \hat{A} = a \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where ϵ_0 has the dimensions of energy.

1. If we measure the energy, what values will we obtain?
2. Suppose that when we measure the energy, we obtain a value of $-\epsilon_0$. Immediately afterwards, we measure $\hat{A}$. What values will we obtain for A and what are the probabilities corresponding to each value?
3. Calculate the uncertainty $\Delta\hat{A}$.

Solution

1. The eigenvalue characteristic equation yields eigenenergies $E_1 = 0, E_2 = -\epsilon_0, E_3 = 2\epsilon_0$. The respective eigenvectors are

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, |\Phi_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, |\Phi_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

these eigenvectors are orthonormal.

If a measurement of the energy yields $-\epsilon_0$, this means that the system is left in the state $|\Phi_2\rangle$. When we measure the next observable, A, the system is in the state $|\Phi_2\rangle$. The result we obtain for A is given by any of the eigenvalues of A. The eigenvalues are $a_1 = -\sqrt{17}a, a_2 = 0, a_3 = \sqrt{17}a$

then

$$|a_1\rangle = \frac{1}{\sqrt{34}} \begin{bmatrix} 4 \\ -\sqrt{17} \\ 1 \end{bmatrix}, |a_2\rangle = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, |a_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ \sqrt{17} \\ 1 \end{bmatrix}$$

Thus when measuring $\hat{A}$ on a system which is in the state $|\Phi_2\rangle$, the probability of finding $-\sqrt{17}$ is given by

$$P_1(a_1) = |\langle a_2|\Phi_2\rangle|^2 = \left| \frac{1}{\sqrt{34}} \begin{bmatrix} 4 & -\sqrt{17} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{34}$$

similarly

$$P_2(a_2) = \frac{1}{16}$$

and

$$P_3(a_3) = \frac{1}{34}$$

Since the system when measuring A is in the state $|\Phi_2\rangle$ the uncertainty ΔA is given by

$$\Delta A = \sqrt{\langle\Phi_2|A^2|\Phi_2\rangle - \langle\Phi_2|A|\Phi_2\rangle^2}$$

which yields the following matrices

$$\langle\Phi_2|A|\Phi_2\rangle = a\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 4 & 0\\ 4 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

similarly

$$\langle\Phi_2|A^2|\Phi_2\rangle = a^2\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 4 & 0\\ 4 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 4 & 0\\ 4 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = a^2$$

[**Problem: Zettili 3.8**] Consider a system whose state and two variables are given by

$$|\Psi(t)\rangle = \begin{bmatrix}-1\\2\\1\end{bmatrix}, \hat{A} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}, \hat{B} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

1. What is the probability that a measurement of A at time t yields -1 ?
2. Let us carry out a set of two measurements where B is measured first and then, imme- diately afterwards, A is measured. Find the probability of obtaining a value of 0 for B and a value of 1 for A.
3. Now we measure A first then, immediately afterwards, B. Find the probability of ob- taining a value of 1 for A and a value of 0 for B.
4. Compare the results of 1 and 2 . Explain.

Solution

1. A measurement of A yields any of the eigenvalues of A which are given by $a_1 = -1, a_2 = 0, a_3 = 1$ the respective eigenstates are

$$|a_1\rangle = \frac{1}{2}\begin{bmatrix}-1\\\sqrt{2}\\-1\end{bmatrix}, |a_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, |a_3\rangle = \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix}$$

The probability of obtaining $a_1 = -1$ is

$$P(-1) = \frac{|\langle a_1|\Psi(t)\rangle|^2}{\langle\Psi(t)|\Psi(t)\rangle} = \frac{1}{6}\left|\frac{1}{2}\begin{bmatrix}-1 & \sqrt{2} & -1\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix}\right|^2 = \frac{1}{3}$$

A measurement of B yields a value which is equal to any of the eigenvalues of B; $b_1 = -1, b_2 = 0, b_3 = 1$ and the eigenvectors are

$$|b_1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}, |b_2\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}, |b_3\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Since the system was in state $|\Psi(t)\rangle$ the probability of obtaining the value $b_2 = 0$ for B is

$$P(b_2) = \frac{|\langle b_2|\Psi(t)\rangle|^2}{\langle\Psi(t)|\Psi(t)\rangle} = \frac{2}{3}$$

We deal now with the measurement of the other observable, A. The observables A and B do not have common eigenstates, since they do not commute. After measuring B (the result is $b_2 = 0$), the system is left, according to Postulate 3, in a state $|phi\rangle$ which can be found by projecting $|\Psi(t)\rangle$ onto $|b_2\rangle$

$$|phi\rangle = |b_2\rangle\langle b_2|\Psi(t)\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}-1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

The probability of finding 1 when we measure A is given by

$$P(a_3) = \frac{|\langle a_3|\phi\rangle|^2}{\langle\phi|\phi\rangle} = \frac{1}{2}$$

In summary, when measuring B then A the probability of finding a value of 0 for B and 1 for A is given by the product of the probabilities

$$P(b_3, a_3) = P(b_3)P(a_3) = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$$

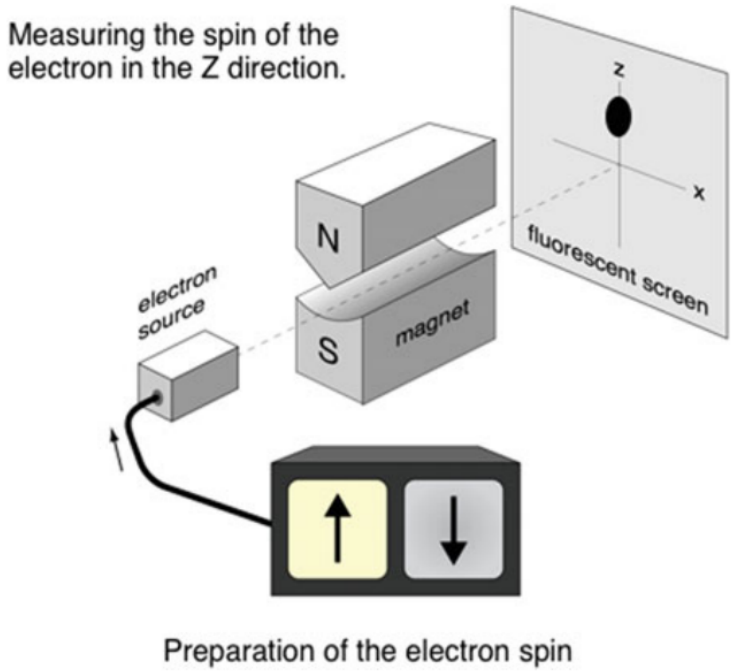


Figure 4.1: Stern garlach experiment

Next we measure A first then B. Since the system is in the state $|\Psi(t)\rangle$, $the probability of measuring$ $a_3 = 1$ for A is given by

$$P'(a_3) = \frac{|\langle a_3|\Psi(t)\rangle|^2}{\langle\Psi(t)|\Psi(t)\rangle} = \frac{1}{3}$$

Then we proceed to measure the measurement of B. The state of the system just after measuring A (with value $a_3 = 0$) is given by a projection of $|\Psi(t)\rangle$ onti $|a_3\rangle$

so

$$|\kappa\rangle = |a_3\rangle\langle a_3|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}$$

So the probability of finding the value of $b_2 = 0$ when measuring B is given by

$$P'(b_2) = \frac{|\langle b_2|\kappa\rangle|^2}{\langle\kappa|\kappa\rangle} = \frac{1}{2}$$

So when measuring A then B, the probability of finding a value of 1 for A and 0 for B is given by the product of the probabilities

$$P(a_3, b_3) = P'(a_3)P'(b_2) = \frac{1}{6}$$

4.2 Spin

Until we do a measurement, microscopic observables dont have any real values. Consider electron spin something like that. But in case of spin we only get two distinct eigen values ($\pm\frac{\hbar}{2}$) When we do not observe they do not have any particular value. The spins we observe in experiments are due to the orientation of the measuring device . It doesnt have any particular value . **So scientists have considered z axis as a formal direction to be used**

The electron source is already measuring the spin in the stern garlach experiment. In the experimental setup no matter how many spins we measure we will always get one value, either all ups or all downs. As we have already made the first measurement. Through the electron source .
But the scenario changes when we rotate the bar magnets to align with the x axis. we will now get 50% in the up state

for this system the most ideal basis is

$$\hat{S}_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{S}_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{-\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solving we get the followings

$$|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$
$$|\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$
$$\hat{S_z}\,|\uparrow\rangle = \frac{\hbar}{2}\,|\uparrow\rangle$$
$$\hat{S_z}\,|\downarrow\rangle = \frac{-\hbar}{2}\,|\downarrow\rangle$$
$$\hat{S_z} = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Let's Shift the position of the magnets anti clock wise by $90°$ But if we do that the input states created is still the same. But the outputs now give 50% possibility.

In a more broad sense we were passing up spins through a magnet oriented in the direction of the up spin, so we got a 100% chance of getting the up spin in the z axis .Since the operator was returning the eigenvalue not changing the states. But when we change the orientation, We are using the same up spin but different operator one whose orientation is in the direction of x axis.Now it is no longer an eigenvalue problem.

to solve this we need to create a new input spin state which will work on this new operator to give us the desired eigenvalues. Lets define these spins as left or right spins. then

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

Here we have written the new basis(x-axis) as the linear combination of the old basis(z axis). Lets define the new operator as $\hat{S_x}$ then

$$\hat{S_x}\,|+\rangle = \frac{\hbar}{2}\,|+\rangle$$
$$\hat{S_x}\,|-\rangle = \frac{-\hbar}{2}\,|-\rangle$$

experimental results show we indeed get 100% in one direction if we do this for one particular state now

Why are the state vectors not zero ? Cause they are in Hilbert space not real space .

$$\frac{1}{2}\Theta_{real} = \Theta_{Hilbert}$$

Since The vectors are $180°$ in real world but $90°$ in Hilbert space we can actually write all the states using the linear combination of $|\downarrow\rangle, |\uparrow\rangle$

4.2.1 $\hat{S_x}$ operator

We need to express in terms of the z basis thats why we are taking $\uparrow, \downarrow$

$$\hat{S_x}\,|+\rangle = \frac{\hbar}{2}\,|+\rangle$$
$$\hat{S_x}\,|-\rangle = \frac{-\hbar}{2}\,|-\rangle$$
$$\hat{S_x} = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$$
$$= \begin{bmatrix}\langle\uparrow|S_x|\uparrow\rangle & \langle\uparrow|S_x|\downarrow\rangle\\ \langle\downarrow|S_x|\uparrow\rangle & \langle\downarrow|S_x|\downarrow\rangle\end{bmatrix}$$
$$= \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

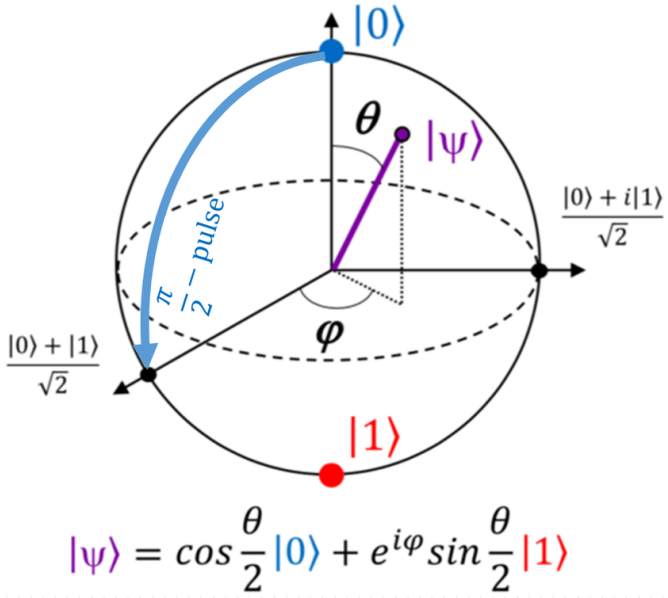


Figure 4.2: Bloch sphere

4.3 Bloch sphere

For any arbitrary direction we can use the state space in the form of a sphere known as the bloch sphere to do Calculation. The angles in the state space corresponds to double the angle in real life.

$$|\Psi(\theta,\phi)\rangle = \begin{bmatrix} cos\frac{\theta}{2} \\ e^{i\phi} sin\frac{\theta}{2} \end{bmatrix}$$

Here θ, ϕ in the Ψ is from the real world value.

4.4 Multiparticle Quantum system

4.4.1 Non interacting Particles

we expect the Hamiltonian of a multi-particle system to take the form

$$H(x_1, x_2, \ldots, x_N, t) = \sum_N \frac{p_i^2}{2m_i} + V(x_1, x_2, \ldots, x_N, t)$$

If the particles do not interact with one another. This implies that each particle moves in a common potential.

$$V(x_1, x_2, \ldots, x_N, t) = \sum_N V(x_i, t)$$

hence we can write the Hamiltonian as

$$H(x_1, x_2, \ldots, x_N, t) = \sum_N H_i(x_i, t)$$

where

$$H_i = \frac{p_i^2}{2m_i} + V(x_i, t)$$

In other words, for the case of non-interacting particles, the multi-particle Hamiltonian of the system can be written as the sum of N independent single-particle Hamiltonians. The multi-particle wavefunction $\psi(x_1, x_2, \ldots x_N, t)$ can be written as the product of N independent single-particle wavefunctions

$$\psi(x_1, x_2, \ldots, x_N, t) = \psi_1(x_1, t)\psi_2(x_2, t)\ldots\psi_N(x_N, t).$$

For a system of N non-interacting particles factorizes into N independent equations of the form

$$\mathrm{i}\hbar\frac{\partial\psi_i}{\partial t} = H_i\psi_i.$$

Assuming that $V(x,t) \equiv V(x)$, the time-independent Schrödinger equation also factorizes to give

$$H_i\psi_{E_i} = E_i\psi_{E_i},$$

Finally we get

$$\psi(x_1, x_2, \ldots, x_n, t) = \psi_E(x_1, x_2, \ldots, x_N)\mathrm{e}^{-\mathrm{i}Et/\hbar},$$

which means

$$\psi_E(x_1, x_2, \ldots, x_N) = \psi_{E_1}(x_1)\psi_{E_2}(x_2)\ldots\psi_{E_N}(x_N)e^{\frac{-it(E_1+E_2+\ldots+E_N)}{\hbar}}$$

For non multi-particle system no matter the size, we always consider the Wave equation of the system as a whole

4.4.2 Interacting particles

For interacting case the Hamiltonian is no longer the previous one.We know have an additional Interacting potential in the equation. For two particle interacting system :

$$H(x_1, x_2) = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1 - x_2).$$

consider $x_1 - x_2$ to be infinity then normal logic suggests that there would be no interacting potential but Calculation from quantum mechanics show that there is still some interaction even at infinity. This is the birth of quantum entanglement paradox.

5. Lecture 11

5.1 Product Space

For two quantum objects we consider two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ with the dimensions N, M The state of the overall system is then determined by the simultaneous specification of the vectors $|\phi\rangle \in \mathscr{H}_1$ and $|\psi\rangle \in \mathscr{H}_2$. These vectors are of dimensions $N * M$. This vector space is called the product space or in a more general sense $\mathscr{H}_1 \otimes \mathscr{H}_2$ The basis are written as

$$|n\rangle \otimes |m\rangle = |n \otimes m\rangle = |nm\rangle$$

so if a vector is in such a space it will be written as

$$|\psi\rangle = \sum_{n,m} a_{nm} |nm\rangle$$

5.1.1 Tensor Product

Tensor product of two vectors are shown below

$$\begin{bmatrix} a_1 & a2 \\ a_3 & a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 B & a2B \\ a_3 B & a_4 B \end{bmatrix}$$

where

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

5.2 Entangled State

Lets consider the following for basis $|\alpha\rangle, |\beta\rangle$

$$\begin{aligned} |\psi_1\rangle &= a_{1\alpha} |\alpha\rangle + a_{2\beta} |\beta\rangle \\ |\psi_2\rangle &= a_{2\alpha} |\alpha\rangle + a_{2\beta} |\beta\rangle \\ |\psi_1\psi_2\rangle &= a_{1\alpha 2\alpha} |\alpha\alpha\rangle + a_{1\alpha 2\beta} |\alpha\beta\rangle + a_{1\beta 2\alpha} |\beta\alpha\rangle + a_{1\beta 2\beta} |\beta\beta\rangle \\ |\psi_1\psi_2\rangle &= a_{\alpha\alpha} |\alpha\alpha\rangle + a_{\alpha\beta} |\alpha\beta\rangle + a_{\beta\alpha} |\beta\alpha\rangle + a_{\beta\beta} |\beta\beta\rangle \end{aligned}$$

if the following condition holds we say their states are in product space

$$a_{\alpha\alpha} a_{\beta\beta} = a_{\alpha\beta} a_{\beta\alpha} = stateisproductspace$$

else

$$a_{\alpha\alpha} a_{\beta\beta} \neq a_{\alpha\beta} a_{\beta\alpha} = stateisentangled$$

Even if we do create a state which is entangled it cant be written as a tensor product of the individual state vectors tensor product

When we can write a state as tensor product it is in product space. Otherwise it's are entangled.

some notations

$$\begin{aligned} |00\rangle = |0\rangle \otimes |0\rangle &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

These are also called quantum bits (qubits) reason is they can be written in terms of same polarized photon states or same spin states

$$|00\rangle = |hh\rangle = |\uparrow\uparrow\rangle$$

5.2.1 Single measurement on entangled states

If we project an horizontal operator onto a state which is made of horizontal and vertical states then the projection operator will give us the part of the horizontal state.

$$\begin{aligned} |\Psi\rangle &= c_1 |h\rangle + c_2 |v\rangle \\ \hat{P} &= |h\rangle \langle h| \\ \hat{P} |\Psi\rangle &= |h\rangle \langle h| (c_1 |h\rangle + |v\rangle) \\ &= \frac{|c_1|}{c_1} |h\rangle \end{aligned}$$

In general

$$\hat{P} = |h_1 \otimes I_1\rangle \langle h_2 \otimes I_2|$$

5.3 Qubits Register

In classical theory to store 8 Numbers we need 8 (states or similar).But a quantum register of three qubits can store this information all in one superposition state.

$$|q\rangle = \sum c_{xyz} |xyz\rangle$$

In general N qubits can be used to store 2^N states or numbers. After the measurement the state collapses to a particular information.But the quantum registers can also be entangled.

5.4 Quantum Gates

A quantum gate acts on certain qubits of a register by means of a specific unitary operation. The final state is measured via a projective measurement. It is possible to perform all sorts of computational operations using only 3 qubits(two 1-qubit gate and one 2-qubit gate)

Summary of gates operations on qubits

- **X-gate** flips bit from $|0\rangle$ to $|1\rangle$ and vice versa.Normal classical not gate.
- **S-Gate** flips $|1\rangle$ to $i |1\rangle$ and vice versa.
- **Z-gate** changes $|1\rangle$ to $-|1\rangle$ and vice versa.
- **Hadamard-Gate** Changes Z-basis state to X basis state.
- **CNOT-gate** changes the state of the target qubit if the controlled qubit is $|1\rangle$ .Creates entangled states if acted on superposition qubits.

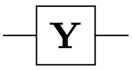
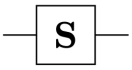
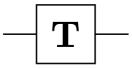
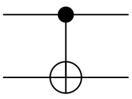
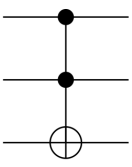
Operator	Gate(s)	Matrix
Pauli-X (X)		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
π/8 (T)		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z (CZ)		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
SWAP		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Figure 5.1: Quantum Logic Gates

6. Additional Topics

6.1 No cloning Theorem

The no-cloning theorem states that it is impossible to create an independent and identical copy of an arbitrary unknown quantum state.

Let's consider a state $|a\rangle$ that we want to clone. By cloning the system all quantum properties will be exactly duplicated. Let's also consider an empty state $|\phi\rangle$ where we will copy our state to.Then the resulting Hilbert space becomes

$$|a\rangle \otimes |\phi\rangle$$

If our machine or system denoted by an unitary operator T acts upon the system then the following cloning occurs

$$T(|a\rangle \otimes |\phi\rangle) = |a\rangle \otimes |a\rangle$$

also the operator or the machine should be state independent meaning it will produce the same results for another state vector $|b\rangle$

$$T(|b\rangle \otimes |\phi\rangle) = |b\rangle \otimes |b\rangle$$

But if we are facing a system which is in superposition state. Then we expect to following to hold from quantum mechanics.

$$\begin{aligned} T(|\Psi\rangle \otimes |\phi\rangle) &= T((c_1 |a\rangle + c_2 |b\rangle) \otimes |\phi\rangle) \\ &= c_1 T(|a\rangle \otimes |\phi\rangle) + c_2 T(|b\rangle \otimes |\phi\rangle) \\ &= c_1 |a\rangle \otimes |a\rangle + c_2 |b\rangle \otimes |b\rangle \end{aligned}$$

But This can also be written as

$$\begin{aligned} T(|\Psi\rangle \otimes |\phi\rangle) &= |\Psi\rangle \otimes |\Psi\rangle \\ &= (c_1 |a\rangle + c_2 |b\rangle) \otimes (c_1 |a\rangle + c_2 |b\rangle) \\ &= c_1^2(|a\rangle \otimes |a\rangle) + c_1 c_2(|b\rangle \otimes |a\rangle) + c_1 c_2(|a\rangle \otimes |b\rangle) + c_2^2(|b\rangle \otimes |b\rangle) \end{aligned}$$

Clearly the two sides don't match up, Which means either the operations are wrong in some way or the cloning process is not possible.

6.2 Quantum Teleportation

Let's suppose we want to teleport a state $|\Psi\rangle$ to a certain bob via alice. Where alice and bob each have an entangled pair $|e\rangle$

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

and

$$|e\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Now the Hilbert space becomes

$$|\Psi\rangle \otimes |e\rangle = \frac{1}{\sqrt{2}}(\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle)$$

Here the first two qubits are with alice and the third one is with bob.The idea is to allow certain operations on the above qubits so that alice's qubits can be separated and written as a product. Then depending on the measurement alice performs on those states.She will be able to get some information about the initial qubits state. If she sends

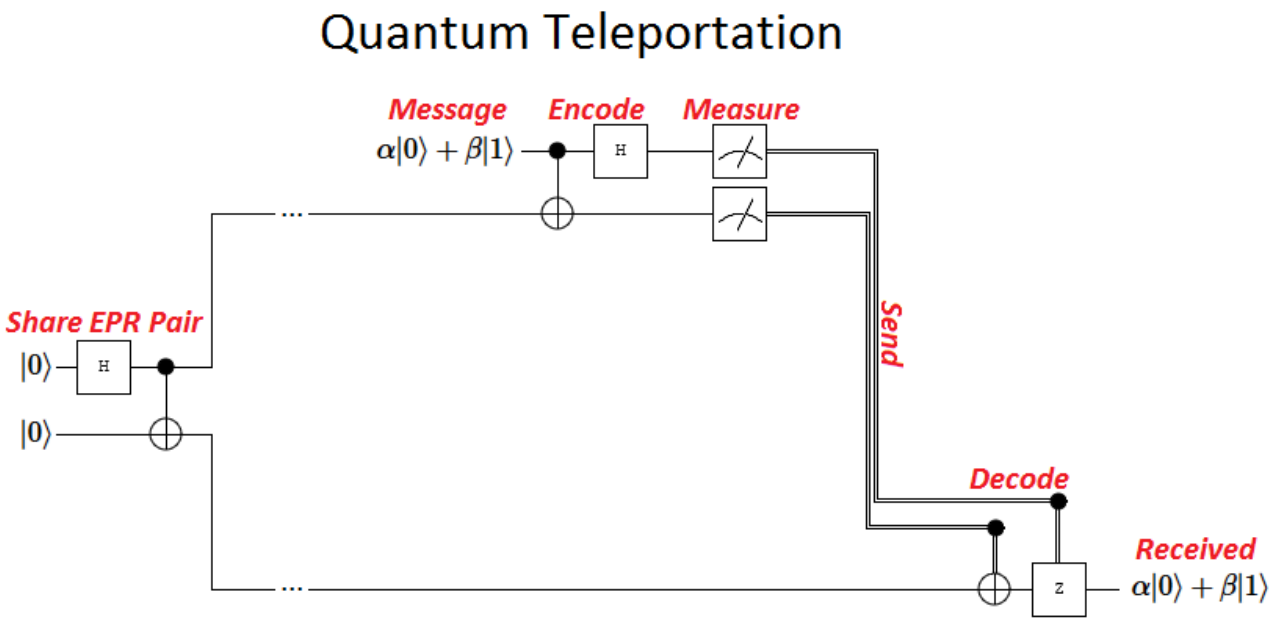


Figure 6.1: Quantum Teleportation Protocol

those information to bob, Then bob can apply certain unitary operations to collapse the product state to the initial quantum state $|Psi\rangle$ that alice wants to send.

$$(H \otimes I \otimes I)(CNOT \otimes I)(|\Psi\rangle \otimes |e\rangle) = \frac{1}{2}(\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle))$$

This can be separated as

$$= \frac{1}{2}(|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) + |11\rangle (\alpha |1\rangle - \beta |0\rangle))$$

The act of applying certain operations (CNOT-gate followed by a Hadamard-gate in our case) had the effect of separating the qubits of alice from bob which also happens to have given us some information on how to transfer the state $|\Psi\rangle$

Now if alice does measurement based upon her separated qubits she will collapse the superposition state. Depending upon her measured basis one of the following will take place.

$$|00\rangle \rightarrow \alpha |0\rangle + \beta |1\rangle$$
$$|01\rangle \rightarrow \alpha |1\rangle + \beta |0\rangle$$
$$|10\rangle \rightarrow \alpha |0\rangle - \beta |1\rangle$$
$$|11\rangle \rightarrow \alpha |1\rangle - \beta |0\rangle$$

Now if alice measures first and then sends the information as to which measurement basis she used, Bob can use that information to extract the state.The reason behind this is that they were entangled , as soon as alice makes her measurement the state on bob's qubit collapses according to the above info. Now if bob knows which state he's dealing with, he can use certain operations to get to the state

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

In this example if the bits received is 00,then no gate needs to be applied. If it's 01 then X gate needs to be applied. Z gate for 10 qubit and X gate followed by a Z gate for 11 .

6.3 Quantum Super Dense Coding

In a quantum teleportation we transmit one qubit using two classical bits.In super dense coding we transmit two classical bits using one qubit. Here, an entangled state is prepared first then one of them is sent to alice and the other one is sent directly to bob.Now Alice encodes some sort of message on the qubit she receives, and then sends it to bob.Bob applies inverse unitary gates to untangle the states.If alice had not encoded anything on her qubit than

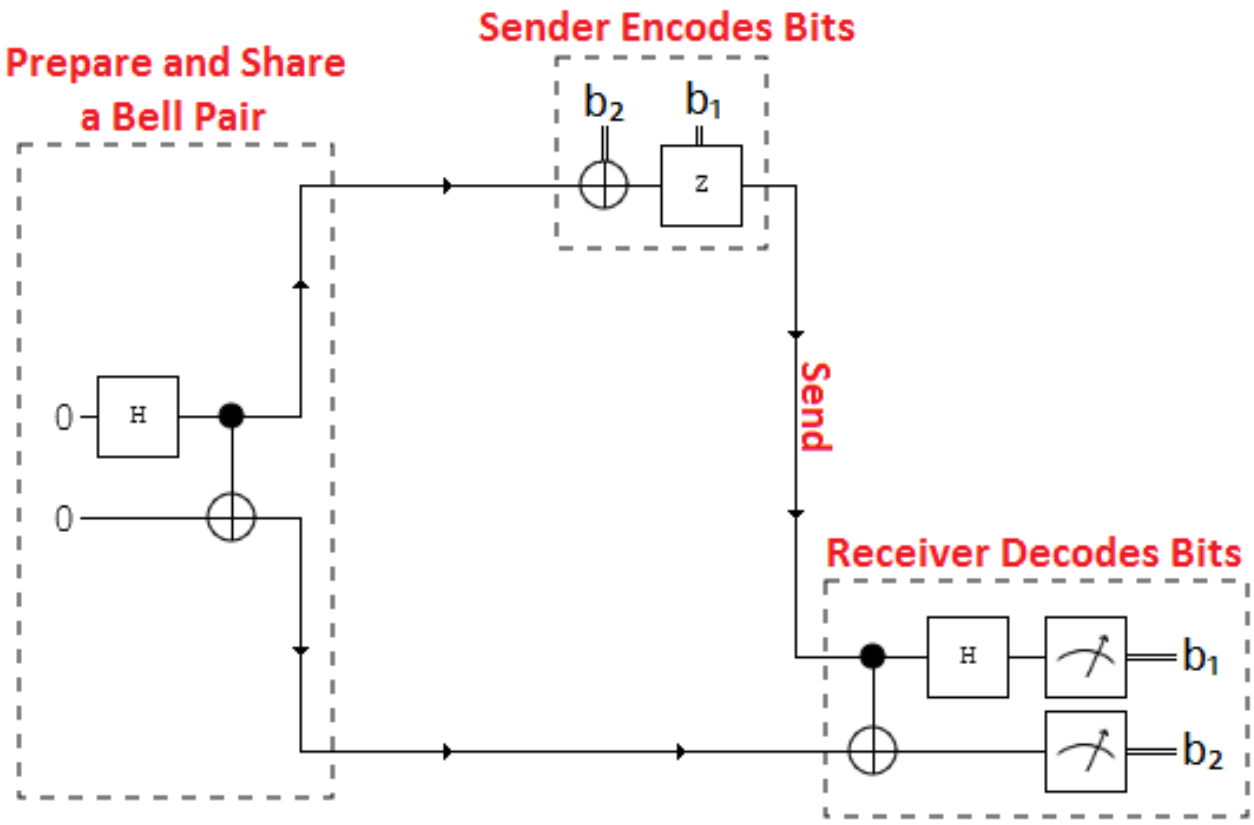


Figure 6.2: Super Dense Coding Protocol

the final output bob would have will be similar to the one created before the entangled pair was created. But since alice had done some tempering, the output will be different, which enables us to send information about the bits to bob.

If we consider a state

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B$$

Now if we create an entangled pair we get to the bell basis state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Here the first bit is sent to alice and the second bit is sent to bob. Now if Alice wants to send bits 00 then she can simply send her qubit to bob without modifying anything.If bob applies inverse unitary operations to untangle the states ,Then makes measurement he will get bits 00.

for bits 01 alice will have to apply a X-gate to her qubit before sending it to bob.Now if bob applies inverse gate operations followed by a measurement he will receive bits 01. The reason behind this is, if the states at the start before the entanglement were 0 and 1 then the entanglement would have created the state

$$CNOT(H(|0\rangle \otimes |1\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

and after dientegration bob would receive the same input states 01. Here Alice's encoding method is making the qubit sent to bob behave in the exact same way.

Similarly for 10 ,alice must use Z-gate for encoding.In the case of 11, alice must use X-gate followed by a Z-gate to send the qubit.

Thus with this method alice can successfully send 00,01,10 and 11 bits to bob.

6.4 Grover's Algorithm

If we have N-entries in a database.Then classically we have to look at N-1 entries at max to get to our desired data. But Grover's algorithm uses only $\sqrt{N}$ steps to get to our desired data.

The idea is to first convert one of the qubits to negative phase.Then apply some sort of amplitude amplification on it such that it's amplitude get's higher.Doing so makes the other state's amplitude go down.Repeating this step allows us to get to our desired data.But the main advantage of this algorithm is that we first superposition the states.The superposition and the phase interference allows our query to be run $\sqrt{N}$ times for N unstructured data.

Outline of the algorithm

First take $|0\rangle$ and apply Hadamard-gates to every initial state vectors to create superposition.

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

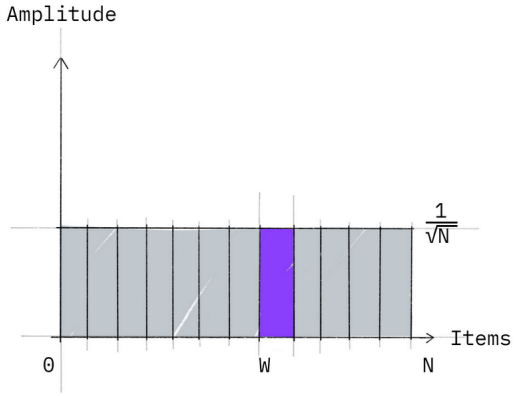


Figure 6.3: State initialization

Now we create an oracle for our desired state ω. which acts as below

$$U_w |x\rangle = \begin{cases} \quad |x\rangle\,, & \text{if } x \neq \omega \\ - \quad |x\rangle\,, & \text{if } x = \omega \end{cases}$$

If we apply this operation to our states, then this will flip the phase of our desired state.

$$U_w |x\rangle = (-1)^{f(x)} |x\rangle$$

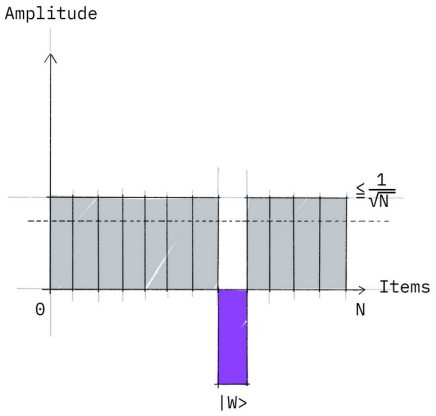


Figure 6.4: Oracle operator operation

Now we apply something known as the grover operator. This flips every amplitude around the average,

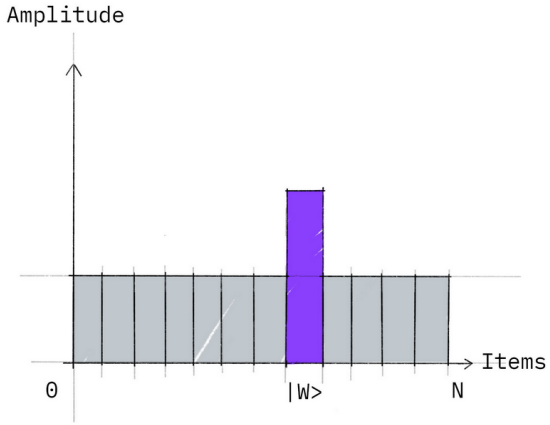


Figure 6.5: Grover operator operation

If we measure the qubits now, we still have a reasonable chance of selecting the wrong answer. So we repeat this process $\sqrt{N}$ times to amplify the amplitude of the right answer $|\omega\rangle$