Rotation in four spatial dimensions

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Problem

Orientation and angular velocity must be handled in four dimensions.

Characteristics

For rotation in four dimensions:

- There is no 'axis of rotation' in the conventional sense. In 3D, a rotation has 1 or 3 real eigenvalues, corresponding to rotation around an axis or the identity/inversion. In 4D, a rotation has 0, 2 or 4 real eigenvalues. These correspond to a rotation that moves every point, a rotation that leaves points in a plane fixed, and the identity/inversion.

Rotation with a pointwise-fixed invariant plane (2 real eigenvalues)

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \textbf{Equation 1}$$

Matrix ***M*** represents a rotation in the *xy* plane by angle θ, keeping the *z* and *w* coordinates the same. The *xy* plane is invariant (because points in that plane stay in that plane), and the *zw* plane is pointwise invariant (because points in it do not move at all).

Rotation without a pointwise-fixed invariant plane (0 real eigenvalues)

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \qquad \textbf{Equation 2}$$

Matrix **M** represents a rotation in the *zw* plane by angle ϕ, keeping the *x* and *y* coordinates the same, followed by a rotation in the *xy* plane by angle θ as above. Generally this operation moves every point except the origin. If $\theta = \phi$ or -ϕ, the rotation is isoclinic, and every point is moved through the same angle with respect to the origin.

Representations

Some possible representations for rotations are:

Type	Pros	Cons
Matrix	• Rapid calculation	• Need to be renormalised periodically due to calculation errors • Difficult to interpolate between two orientations • Cannot represent angular velocity >π
Quaternion pairs	• Easily converted to matrix form • Easy to interpolate.	• Moderately difficult to calculate • Cannot represent angular velocity >π
Quaternion pair + power	• Easily converted to matrix form • Can represent angular velocity >π	• Difficult to calculate • Difficult to interpolate
Angular velocity	• Potentially simple	• In 4D objects do not rotate about an axis. Two velocities and two orthogonal rotation planes are required
Euler angles	• Simple to generate a matrix representation	• Difficult to combine rotations • Difficult to interpolate – gimbal lock

Quaternion representation and conversion

A unit quaternion is defined as

$$\mathbf{q} = x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k} \quad \textbf{Equation 3}$$

where

$$x^2 + y^2 + z^2 + w^2 = 1 \quad \textbf{Equation 4}$$

and Hamilton's rules apply

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \quad \textbf{Equation 5}$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} \quad \textbf{Equation 6}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \quad \textbf{Equation 7}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j} \quad \textbf{Equation 8}$$

Converting from quaternion pair to matrix representation

From this definition, quaternion multiplication can be expanded and represented as a matrix.

$\mathbf{q_3} = \mathbf{q_1 q_2}$ **Equation 9**

Expanding the multiplication yields:

$$(x_3 + y_3\mathbf{i} + z_3\mathbf{j} + w_3\mathbf{k}) = (x_1 + y_1\mathbf{i} + z_1\mathbf{j} + w_1\mathbf{k})(x_2 + y_2\mathbf{i} + z_2\mathbf{j} + w_2\mathbf{k})$$ **Equation 10**

Collecting terms:

$$\begin{aligned}
&(x_3 + y_3\mathbf{i} + z_3\mathbf{j} + w_3\mathbf{k}) = \\
&(x_1x_2 - y_1y_2 - z_1z_2 - w_1w_2) + \\
&(x_1y_2 + y_1x_2 + z_1w_2 - w_1z_2)\mathbf{i} + \\
&(x_1z_2 + z_1x_2 - y_1w_2 + w_1y_2)\mathbf{j} + \\
&(x_1w_2 + w_1x_2 + y_1z_2 - z_1y_2)\mathbf{k}
\end{aligned}$$

Equation 11

The equivalent matrix representation is:

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 & -z_1 & -w_1 \\ y_1 & x_1 & -w_1 & z_1 \\ z_1 & w_1 & x_1 & -y_1 \\ w_1 & -z_1 & y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$$

Equation 12

In three dimensions a unit quaternion can represents any rotation about the origin if points are transformed thus:

$\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^{-1}$ **Equation 13**

NB: Many published formulae for *quaternion to matrix conversion* result in a matrix that encapsulates the effect of *both* quaternion multiplications (which is shown below in Equation 19). The matrix above encapsulates just one.

For unit quaternions, the inverse is equivalent to the conjugate

$\mathbf{q}^{-1} = x - y\mathbf{i} - z\mathbf{j} - w\mathbf{k}$ **Equation 14**

In four dimensions, a pair of unit quaternions can represent any rotation about the origin[1]

$\mathbf{v}' = \mathbf{q_l}\mathbf{v}\mathbf{q_r}$ **Equation 15**

As shown in Equation 12, left multiplication by a quaternion can be represented by a 4×4 matrix:

$$\mathbf{M}_L = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$ **Equation 16**

A similar procedure yields the matrix for right multiplication:

$$\mathbf{M}_R = \begin{pmatrix} e & -f & -g & -h \\ f & e & h & -g \\ g & -h & e & f \\ h & g & -f & e \end{pmatrix}$$ **Equation 17**

such that

$$\mathbf{v}' = \mathbf{q_l v q_r} = \mathbf{M_L M_R v} = \mathbf{Mv}$$ **Equation 18**

$$\mathbf{M} = \begin{pmatrix} ae-bf-cg-dh & -af-be+ch-dg & -ag-bh-ce+df & -ah+bg-cf-de \\ be+af-dg+ch & -bf+ae+dh+cg & -bg+ah-de-cf & -bh-ag-df+ce \\ ce+df+ag-bh & -cf+de-ah-bg & -cg+dh+ae+bf & -ch-dg+af-be \\ de-cf+bg+ah & -df-ce-bh+ag & -dg-ch+be-af & -dh+cg+bf+ae \end{pmatrix}$$

Equation 19

Matrix **M** can be used directly for rotation calculations.

There are two quaternion pair representations for each rotation[1], one obtained from the other by negating both quaternions. Negating all values *a* to *h* in Equation 19 leaves the matrix unchanged. This is relevant when interpolating between two orientations.

Converting from matrix to a quaternion pair representation

Determining quaternion values from a given matrix can be achieved as follows[2]. For a rotation matrix **M**

$$\mathbf{M_A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$ **Equation 20**

Let

$$\mathbf{q_s} = a_{00} + a_{10}\mathbf{i} + a_{20}\mathbf{j} + a_{30}\mathbf{k}$$ **Equation 21**

Let $\mathbf{M_S}$ be the matrix which implements premultiplication by $\mathbf{q_s}$.

$$\mathbf{M_S} = \begin{pmatrix} a_{00} & -a_{10} & -a_{20} & -a_{30} \\ a_{10} & a_{00} & -a_{30} & a_{20} \\ a_{20} & a_{30} & a_{00} & -a_{10} \\ a_{30} & -a_{20} & a_{10} & a_{00} \end{pmatrix}$$ **Equation 22**

Since the inverse is the conjugate

$$\mathbf{q_s}^{-1} = a_{00} - a_{10}\mathbf{i} - a_{20}\mathbf{j} - a_{30}\mathbf{k}$$ **Equation 23**

$$\mathbf{M_S}^{-1} = \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ -a_{10} & a_{00} & a_{30} & -a_{20} \\ -a_{20} & -a_{30} & a_{00} & a_{10} \\ -a_{30} & a_{20} & -a_{10} & a_{00} \end{pmatrix} \quad \textbf{Equation 24}$$

Calculate **P** such that

$$\mathbf{M_A v} = \mathbf{M_S P v} \quad \textbf{Equation 25}$$

$$\mathbf{P} = \mathbf{M_S}^{-1}\mathbf{M_A} = \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ -a_{10} & a_{00} & a_{30} & -a_{20} \\ -a_{20} & -a_{30} & a_{00} & a_{10} \\ -a_{30} & a_{20} & -a_{10} & a_{00} \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \textbf{Equation 26}$$

Therefore P has the form

$$\mathbf{P} = \mathbf{M_S}^{-1}\mathbf{M_A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & p_{13} \\ 0 & p_{21} & p_{22} & p_{23} \\ 0 & p_{31} & p_{32} & p_{33} \end{pmatrix} \quad \textbf{Equation 27}$$

The top row is 1,0,0,0 because the rotation matrix is orthonormal and has the property

$$a_{i0}a_{j0} + a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = 1 \text{ if } i = j,\ 0 \text{ if } i \neq j \quad \textbf{Equation 28}$$

The zeros in the first column can be shown by inspection. We then construct a quaternion $\mathbf{q_p}$ that implements the 3D rotation **P** (by substitution into Equation 19, or as Salamin[3]).

$$\mathbf{q_p} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad \textbf{Equation 29}$$

$$q_0^{\ 2} = \frac{1}{4}(1 + p_{11} + p_{22} + p_{33}) \quad \textbf{Equation 30}$$

$$q_1^{\ 2} = \frac{1}{4}(1 + p_{11} - p_{22} - p_{33}) \quad \textbf{Equation 31}$$

$$q_2^{\ 2} = \frac{1}{4}(1 - p_{11} + p_{22} - p_{33}) \quad \textbf{Equation 32}$$

$$q_3^{\ 2} = \frac{1}{4}(1 - p_{11} - p_{22} + p_{33}) \quad \textbf{Equation 33}$$

These equations determine the magnitude but not the sign of values within the quaternion. It is adequate to choose one root (usually the largest) with arbitrary sign and calculate the rest using the following.

$$q_0 q_1 = \frac{1}{4}(p_{32} - p_{23}) \quad \textbf{Equation 34}$$

$$q_0 q_2 = \frac{1}{4}(p_{13} - p_{31}) \quad \textbf{Equation 35}$$

$$q_0 q_3 = \frac{1}{4}(p_{21} - p_{12})$$ **Equation 36**

$$q_1 q_2 = \frac{1}{4}(p_{12} + p_{21})$$ **Equation 37**

$$q_1 q_3 = \frac{1}{4}(p_{13} + p_{31})$$ **Equation 38**

$$q_2 q_3 = \frac{1}{4}(p_{23} + p_{32})$$ **Equation 39**

Returning to

$$\mathbf{M_A v} = \mathbf{M_S P v}$$ **Equation 25**

$$\mathbf{M_A v} = \mathbf{M_S q_p v q_p}^{-1}$$ **Equation 40**

$$\mathbf{M_A v} = \mathbf{q_s q_p v q_p}^{-1}$$ **Equation 41**

The quaternions performing the rotation are

$$\mathbf{v}' = \mathbf{q_l v q_r}$$ **Equation 42**

$$\mathbf{q_l} = \mathbf{q_s q_p}$$ **Equation 43**

$$\mathbf{q_r} = \mathbf{q_p}^{-1}$$ **Equation 44**

Summary

Quaternion values can be substituted into Equation 19 to generate the corresponding matrix. Below, the plane numbering is arbitrarily chosen. Trigonometric identities

$$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

Equation 45

and

$$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

Equation 46

are used to simplify the results.

The choice of direction of rotation and plane numbering below is arbitrary, and in this case has been chosen such that, for a rotation of $\pi/2$, the first axis (appearing in the order *xyzw*) moves to the second. One alternative scheme is to ensure that a sequence of $\pi/2$ rotations such as $xy \to yz \to zx$ will lead will leave the x axis pointing in the same direction (in this scheme here it would lie along $-x$). This can be achieved by negating the sine components in planes 2 and 3, and may be more consistent with group theory interpretations.

Rotation in plane 0: *xy* plane

A rotation through angle θ in the *xy* plane, where $\theta=\pi/2$ takes **x** to **y** and **y** to **–x**.

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equation 47

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}}, \sqrt{\frac{1-\cos\theta}{2}}, 0, 0 \right) = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right), 0, 0 \right)$$

Equation 48

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}}, \sqrt{\frac{1-\cos\theta}{2}}, 0, 0 \right) = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right), 0, 0 \right)$$

Equation 49

Rotation in plane 1: *zw* plane

A rotation through angle θ in the *zw* plane, where $\theta=\pi/2$ takes **z** to **w** and **w** to **–z**.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Equation 50

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}}, \sqrt{\frac{1-\cos\theta}{2}}, 0, 0 \right) = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right), 0, 0 \right)$$

Equation 51

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}}, -\sqrt{\frac{1-\cos\theta}{2}}, 0, 0 \right) = \left(\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right), 0, 0 \right)$$

Equation 52

Rotation in plane 2: *xz* plane

A rotation through angle θ in the *xz* plane, where $\theta=\pi/2$ takes **x** to **z** and **z** to **–x**.

$$\mathbf{M} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equation 53

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,\sqrt{\frac{1-\cos\theta}{2}},0\right) = \left(\cos\left(\frac{\theta}{2}\right),0,\sin\left(\frac{\theta}{2}\right),0\right)$$

Equation 54

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,\sqrt{\frac{1-\cos\theta}{2}},0\right) = \left(\cos\left(\frac{\theta}{2}\right),0,\sin\left(\frac{\theta}{2}\right),0\right)$$

Equation 55

Rotation in plane 3: *yw* plane

A rotation through angle θ in the *yw* plane, where $\theta=\pi/2$ takes **y** to **w** and **w** to **–y**. Note that the negated sine value is now in the left hand side quaternion.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Equation 56

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,\sqrt{\frac{1-\cos\theta}{2}},0\right) = \left(\cos\left(\frac{\theta}{2}\right),0,-\sin\left(\frac{\theta}{2}\right),0\right)$$

Equation 57

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,-\sqrt{\frac{1-\cos\theta}{2}},0\right) = \left(\cos\left(\frac{\theta}{2}\right),0,\sin\left(\frac{\theta}{2}\right),0\right)$$

Equation 58

Rotation in plane 4: *xw* plane

A rotation through angle θ in the *xw* plane, where $\theta=\pi/2$ takes **x** to **w** and **w** to **–x**.

$$\mathbf{M} = \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

Equation 59

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,0,\sqrt{\frac{1-\cos\theta}{2}}\right) = \left(\cos\left(\frac{\theta}{2}\right),0,0,\sin\left(\frac{\theta}{2}\right)\right)$$

Equation 60

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}},0,0,\sqrt{\frac{1-\cos\theta}{2}}\right) = \left(\cos\left(\frac{\theta}{2}\right),0,0,\sin\left(\frac{\theta}{2}\right)\right)$$

Equation 61

Rotation in plane 5: *yz* plane

A rotation through angle θ in the *yz* plane, where $\theta=\pi/2$ takes **y** to **z** and **z** to **–y**.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equation 62

$$\mathbf{q_l} = \left(\sqrt{\frac{1+\cos\theta}{2}}, 0, 0, \sqrt{\frac{1-\cos\theta}{2}} \right) = \left(\cos\left(\frac{\theta}{2}\right), 0, 0, \sin\left(\frac{\theta}{2}\right) \right)$$

Equation 63

$$\mathbf{q_r} = \left(\sqrt{\frac{1+\cos\theta}{2}}, 0, 0, -\sqrt{\frac{1-\cos\theta}{2}} \right) = \left(\cos\left(\frac{\theta}{2}\right), 0, 0, -\sin\left(\frac{\theta}{2}\right) \right)$$

Equation 64

Combining rotations

$$\mathbf{M} = \mathbf{BA}$$

Equation 65

The quaternion pair applying rotation A followed by rotation B is

$$\mathbf{q_l} = \mathbf{b_l a_l}$$

Equation 66

$$\mathbf{q_r} = \mathbf{a_r b_r}$$

Equation 67

Interpolation

Quaternion representations of rotations can be interpolated using the slerp formula[4]

$$\text{Slerp}(\mathbf{q_1}, \mathbf{q_2}, u) = \frac{\sin(1-u)\theta}{\sin\theta}\mathbf{q_1} + \frac{\sin u\theta}{\sin\theta}\mathbf{q_2}$$

Equation 68

Where

$$\mathbf{q_1} \cdot \mathbf{q_2} = \cos\theta$$

Equation 69

It is likely that this will extend to 4D rotations, using slerp on each quaternion independently, but no proof is presented here.

Version History

1. 2004-12-01: Created with filename rotation1-2004-12-01
2. 2005-02-28: Draft

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