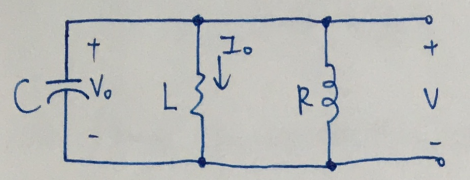
Natural Response of a Parallel RLC circuit



V_0: initial voltage

I_0: initial current

☆ intend to find $V(t)$

Node voltage method: $\frac{V}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$

By solving differential equation: $\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{V}{LC} = 0$,

(assume $v(t)$ has the form Ae^{st})

we can get

① $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \Rightarrow$ characteristic equation

$$\Rightarrow S_{1,2} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{(2RC)^2} - \frac{1}{LC}}$$

② parameters

☆ $\alpha = \frac{1}{2RC}$ (Neper frequency)

☆ $W_0 = \frac{1}{\sqrt{LC}}$ (Resonant radian frequency)

☆ $S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - W_0^2}$ (characteristic roots)

③ ☆ Three cases

$\alpha^2 - W_0^2 > 0$: overdamped

$\alpha^2 - W_0^2 = 0$: critically damped

$\alpha^2 - W_0^2 < 0$: underdamped

④ Solution

☆☆ $V(t) = \underline{A_1}e^{S_1 t} + \underline{A_2}e^{S_2 t}$, where $V(0^+) = A_1 + A_2 = V_0$

$$\frac{dV}{dt}(0^+) = S_1A_1 + S_2A_2 = \frac{1}{C}i_C(0^+)$$

$$= \frac{1}{C}\left(-\frac{V_0}{R} - I_0\right)$$

How to solve the problem?

1. Find the parameters: $\alpha = \frac{1}{2RC}$, $W_0 = \frac{1}{\sqrt{LC}}$

2. Find characteristic roots: $S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - W_0^2}$ determine the case of voltage response

3. Find A_1 and A_2 $\begin{cases} A_1 + A_2 = V(0^+) = V_0 \\ S_1 A_1 + S_2 A_2 = \frac{1}{C} i_c(0^+) = \frac{1}{C}\left(-\frac{V_0}{R} - I_0\right) \end{cases}$

4. $V(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

* Underdamped voltage response

$\alpha^2 - W_0^2 < 0 \Rightarrow S_{1,2}$ are complex

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$\Rightarrow V(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

$$= \boxed{e^{-\alpha t}(B_1 \cos W_d t + B_2 \sin W_d t) \quad \text{where } B_1 = A_1 + A_2,\ B_2 = (A_1 - A_2)j,\ W_d = \sqrt{W_0^2 - \alpha^2}}$$

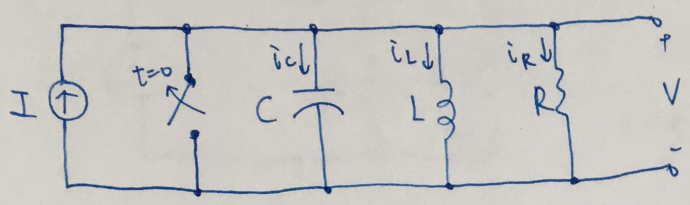
$$\begin{cases} V(0^+) = B_1 = V_0 \\ \frac{dV}{dt}(0^+) = \frac{1}{C} i_c(0^+) = -\alpha B_1 + W_d B_2 = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right) \end{cases}$$

* Critically damped voltage response

$\alpha^2 = W_0^2 \Rightarrow S_1 = S_2 = -\alpha = -\frac{1}{2RC}$

Solution should be: $\boxed{\begin{array}{l} V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \\ \text{where } V(0^+) = V_0 = D_2 \\ \qquad D_1 - \alpha D_2 = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right) \end{array}}$

Step Response of a Parallel RLC circuit



☆ intend to find $V(t)$

KCL: $I = i_C + i_L + i_R \Rightarrow \frac{1}{L}\int_0^t v\,d\tau + \frac{V}{R} + C\frac{dV}{dt} = I$

By solving differential equation: $\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V}{LC} = 0$

$\Rightarrow$ $v(t)$ has the same form as in natural response

Three cases:

* Overdamped system: $\alpha^2 - \omega_0^2 > 0$

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad \begin{cases} A_1 + A_2 = V(0^+) = V_0 \\ s_1 A_1 + s_2 A_2 = \frac{dV}{dt}(0^+) = \frac{1}{C}\left(I - I_0 - \frac{V_0}{R}\right) \end{cases}$$

* Underdamped system: $\alpha^2 - \omega_0^2 < 0$

$$V(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t), \quad \begin{cases} B_1 = A_1 + A_2 \\ B_2 = (A_1 - A_2)j \\ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \end{cases}$$

* Critically damped system: $\alpha^2 - \omega_0^2 = 0$

$$V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad \begin{cases} D_2 = V(0^+) = V_0 \\ D_1 - \alpha D_2 = \frac{1}{C}\left(I - I_0 - \frac{V_0}{R}\right) \end{cases}$$

Branch Current

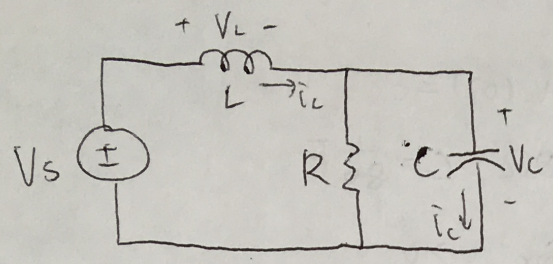
$$V = L\frac{di}{dt} \Rightarrow i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$i_L = I + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

use $i_L(0)$ and $V_L(0) = L\frac{di_L(0)}{dt}$ to find $A_1', A_2', B_1', B_2', D_1', D_2'$

①



$V_s(t) = 20u(t)$ V

$L = 2H$, $R = 5\Omega$, and $C = \frac{1}{50}$ F

Find i_L for $t > 0$

(i) Find α, w_0, $S_{1,2}$

$$\alpha = \frac{1}{2RC} = 5, \quad w_0 = \frac{1}{\sqrt{LC}} = 5, \quad S_1, S_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -5$$

(ii) Check $\alpha^2 - w_0^2$

$\alpha^2 - w_0^2 = 0 \Rightarrow$ Critically damped

(iii) Equation

$$i_L = I_L + D_1 t e^{st} + D_2 e^{st}$$

(iv) Use initial value and first-order derivative to find the coefficients

$$I_L = i_L(\infty) = \frac{V_s}{R} = 4 \text{ A (steady state)}$$

$$i_L(0^+) = i_L(0^-) = 0 = I_L + D_2 \Rightarrow D_2 = -4$$

$$V_L(0^+) = V_s(0^+) - V_c(0^+) = V_s(0^+) - V_c(0^-) = 20 - 0 = 20$$

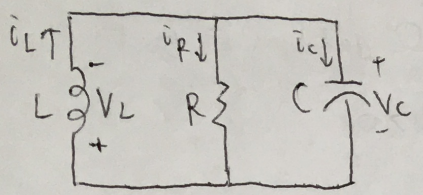
$$= L\frac{di_L(0^+)}{dt} = 2 \cdot \left[-5D_1 t e^{-5t} + D_1 e^{-5t} - 5D_2 e^{-5t}\right]\Big|_{t=0^+}$$

$$= 2 \cdot (D_1 + 20)$$

$$\Rightarrow D_1 = -10$$

$$\therefore i_L = 4 - 4e^{-5t} - 10te^{-5t}, \quad t > 0$$

②



$i_L(0^-) = 3A$, $V_C(0^-) = 0$

$L = 5H$, $R = 8\Omega$, $C = \frac{1}{80}F$

Find $V_C(t)$ for $t > 0$

(i) Find α and ω_0

$$\alpha = \frac{1}{2RC} = 5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 4$$

(ii) Check $\alpha^2 - \omega_0^2$

$$\alpha^2 - \omega_0^2 = 9 > 0 \Rightarrow \text{overdamped}$$

$$S_{1,2} = -5 \pm 3 \Rightarrow S_1 = -2, \; S_2 = -8$$

(iii) Equation

$$V_C = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

(iv) Initial value and first-order derivative

$$V_C(0^+) = V_C(0^-) = 0 = A_1 + A_2$$

$$i_C(0^+) = i_L(0^+) - i_R(0^+) = i_L(0^-) - \frac{V_C(0^+)}{R} = 3$$

$$= C\frac{dV_C(0^+)}{dt} = \frac{1}{80} \cdot \left[-2A_1 e^{-2t} - 8A_2 e^{-8t}\right]\Big|_{t=0^+} = \frac{1}{80}(-2A_1 - 8A_2)$$

$$\Rightarrow \begin{cases} A_1 + A_2 = 0 \\ A_1 + 4A_2 = -120 \end{cases} \Rightarrow A_1 = 40, \; A_2 = -40$$

$$\therefore V_C(t) = 40e^{-2t} - 40e^{-8t}, \quad t > 0$$