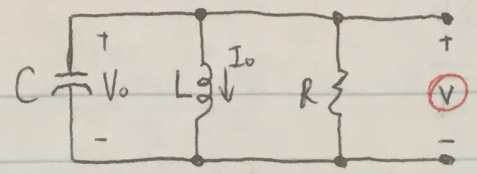
# Natural Response of a Parallel RLC circuit




$V_0$ : initial voltage

$I_0$ : initial current

☆ intend to find $v(t)$

By solving the differential equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0,$$

(assume $v(t)$ has the form $Ae^{st}$)

we can get

① ☆ $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \Rightarrow$ $\boxed{\text{characteristic equation}}$

$$\Rightarrow s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{(2RC)^2} - \frac{1}{LC}}$$

② parameters:

i. Neper frequency: ☆ $\alpha = \frac{1}{2RC}$

ii. Resonant radian frequency: ☆ $\omega_0 = \frac{1}{\sqrt{LC}}$

iii. Characteristic roots: ☆ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

③ ☆ three cases

$\alpha^2 - \omega_0^2 > 0$ : overdamped

$\alpha^2 - \omega_0^2 = 0$ : critically damped

$\alpha^2 - \omega_0^2 < 0$ : underdamped

④ solution

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$,

where ☆ $v(0^+) = A_1 + A_2 = V_0$

☆ $\frac{dv}{dt}(0^+) = s_1 A_1 + s_2 A_2 = \frac{1}{C} i_C(0^+) = \frac{1}{C}\left(-\frac{V_0}{R} - I_0\right)$

Solve the problem

1. Find the parameters $\alpha = \frac{1}{2RC}$

$$W_0 = \frac{1}{\sqrt{LC}}$$

2. Find characteristic roots: $S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - W_0^2}$

determine the case of voltage response

3. Find $A_1$ and $A_2$

$$\begin{cases} A_1 + A_2 = V(0^+) = V_0 \\ S_1 A_1 + S_2 A_2 = \frac{1}{C} i_C(0^+) = \frac{1}{C}\left(-\frac{V_0}{R} - I_0\right) \end{cases}$$

4. $v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

* Underdamped voltage response

$\alpha^2 - W_0^2 < 0 \Rightarrow S_{1,2}$ are complex

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\therefore V(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$= \boxed{e^{-\alpha t}(B_1 \cos W_d t + j B_2 \sin W_d t)}$$

where $B_1 = A_1 + A_2$, $B_2 = A_1 - A_2$, $W_d = \sqrt{W_0^2 - \alpha^2}$

$$\begin{cases} v(0^+) = B_1 = V_0 \\ \frac{dV}{dt}(0^+) = \frac{1}{C} i_C(0^+) = -\alpha B_1 + W_d B_2 = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right) \end{cases}$$

* Critically damped voltage response

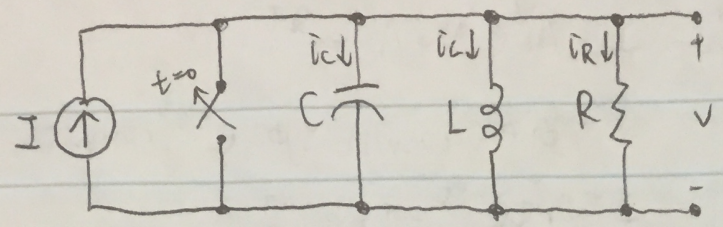
$\alpha^2 = W_0^2 \Rightarrow S_1 = S_2 = -\alpha = -\frac{1}{2RC}$

solution should be: $\boxed{V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$

where $V(0^+) = V_0 = D_2$

$$D_1 - \alpha D_2 = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

# Step Response of a Parallel RLC Circuit




* intend to find v(t)

KCL: $I = i_C + i_L + i_R \Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$

$\Rightarrow$ v(t) has the same solution form as in natural response

* Overdamped system

$\alpha^2 - \omega_0^2 > 0$

$$\boxed{\begin{aligned} &v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ &\begin{cases} A_1 + A_2 = v(0^+) = V_0 \\ s_1 A_1 + s_2 A_2 = \frac{dv}{dt}(0^+) = \frac{1}{C}\left(I - I_0 - \frac{V_0}{R}\right) \end{cases} \end{aligned}}$$

* Underdamped system

$\alpha^2 - \omega_0^2 < 0$

$$\boxed{\begin{aligned} &v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + j B_2 \sin\omega_d t) \\ &\text{where } B_1 = A_1 + A_2,\ B_2 = A_1 - A_2,\ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \end{aligned}}$$

* Critically damped system

$\alpha^2 - \omega_0^2 = 0$

$$\boxed{\begin{aligned} &v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \\ &\begin{cases} D_2 = v(0^+) = V_0 \\ D_1 - \alpha D_2 = \frac{1}{C}\left(I - I_0 - \frac{V_0}{R}\right) \end{cases} \end{aligned}}$$

Branch current:

$$V = L\frac{di}{dt} \Rightarrow \quad i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$i_L = I + B_1' e^{-\alpha t}\cos w_d t + jB_2' e^{-\alpha t}\sin w_d t$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

use $i_L(0)$ and $V_L(0) = L\frac{d i_L(0)}{dt}$ to find $A_1', A_2', B_1', B_2', D_1', D_2'$