Kalman Filter Theory and Applications
Equation Drilldown

https://github.com/musicarroll/kalman_course

Michael L. Carroll

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- Understanding the Equations: Heuristic Introduction

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- **Equation Drilldown: Taking the Equations Apart**

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$K(k) = P^{-}(k)\, H(k)^{\top} \left[H(k) P^{-}(k)\, H(k)^{\top} + R(k) \right]^{-1}$
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  $\hat{x}^{+}(k) = \hat{x}^{-}(k) + K(k)\left[z(k) - H(k)\hat{x}^{-}(k)\right]$ becomes

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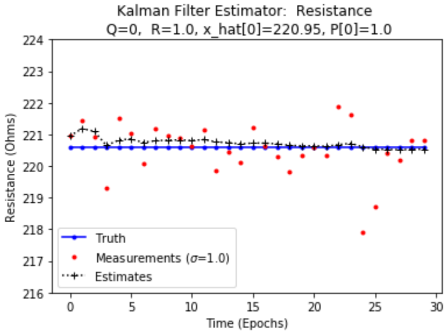

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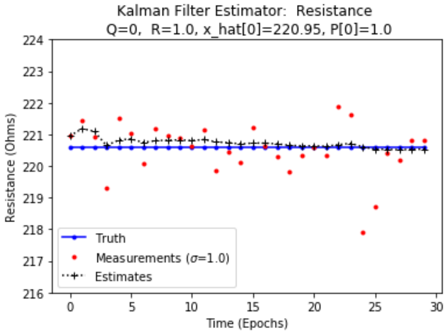

Static, Scalar Example: Resistor Revisited: Comparing KF and Running Average

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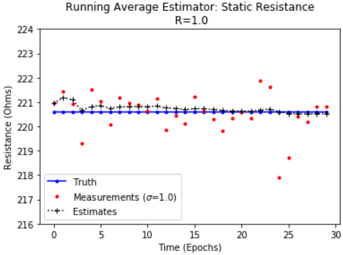
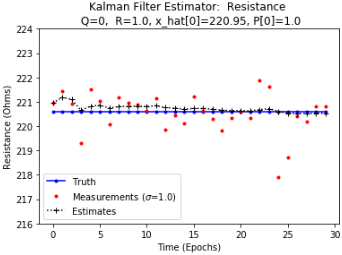


(a) Simple Running Average Estimator



(b) Kalman Estimator

Figure: Comparing Running Average and Kalman Estimators

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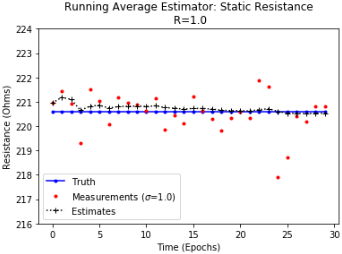

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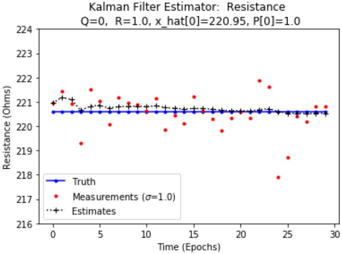

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Kalman Sawtooth Plots

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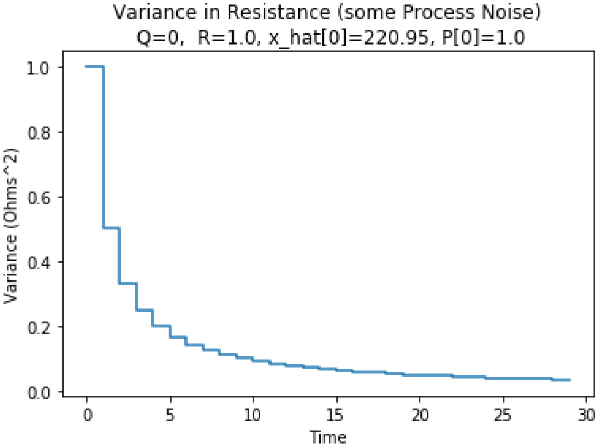

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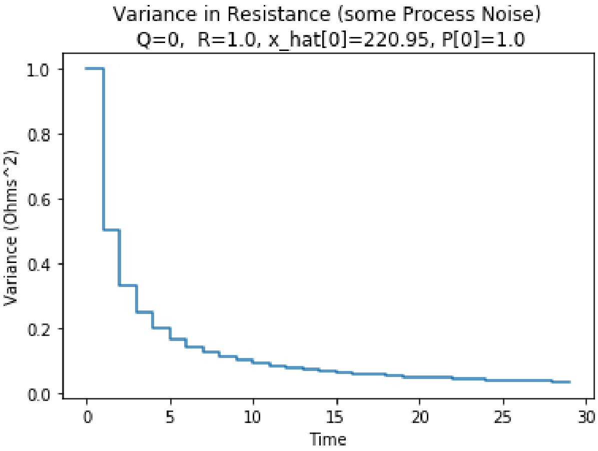

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Vector Check

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