Kalman Filter Theory and Applications
Equation Drilldown

https://github.com/musicarroll/kalman_course

Michael L. Carroll

June 18, 2023

Part I

The Five Basic Kalman Equations

Topics

- Understanding the Equations: Heuristic Introduction

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- **Equation Drilldown: Taking the Equations Apart**

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Equation Drilldown: Taking the Equations Apart

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Non-accelerated, kinematic motion in 1 dimension

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Filter Performance (without Process Noise)

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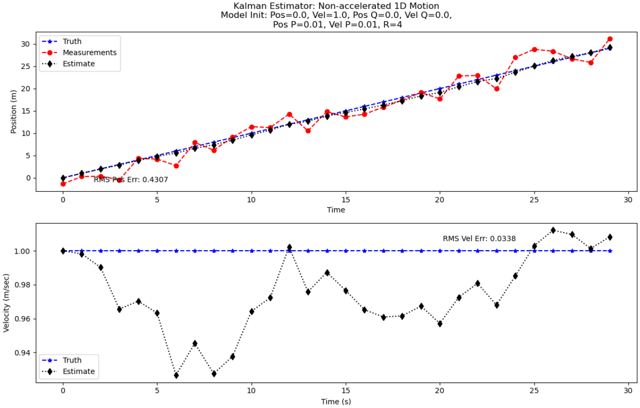

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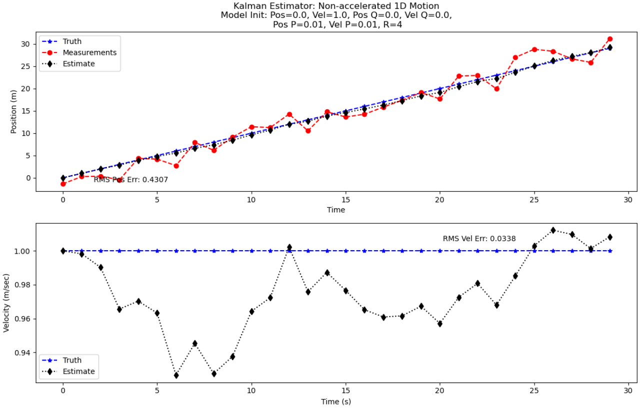

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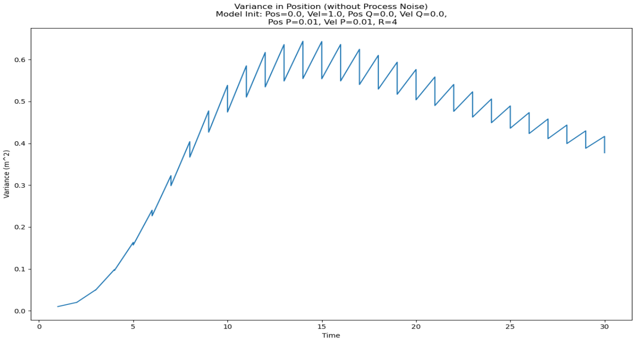

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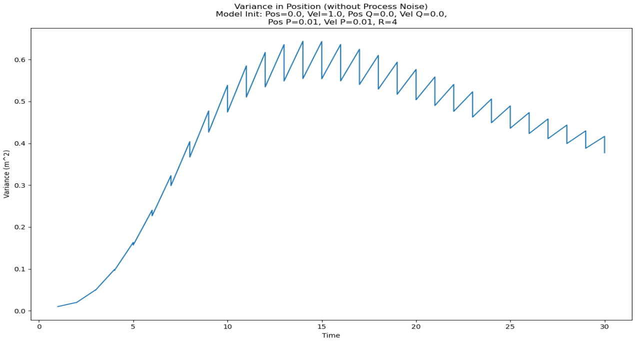

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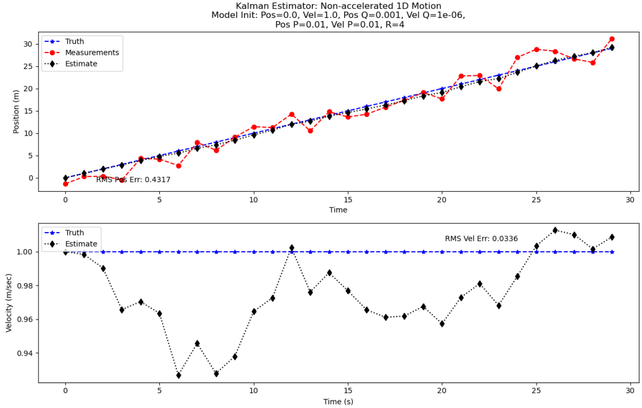

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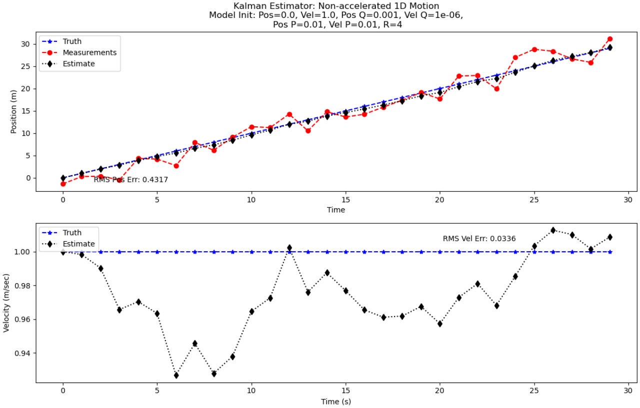

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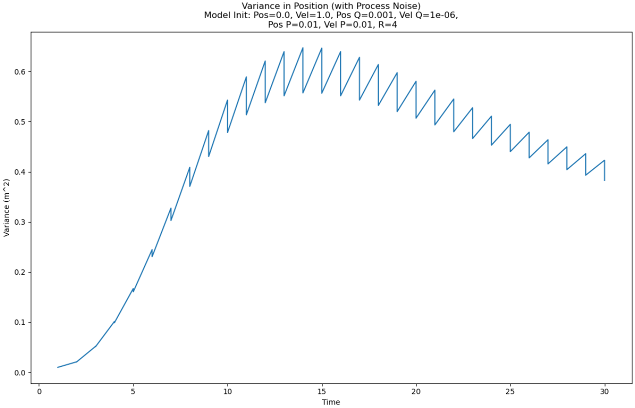

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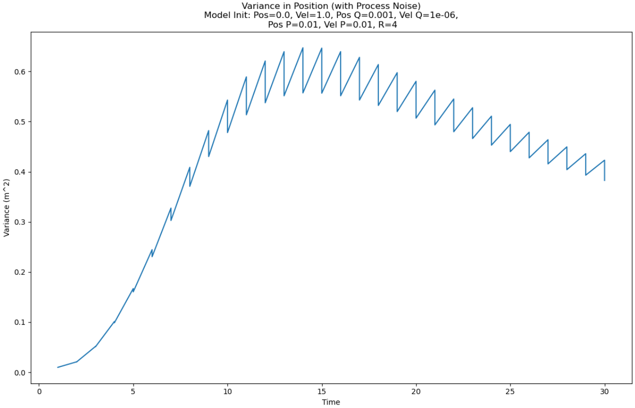

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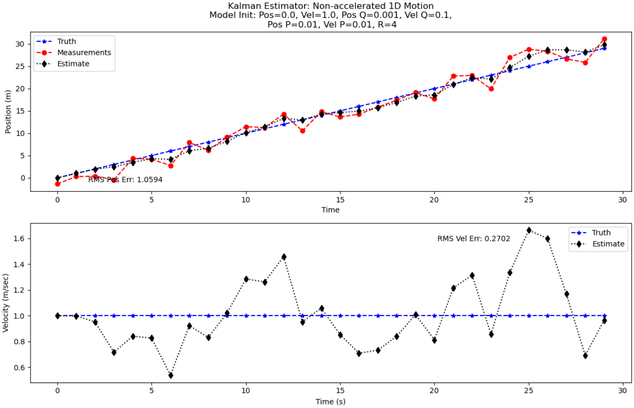

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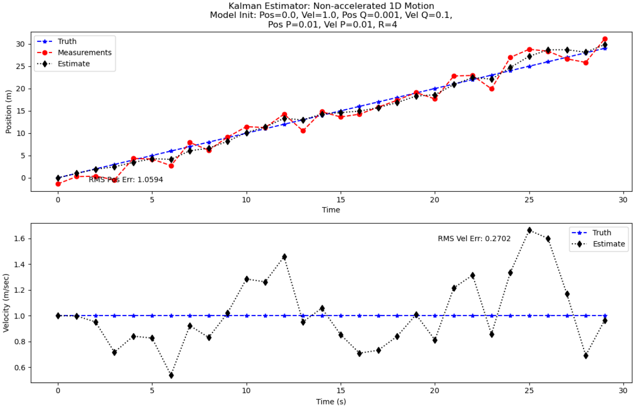

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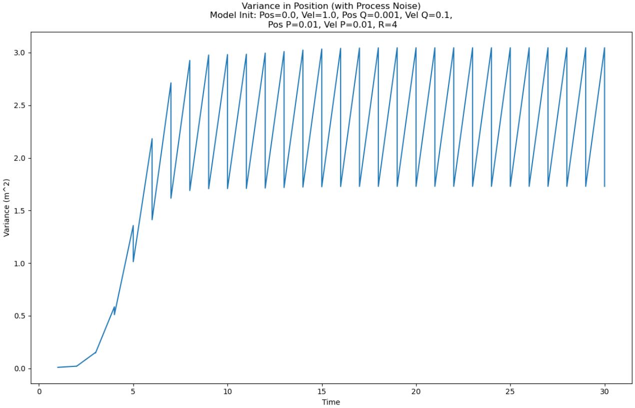

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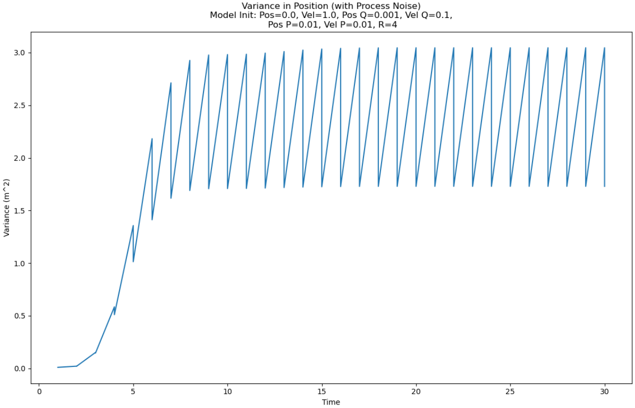

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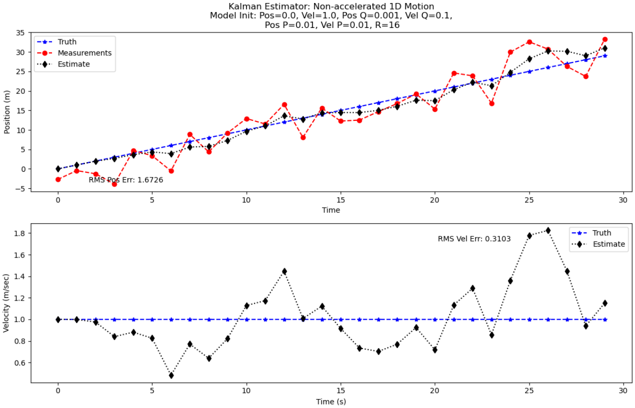

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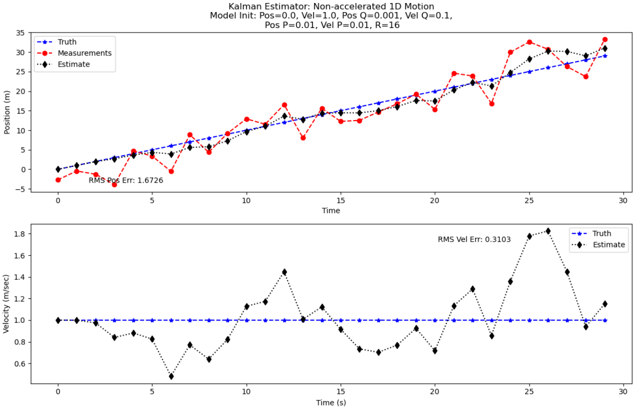

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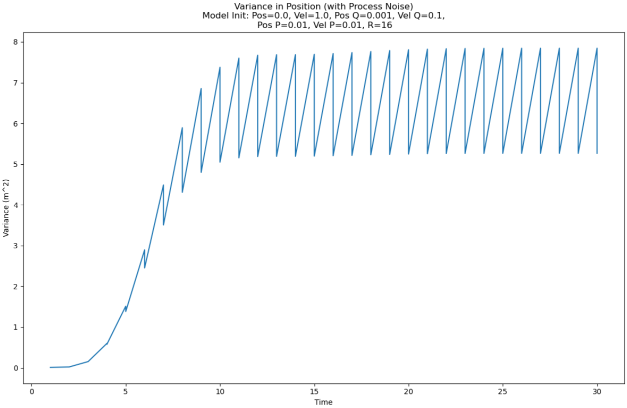

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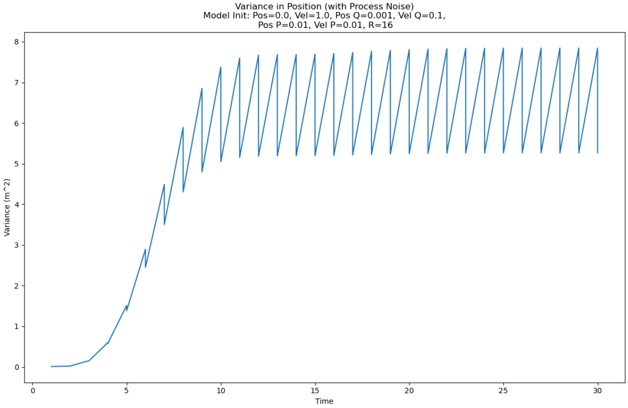

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