Kalman Filter Theory and Applications Equation Drilldown

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- Drilldown on State Dynamics and Covariance Extrapolation Equations

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where $R(k)$ is the measurement noise covariance matrix (to be defined later) governing the white measurement noise $v(k)$ in the measurement model:

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State Update Equation

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