Auralizing Tuning Systems with PureData

Stimmungssysteme und Temperamente sind durch ihren Platz in der Musikgeschichte und -praxis miteinander verbunden. Die spekulative Tradition der Musiktheorie bezieht Stimmungssysteme in ihren Diskurs ein. Die Subjektweise kann unzugänglich erscheinen, wenn die Mathematik hinter diesen Stimmungsanordnungen nicht gehört, erlebt und differenziert werden kann. Idealerweise sollten mehrere historische und aktuelle Instrumente verwendet werden, um verschiedene Stimmsysteme im Unterricht zu demonstrieren. Allerdings haben Musiktheoretiker nicht immer Zugang zu großen Instrumentensammlungen, aber oft haben sie Zugang zu Computern. Die hier vorgestellte Studie stellt ein Stimmungssystem-Framework namens *Xenharmonium* vor, das mit dem Open-Source- und frei verfügbaren Computermusikprogramm PureData (Pd) erstellt wurde.

Tuning systems and temperaments are bound together by their locations within music history and practice. The speculative tradition of music theory incorporates tuning systems in its discourse. The subject manner can seem inaccessible if the mathematics behind these tuning arrangements cannot be heard, experienced, and differentiated. Ideally, multiple historical and current instruments should be used to demonstrate different tuning systems in the classroom. However, not always do music theorists have access to large instrument collections, but many times do have access to computational devices. The study presented here introduces a tuning-system framework called *Xenharmonium* built with the open-source and freely available computer music tool PureData Pd.[1]

> "There is geometry in the humming of the strings, there is music in the spacing of the spheres."
> Pythagoras

Introduction

[Listening to *Messe de Notre Dame: II Gloria* (1364) by Guillaume Machaut (1300-1397)] sung in just intonation beckons us to experience all kinds of musics in different tuning systems.[2] Pd[3] and *Xenharmonium* aide in the creation of any tuning system including the natural overtone series, ratio based tunings like just intonation, Pythagorean tuning, meantone temperament, etc. The paper discusses logarithmically generated equal temperaments (dividing the octave into any equal parts), the Bohlen–Pierce scale (not consisting of octaves, but tritaves), how to implement these systems and their corresponding mathematical formulas in Pd utilizing basic musical acoustics knowledge. We provide demonstrations on how to listen to discrete intervals (e.g., major third, minor sixth, perfect fifth, etc.), melodies, and harmonies (e.g., major/minor and their inversions, or diminished, and augmented triads) in the different tuning systems with Pd. The paper also demonstrates how to import existing tuning schemes expressed in cents or ratios like the Slendro (5-note) Gamelan tuning system, the Pelog (7-notes) Gamelan tuning system, among others. The framework includes systems that enable listeners to hear monophonic, polyphonic, homophonic, and heterophonic textures using any one of the tuning systems individually or in combination.

Various platforms are available for demonstrating tuning systems besides Pd. For example many synthesizers can be tuned to non-equal-temperament tunings, or we could tune a piano for many hours.

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[1]Pd is available at http://msp.ucsd.edu; the Xenharmonium framework is available at https://github.com/musicus/Xenharmonium.

[2] The Hilliard Ensemble *Machaut: Messe de Notre Dame*, Hyperion Records 1989.

[3] Everything mentioned in this paper can be translated into Max8 patches with relative ease. https://cycling74.com

However, there are numerous advantages to using Pd. Many music school labs may already have Pd or Max 8 installed for integration into multimedia courses or computer/electronic music programs, and therefore students may already have experience working with Pd. Pd is free and can be installed on most computer platforms, like Windows, MacOS, Linux, etc. By using Pd, we are not bound to A = 440 Hz. Tuning systems can be built on the fly with Pd, reinforcing foundational mathematical and musical acoustics knowledge acquisitions.[4] Pd can also read Scala files (.scl) from the scale archive at the Huygens-Fokker Foundation.[5] Anything built in this paper is "hackable" and everyone can experiment or extend any of the tools provided in Pd.

Ratio Based Tuning

Contemplating tuning systems has been popular since antiquity. The Pythagorean tradition is one of the three basic ancient Greek music theory traditions.[6] The two basic creeds of its followers are that "numbers are constituent elements of reality," and that " numbers and their ratios provide the key to explaining the order of nature and the universe."[7] Furthermore, "the metaphysical significance of numbers transcends their computational utility."[8] Pythagoras's thoughts were part of many treatises from antiquity to the middle ages.[9] Any scholastic work dealing with music had to include a rendition of the *Pythagorean myth*.

The *Pythagorean Myth*, appearing in Nichomachus's *Manual of Harmonics*, describes the genesis of how Pythagoras "discovered" using the ratios to construct certain intervals.[10] The *Τετρακτύς* of the decad is a Pythagorean symbol of this procedure, constructed by arranging ten points to form a triangle.[11] By utilizing the *Τετρακτύς* we can construct the ratios of the "harmonious intervals" or "consonances."[12] Figure 1 shows how reading the *Τετρακτύς* from the bottom up yields the 3 basic ratios of 4:3 (perfect 4th), 3:2 (perfect 5th), 2:1 (perfect octave).

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[4] Knowing how to build foundational tuning principles liberates us from rapidly changing or evolving platforms, operating systems, and programming languages.

[5] http://www.huygens-fokker.org/docs/scalesdir.txt; currently limited to 12-note scales.

[6] Thomas J. Mathiesen, "Greek Music Theory," in *The Cambridge History of Western Music Theory*, ed. Thomas Christensen(New York: Cambridge University Press, 2002), 114. The other traditions were the *Harmonicists* and the *Aristoxenians.* Ibid., 117, 120.

[7] Catherine Nolan, "Music Theory and Mathematics," in *The Cambridge History of Western Music Theory*, ed. Thomas Christensen(New York: Cambridge University Press, 2002), 273.

[8] Ibid.

[9] Antiquarian writings such as the *Divisions of the Canon*, Plato's *Timaeus*, Nichmachus's *Introduction to Arithmetic* and *Manual of Harmonics*, Theon of Smyrna's *On Mathematics Useful for the Understanding of Plato*, Ptolemy's *Harmonics*, Gaudentius's *Harmonic Introduction*, and writings ranging from early medieval thinkers like Boethius (*Fundamentals of Music*) to Jacobus de Liège (*Speculum Musicae*) discuss Pythagorean philosophy on music.

[10] Mathiesen, 117.

[11] Nolan, 273. The *Τετρακτύς* can be represented by a Λ.

[12] Ibid. Nolan also connects certain meanings to the numbers, e.g.: 4 ≈ earth, water, air and fire; 1 + 2 + 3 + 4 = 10; or 10 ≈ "concomitant principal of cyclical renewal"; etc.

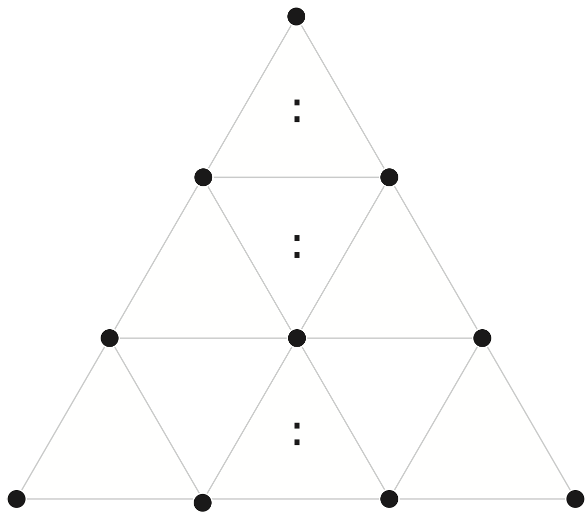
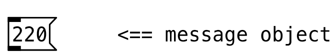


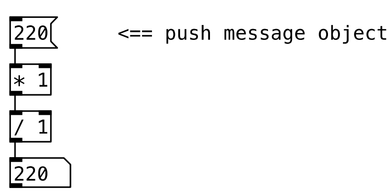
Figure 1: ***Τετρακτύς*.**

The ratios of the *Τετρακτύς* enable us to devise pitch class members of distinct pitch class collections. To maintain simplicity in the initial stages of devising a pitch class collection tuned to the ratios of the *Τετρακτύς*, we start with the blues major pentatonic scale. We can build the blues major pentatonic scale (a major scale with the omission of scale degrees 3 and 7) by only utilizing the ratios of the perfect fifth and its inversion the perfect fourth.[13] It only takes seven steps to represent the blues major pentatonic scale in ratios, as outlined by Loy[14], utilizing PureData (Pd) to visualize and auralize ratio divisions of a monochord.

1. Establish a reference pitch or frequency (here 220 Hz , or A3, but any other frequency can be used, and should be encouraged) with a message box in a Pd patch (a small Pd program).[15]



2. Multiplying the reference frequency of 220 Hz by the ratio of 1:1 creates a perfect unison, which is equivalent to $\left(\frac{3}{2}\right)^0$. Listen.



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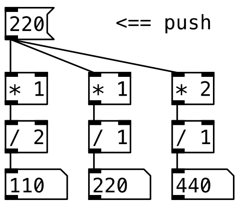
[13] Gareth Loy, *Musimathics*, 2 vols., vol. 1 (Cambridge, MA: MIT Press, 2006), 44.

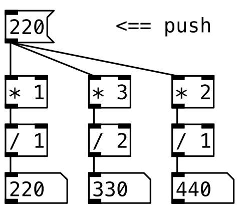
[14] Ibid., 44-45. Pd–developed by Miller Puckette at the University of California in San Diego–a freely-available open-source visual dataflow computer music programming tool.

[15] Consult basic Pd operations at http://www.pd-tutorial.com. All patches in this paper can be downloaded from https://github.com/musicus/Xenharmonium/Paper.

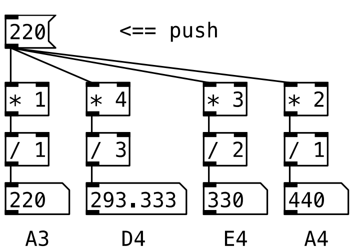
3. Applying the ratio of 2:1 to 220 Hz results in 440 Hz (A4), sounding a prefect octave higher. The inverse ratio of 2:1 is 1:2 and applied to 220 Hz results in 110 Hz (A2), sounding a perfect octave lower. [Listen](#).



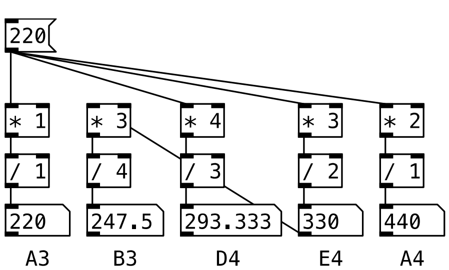
4. The perfect fifth resides between the reference frequency and the perfect octave. So if the reference frequency is 220 Hz and the perfect octave is 440 Hz, the frequency in between is 330 Hz (E4), resulting from the 3:2 ratio, or $\left(\frac{3}{2}\right)^1$. The brightness and clarity of the Pythagorean perfect fifth become immediately apparent, especially to ears accustomed to the perfect fifth on a piano. [Listen](#).



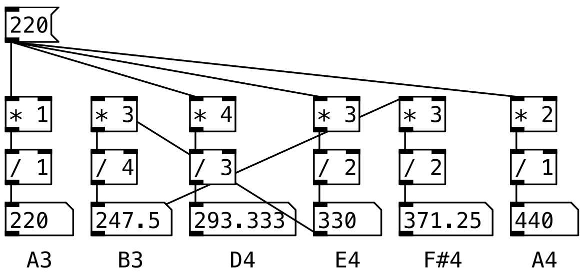
5. Inverting the ascending perfect fifth ratio from 3:2 to 2:3 creates a descending perfect fifth. The new pitch is 146.666 Hz (D3), which is a perfect fourth above 110 Hz (A2). However, the desired pitch is D4, which is an octave above. Multiplying the descending perfect fifth ratio of 2:3 by the octave ratio of 2:1 results in the ratio of 4:3, generating D4 at 293.333 Hz. [Listen](#).



6. To get the second note of the blues major pentatonic scale (B3), descend a perfect fourth–the inverse from the 4:3 ratio, or 3:4–from the perfect fifth, multiplying the reference pitch of 220Hz by (3:2 x 3:4), resulting in the ratio of 9:8. [Listen](#).



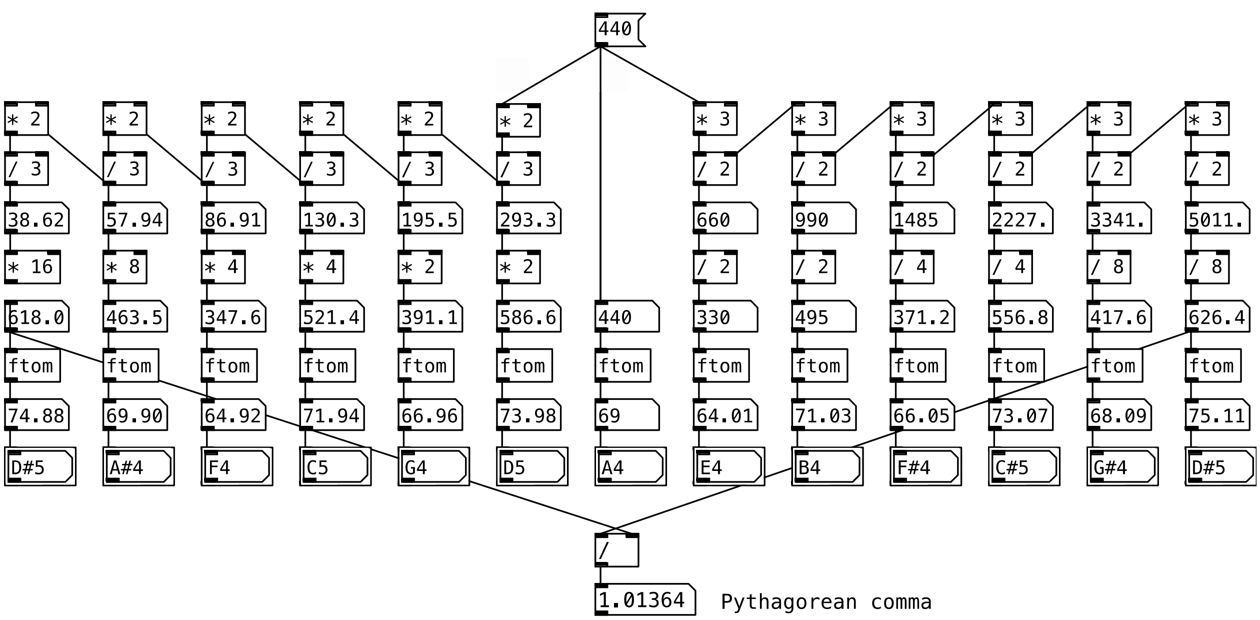
7. Ascending a perfect fifth from B3 creates the fifth note of the blues major pentatonic scale (F#4). Multiplying the reference pitch of 220Hz by the ratios (9:8 x 3:2) or the simplified ratio of 27:16 results in 371.25 Hz, representing F#4. [Listen](#).



Building the blues pentatonic scale was completed with relative ease. Building a Pythagorean dodecaphonic scale requires a few additional steps.[16]

$\left(\frac{3}{2}\right)^{-6}$	$\left(\frac{3}{2}\right)^{-5}$	$\left(\frac{3}{2}\right)^{-4}$	$\left(\frac{3}{2}\right)^{-3}$	$\left(\frac{3}{2}\right)^{-2}$	$\left(\frac{3}{2}\right)^{-1}$	$\left(\frac{3}{2}\right)^{0}$	$\left(\frac{3}{2}\right)^{1}$	$\left(\frac{3}{2}\right)^{2}$	$\left(\frac{3}{2}\right)^{3}$	$\left(\frac{3}{2}\right)^{4}$	$\left(\frac{3}{2}\right)^{5}$	$\left(\frac{3}{2}\right)^{6}$
$\frac{64}{729}$	$\frac{32}{243}$	$\frac{16}{81}$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$	$\frac{81}{16}$	$\frac{243}{32}$	$\frac{729}{64}$
Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	D#

We create a series of ascending fifths and a series of descending fifths from A. The ascending fifths are positive powers, while the descending fifths are negative powers. Recreating the approach in Pd looks almost identical, but instead we are using the inverse ratio (2:3) of the ascending fifths (3:2) to create the series of descending fifths. The Pd patch, found in the *Xenharmonium* library as *Pythagorean-Comma.pd*, additionally transposes all resulting pitches into the span of a single octave in order to stay in a comfortable hearing range.



[16] Ibid., 49-50.

When we follow through with our scheme of all perfect fifths we end up with a D# on the upper end and an Eb on the lower end. Although the two pitches are enharmonically the same in 12-ET they are not the same in Pythagorean tuning. In fact the difference between the two notes creates the *Pythagorean Comma*. Listen to the resulting series of perfect fifths, a major scale, and the chromatic scale. At the end of each example, we audibly present the Pythagorean comma.

Because the Pythagorean tuning system creates a spiral system, the necessity for other tuning systems arises. Building other ratio based tuning systems like the just intonation tuning system based on super particular ratios (Table 1) as proposed by the second century Greek scientist, mathematician, and geographer Claudius Ptolemy can be accomplished in similar ways as previously described with the Pythagorean tuning systems. Auralization of the differences and similarities leads to the recognition of nuances in perceptibility of, for example, the *syntonic comma*.

A	B	C#	D	E	F#	G#	A
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Table 1: Just intonation proposed by Ptolemy featuring super particular ratios

No actual audio synthesis happens until a unit generator such as an oscillator or a phasor connects with one of the frequency in Hz outlets. An envelope generator enhances the synthesis experience greatly. The *Xenharmonium* framework includes both unit generators with different modeled sound possibilities and adjustable envelope generators.[17] *Xenharmonium* includes objects to listen to the intervals melodically and harmonically, making detailed interval listening comparisons possible. The framework also includes objects that enable the user to use the mouse, computer keyboard, or a MIDI keyboard to play the created tuning systems. To create a more user friendly listening experience a few physically modeled instrument sounds are also included in the framework.[18] The listening examples of the pitch class collections tuned to the Pythagorean system use these instrument sounds.

Logarithmic Tuning

Equal temperament is a logarithmic tuning system. Zhu Zaiyu was the first person to mathematically solve twelve-tone equal temperament, which he describes in his *Fusion of Music and Calendar* in 1580 and the *Complete Compendium of Music and Pitch* in 1584. In Europe Simon Stevin was the first to develop 12-ET based on the twelfth root of two, which he described in *Van de spiegheling der singconst* (ca. 1605), although Marin Mersenne has been previously credited with the task.

[17] https://github.com/musicus/Xenharmonium

[18] The four available complex sounds (harp, marimba, piano, and rds) originate from Miguel Moreno's pd-mkmr library https://github.com/MikeMorenoDSP/pd-mkmr. The pd-mkmr library has many more instrument sounds available. Certainly, one can always create an FM synth instrument from scratch in Pd.

Equal temperament comes standard in Pd, because of its MIDI compatibility tools, e.g.: (1) the [mtof] object that converts MIDI pitch 69 (A4) and assigns it to the frequency of 440 Hz, or (2) the [ftom] object that converts a frequency in Hz to its closest MIDI pitch equivalent. [19]

However, when using MIDI pitch designations, the user operates within an enclosed system of predetermined Hz values that is limited to the division of the octave by 12 equal parts. There are more possibilities when any equal temperament system can be used, and fortunately this is also possible with Pd. Consider the following formula for the 12-ET system:

$$\frac{2^{1/12}}{1} = \sqrt[12]{2} \equiv 1.05946$$

The formula shows how to calculate one semitone, meaning if the reference or starting pitch started at 220 Hz (A3; again, any other frequency value can be used), one semitone up would be 220 Hz x 1.05946. Figure 2 shows a representation of the equal temperament formula in the *12-ET.pd* patch.

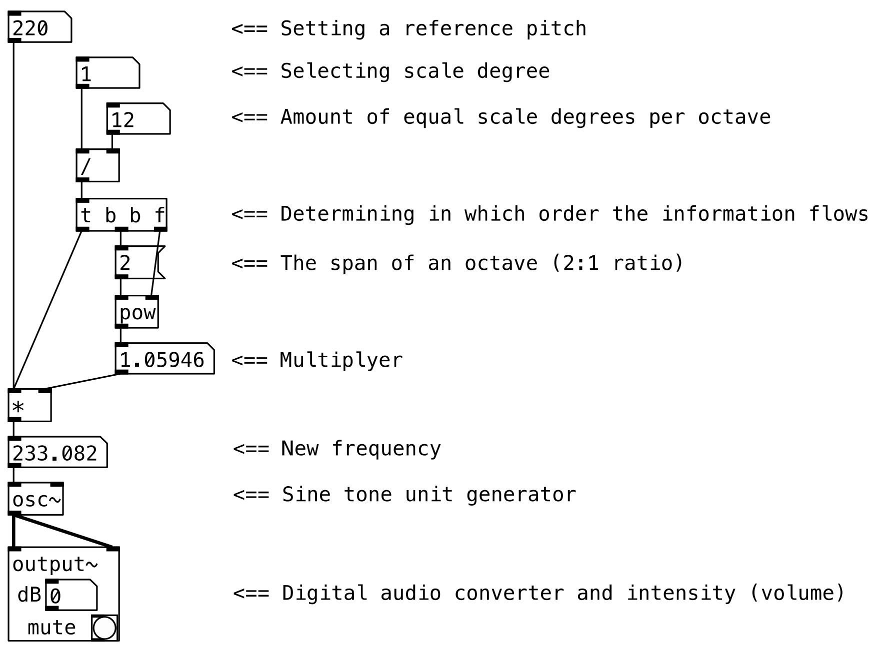


Figure 2: Sounding representation of the equal temperament formula in Pd.

Beginning at the top, set a reference frequency (220 Hz). Next, determine the desired scale degree. The scale degree count starts at 0 (base 12) for a perfect unison interval. Setting the scale degree to 1 results in an ascending minor second. Then specify how many equal steps the octave consists of (12, but any number can be entered, e.g.: entering 4 would divide the octave into 4 equal parts, resulting in a maximally even fully diminished seventh chord, etc.). The [t b b f] object looks slightly alien, but its use determines in which order and where data flows to. The span of the octave is set to 2, representing the 2:1 ratio. The multiplier becomes 1.05946, because of the ascending minor second choice, and when multiplied with the reference frequency of 220 Hz, the minor second away from A3 becomes Bb3 (or 233.082 Hz). Connecting

[19] Feeding the [mtof] object MIDI pitches as integers creates 12-ET. Providing 60, 60.5, 61, 61.5 creates 24-ET. However, any floating point number can be provided as a MIDI number, up to a ten thousandths decimal value (.0001), making Pd compatible to receive Multidimensional Polyphonic Expression (MPE) MIDI messages. The [ftom] object creates the inverse of the [mtof] object.

the result to a sine tone unit generator [osc~] and a digital audio converter [output~] auralizes the process. Using one of the complex sounds paints a much richer soundscape (listen to the patch with a reference pitch of A = 438 Hz using a physically modelled harp sound).

Playing with the patch gives the user a greater understanding of what equal temperament is, how to use it, and how it differs from, for example, Pythagorean tuning. Setting the scale degree to 7 generates a perfect fifth, but the resulting multiplier (1.49831) falls short of the perfect fifth ratio of 3:2 (or 1.5) and sounds flat. Setting the amount of scale degrees per octave to 3 results in the augmented triad, actually sounding maximally even. Or, try to set the amount of scale degrees to 8 and discover what a maximally even octatonic scale would sound like.

The same patch with slightly altered settings can be used to demonstrate the Bohlen-Pierce temperament. The American composer Curtis Roads uses the temperament in his composition *Purity* (listen), and using Pd can assist in analyzing pitch material in the composition. The temperament uses a tritave in lieu of an octave that is divided into 13 equal parts. The formula for the Bohlen-Pierce temperament shows how to create the first step multiplier of the resulting scale:

$$3^{1/13} = \sqrt[13]{3} \equiv 1.08818$$

Figure 3 shows how to implement the formula in Pd.

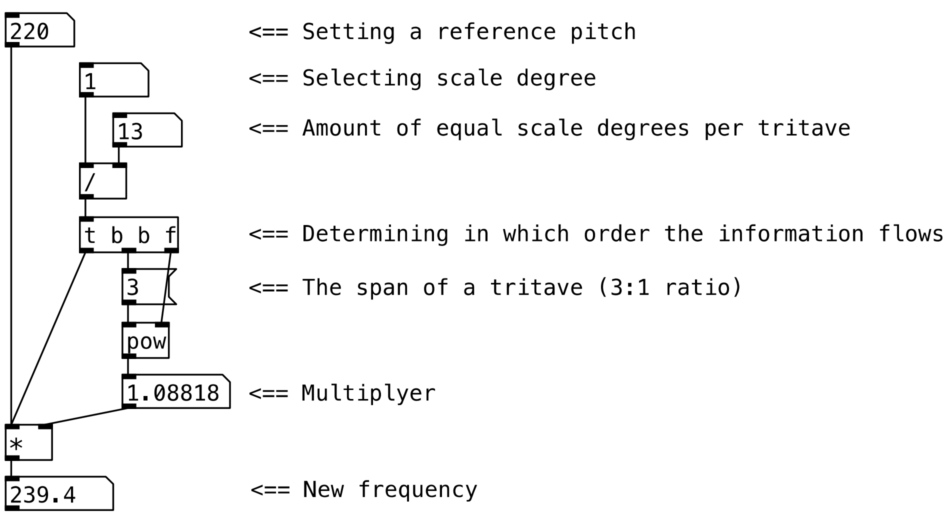


Figure 3: Bohlen-Pierce temperament in Pd.

Listen to the *Bohlen-Pierce.pd* patch.

Cent Tables

Many resources do not provide ratios of intervals or logarithmic tuning schemes, but represent tuning systems within cent tables, while some tuning system tables also may have mixed formats such as ratios and cents. One such resource is the website of the Huygens-Fokker Foundation's center for microtonal music based in the Netherlands, which has compiled a collection of tuning systems in Scala (.scl) files.[20] Pd can parse these text based files in different ways, (1) by embedding another programming language, such as JavaScript, Lua, or Python within Pd, (2) by forging a new Pd object in the C programming language, (3)

[20] https://www.huygens-fokker.org/microtonality/scales.html

within Pd itself, or (4) by combining any of the three methods previously mentioned. Figure 4 shows a Pd patch that reads five note tuning schemes from a Slendro (5-note) Gamelan tuning scheme derived by omitting two notes from a Pelog (7-note) Gamelan tuning scheme.

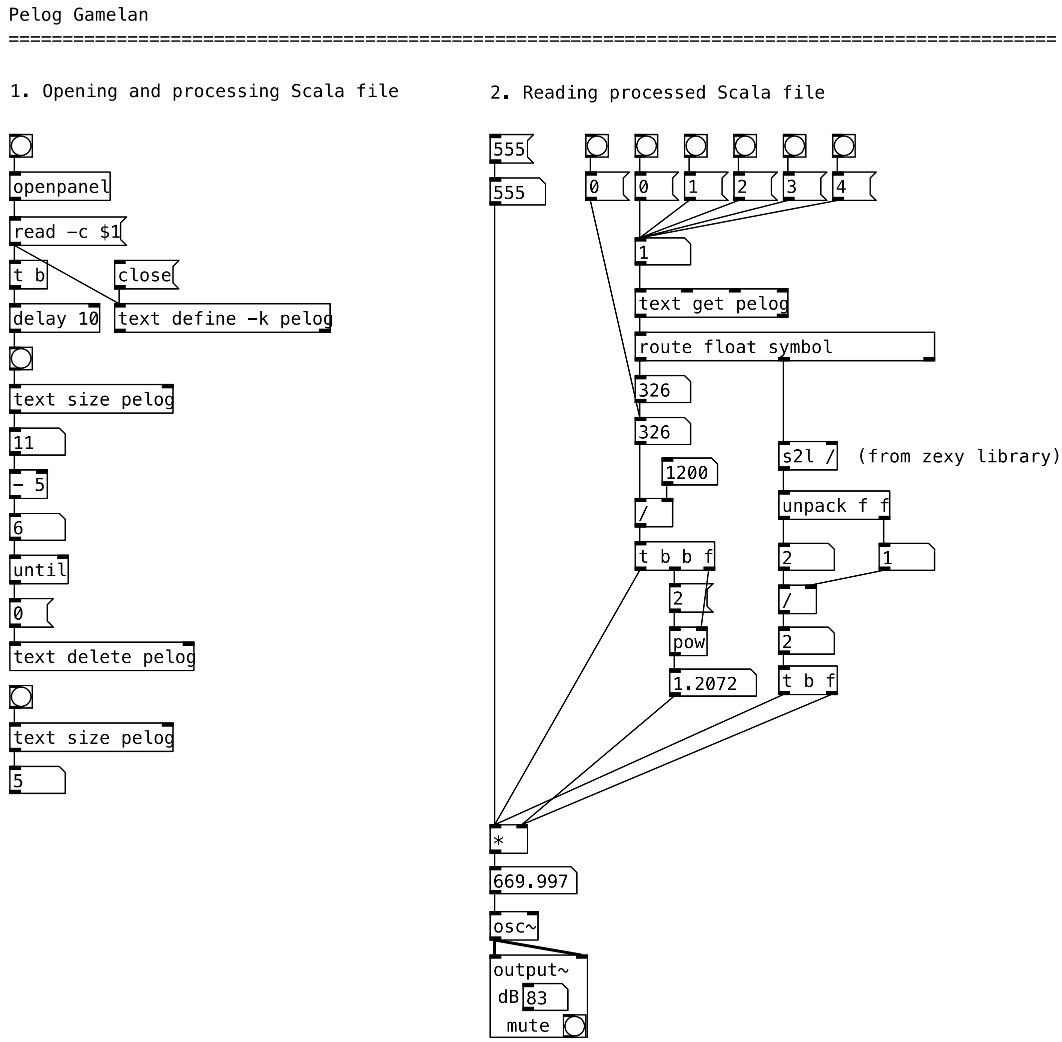


Figure 4: Parsing a Scala file that contains tuning system information with a Pd patch.

First open a Scala file (for example "pelog3.scl" which is one of four 5-note Slendro Gamelan tuning schemes included in the *Xenharmonium* framework at GitHub)[21] by pushing the top button in step 1 of the Pd patch. Once the file has loaded (as soon as the second button flashes) the tuning information is accessible within the patch. The contents of the Scala file can be viewed by clicking on the [text define -k pelog] object.

In step 2 of the Pd patch, specify a reference pitch of 555 Hz. Push the message box to initialize the 555 Hz. The six buttons next to the message box play the five notes from left to right (lower to higher pitches) including its octave replication, the sixth button. Parts of the equal temperament patch have been recycled, because the logarithm that creates equal temperament is the same logarithm that creates multipliers for the reference note, except in increments of cents or 100, meaning that each half step consists of 100 finely tuned steps. Not all values in the Scala file are cent values. The last value, the octave

[21] https://github.com/musicus/Xenharmonium/Paper

duplication at 1110 Hz, includes a 2:1 ratio. The [s2l /] object, forged in the C programming language, and part of the *zexy* library (an extension that can be downloaded by selecting "Help" > "Find externals") parses an encountered ratio and uses the result of the ratio as a multiplier. The results of the multiplier operation can be directly connect to an [osc~] object, and an [output~] to auralize the Balinese 5-note Slendro Gamelan tuning system. The *Slendro - Pelog Gamelan Tuning Scheme Import.pd* patch features a keyboard sound (listen).

Listening

After knowing how to assemble and program different types of tuning systems, it is time to listen to some intervals and chords. The listening example features the following intervals and chords:

1. Thirds (minor/major)
2. Chromatic Thirds
3. Sixths (minor/major)
4. Fifths
5. Minor/Major Triads
6. Diminished Chords
7. Diminished Chords in first inversion
8. Augmented Triads
9. Diatonic Chords
10. Major Scale
11. Minor Scale

Conclusion

Hopefully this short paper has whetted appetites to demonstrate or auralize tuning systems with Pd and *Xenharmonium*. To be able to construct and to auralize tuning systems in Pd should be an essential part of musicians' training paths in music theory. Pd is open source and free, and may already be installed at many music school computer labs. Pd is also compatible on a wide variety of computer systems, and offers tools for expansion, yet also has already many pre-built libraries. The paper includes how to create ratio based tuning systems from scratch in Pd. Also, the paper shows how to create equal tempered tuning systems in Pd. And last, methods of how to import tuning schemes from Scala files represented with ratios and cents are shown. The process of building and auralizing tuning schemes in Pd highlights the fluidity of tuning systems within time, but also across cultural boundaries.

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