Advanced ML - Q3

1. probability distribution of xi

$$P(x_i) = \begin{cases} q & \text{for } x=0 \\ (1-q) & \text{for } x=1 \end{cases}$$

The probability mass function of each xi when the xi are independent Bernoulli random variables with unknown parameter q:

$$f(x_i, q) = q^{1-x_i}(1-q)^{x_i}$$

$$\text{for } x_i = 0 \text{ or } 1 \quad \text{and } 0 < q < 1$$

The max likelihood function L(q) is:

$$L(q) = \prod_{i=1}^{n} f(x_i, q) = q^{1-x_1}(1-q)^{x_1} \times q^{1-x_2}(1-q)^{x_2} \times \cdots\cdots \times q^{1-x_n}(1-q)^{x_n}$$

$$L(q) = q^{n-\Sigma x_i}(1-q)^{\Sigma x_i}$$

2. natural log of the likelihood function

$$\log L(q) = (n - \Sigma x_i)\log(q) + (\Sigma x_i)\log(1-q)$$

Setting derivative of the log likelihood to 0

$$\frac{\partial \log L(q)}{\partial q} = \frac{(n - \Sigma x_i)}{q} - \frac{\Sigma x_i}{(1-q)} = 0$$

$$(1-q)(n - \Sigma x_i) - q\Sigma x_i = 0$$

$$n - \Sigma x_i - qn + q\Sigma x_i - q\Sigma x_i = 0$$

$$\hat{q} = \frac{n - \sum_{i=1}^{n} x_i}{n} \quad \ldots \text{ maximum likelihood estimator}$$

3. max likelihood of estimator of q

$$\hat{q} = \frac{n - \sum_{i=1}^{n} x_i}{n}$$

$$n = 30 \quad \& \quad \sum_{i=1}^{n} x_i = 17$$

Values in the formula above the maximum likelihood estimate of q

$$\hat{q} = \frac{30 - 17}{30} = \frac{13}{30} = 0.43$$

4. By definition, we need to confirm that

$$E[\hat{q}] = q$$

$$E[\hat{q}] = E\left[\frac{1}{n}\left(n - \sum_{i=1}^{n} x_i\right)\right]$$

$$= E\left[1 - \frac{\sum_{i=1}^{n} x_i}{n}\right]$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} \left(1 \cdot P(x_i = 1) + 0 \cdot P(x_i = 0)\right)$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} (1-q)$$

$$= 1 - \frac{n(1-q)}{n} = 1 - (1-q)$$

$$= q$$

Hence MLE gives an unbiased estimate