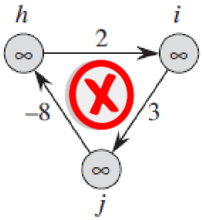
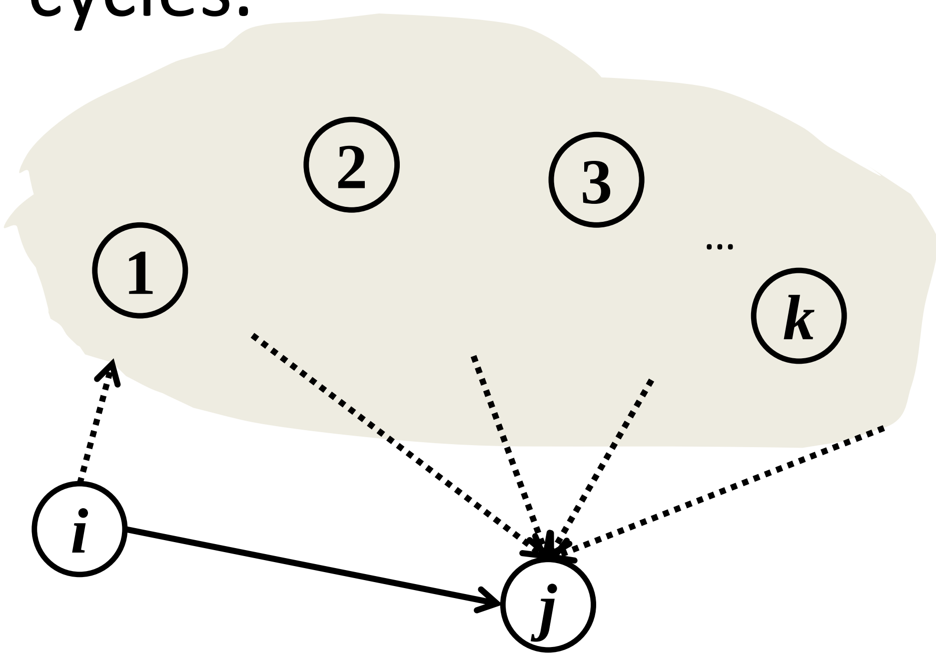
Floyd-Warshall Algorithm



- It a dynamic-programming formulation to solve the all-pairs shortest-paths problem on a directed graph, which may have negative-weight edges, but it is assumed that there are no negative-weight cycles.



$$d_{ij}^{(0)} = w_{ij}$$

$$d_{ij}^{(1)} = min(d_{ij}^{(0)}, d_{i1}^{(0)} + d_{1j}^{(0)})$$

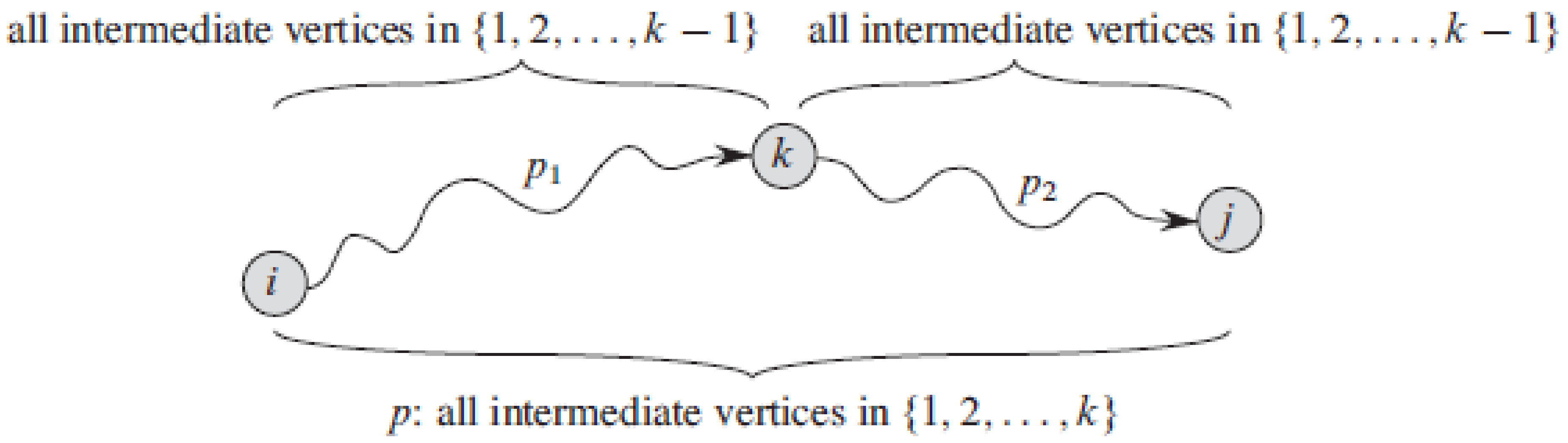
$$d_{ij}^{(2)} = min(d_{ij}^{(1)}, d_{i2}^{(1)} + d_{2j}^{(1)})$$

$$d_{ij}^{(3)} = min(d_{ij}^{(2)}, d_{i3}^{(2)} + d_{3j}^{(2)})$$

...

$$d_{ij}^{(k)} = min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

Contd...



$$d_{ij}^{(k)} = min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

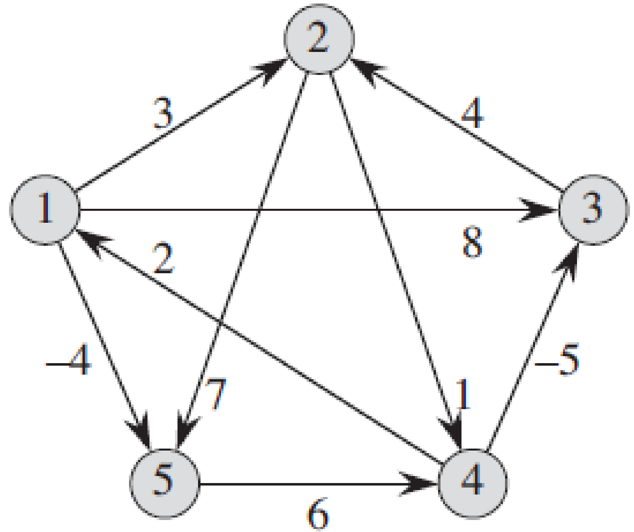
Contd...

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

Example



$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0\,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1\,. \end{cases}$$

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty\,, \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty\,. \end{cases}$$

$D^{(0)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	∞	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	∞	∞
V_4	2	∞	-5	0	∞
V_5	∞	∞	∞	6	0

$\pi^{(0)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	NIL	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	NIL	NIL
V_4	4	NIL	4	NIL	NIL
V_5	NIL	NIL	NIL	5	NIL

Contd…

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

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$D^{(0)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	∞	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	∞	∞
V_4	2	∞	-5	0	∞
V_5	∞	∞	∞	6	0

$\pi^{(0)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	NIL	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	NIL	NIL
V_4	4	NIL	4	NIL	NIL
V_5	NIL	NIL	NIL	5	NIL

$D^{(1)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	∞	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	∞	∞
V_4	2	5	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(1)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	NIL	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	NIL	NIL
V_4	4	1	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

Contd...

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0\,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1\,. \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\,, \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\,. \end{cases}$$

$D^{(1)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	∞	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	∞	∞
V_4	2	5	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(1)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	NIL	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	NIL	NIL
V_4	4	1	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

$D^{(2)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	4	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	5	11
V_4	2	5	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(2)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	2	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	2	2
V_4	4	1	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

Contd...

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0\,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1\,. \end{cases}$$

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$D^{(2)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	4	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	5	11
V_4	2	5	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(2)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	2	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	2	2
V_4	4	1	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

$D^{(3)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	4	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	5	11
V_4	2	-1	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(3)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	2	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	2	2
V_4	4	3	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

Contd…

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

$D^{(3)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	8	4	-4
V_2	∞	0	∞	1	7
V_3	∞	4	0	5	11
V_4	2	-1	-5	0	-2
V_5	∞	∞	∞	6	0

$\pi^{(3)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	1	2	1
V_2	NIL	NIL	NIL	2	2
V_3	NIL	3	NIL	2	2
V_4	4	3	4	NIL	1
V_5	NIL	NIL	NIL	5	NIL

$D^{(4)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	-1	4	-4
V_2	3	0	-4	1	-1
V_3	7	4	0	5	3
V_4	2	-1	-5	0	-2
V_5	8	5	1	6	0

$\pi^{(4)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	4	2	1
V_2	4	NIL	4	2	1
V_3	4	3	NIL	2	1
V_4	4	3	4	NIL	1
V_5	4	3	4	5	NIL

Contd...

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1. \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}. \end{cases}$$

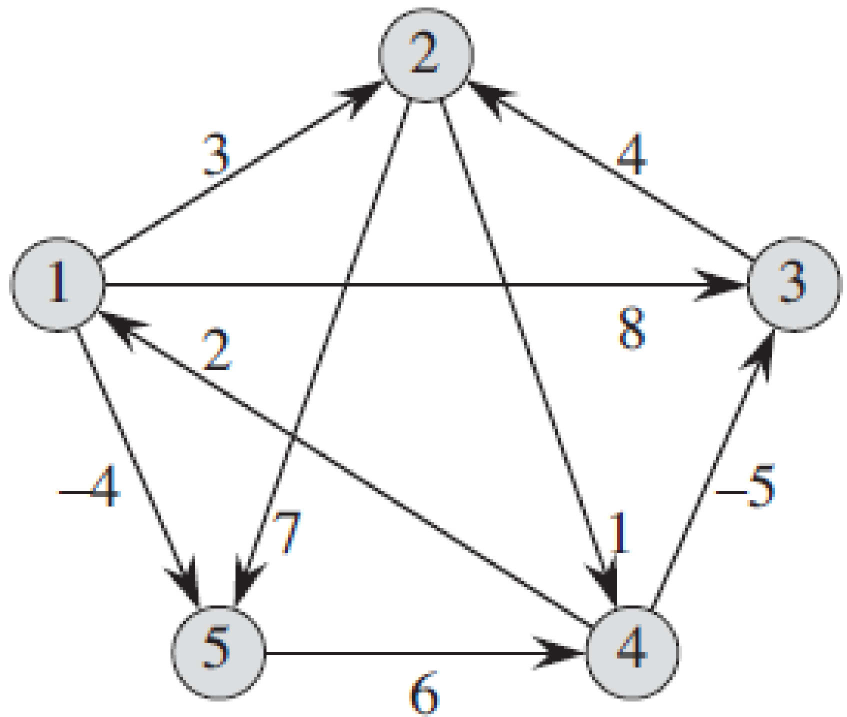
$D^{(4)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	3	-1	4	-4
V_2	3	0	-4	1	-1
V_3	7	4	0	5	3
V_4	2	-1	-5	0	-2
V_5	8	5	1	6	0

$\pi^{(4)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	1	4	2	1
V_2	4	NIL	4	2	1
V_3	4	3	NIL	2	1
V_4	4	3	4	NIL	1
V_5	4	3	4	5	NIL

$D^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	1	-3	2	-4
V_2	3	0	-4	1	-1
V_3	7	4	0	5	3
V_4	2	-1	-5	0	-2
V_5	8	5	1	6	0

$\pi^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	3	4	5	1
V_2	4	NIL	4	2	1
V_3	4	3	NIL	2	1
V_4	4	3	4	NIL	1
V_5	4	3	4	5	NIL

Contd...



$D^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	1	-3	2	-4
V_2	3	0	-4	1	-1
V_3	7	4	0	5	3
V_4	2	-1	-5	0	-2
V_5	8	5	1	6	0

$\pi^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	3	4	5	1
V_2	4	NIL	4	2	1
V_3	4	3	NIL	2	1
V_4	4	3	4	NIL	1
V_5	4	3	4	5	NIL

Implementation

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

n= W.rows
$D^0 = W$
π^0 is a matrix with nil in every entry
for i=1 to n **do**
    for j = 1 to n **do**
        if $i \neq j$ and $D^0_{i,j} < \infty$ **then**
            $\pi^0_{i,j} = i$
        end if
    end for
end for
for k=1 to n **do**
    let D^k be a new $n \times n$ matrix.
    let π^k be a new $n \times n$ matrix
    for i=1 to n **do**
        for j = 1 to n **do**
            if $d^{k-1}_{ij} \leq d^{k-1}_{i,k} + d^{k-1}_{k,j}$ **then**
                $d^k_{i,j} = d^{k-1}_{i,j}$
                $\pi^k_{i,j} = \pi^{k-1}_{i,j}$
            else
                $d^k_{i,j} = d^{k-1}_{i,k} + d^{k-1}_{k,j}$
                $\pi^k_{i,j} = \pi^{k-1}_{k,j}$
            end if
        end for
    end for
end for

Print-All-Pair-Shortest-Path

```
PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)
if i == j
    print i
elseif π_ij == NIL
    print "no path from" i "to" j "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_ij)
    print j
```

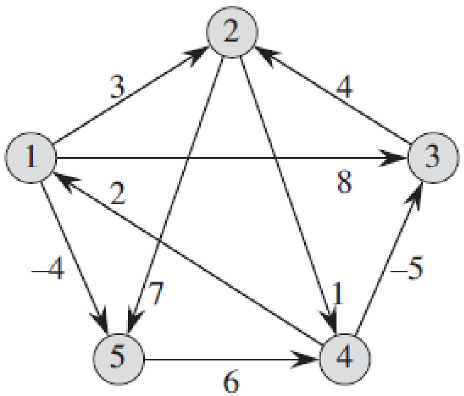
Example:

? 1->2

? 1->3

? 1->4

? 1->5




```
PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)
if i == j
    print i
elseif π_ij == NIL
    print "no path from" i "to" j "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_ij)
    print j
```

Example:

? 1->2 : 1, 5, 4, 3, 2 : (-4)+(6)+(-5)+(4) = 1

? 1->3 : 1, 5, 4, 3 : (-4)+(6)+(-5) = -3

? 1->4 : 1, 5, 4 : (-4)+(6) = 2

? 1->5 : 1, 5 : (-4) = -4

$D^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	0	1	-3	2	-4
V_2	3	0	-4	1	-1
V_3	7	4	0	5	3
V_4	2	-1	-5	0	-2
V_5	8	5	1	6	0

$\pi^{(5)}$	V_1	V_2	V_3	V_4	V_5
V_1	NIL	3	4	5	1
V_2	4	NIL	4	2	1
V_3	4	3	NIL	2	1
V_4	4	3	4	NIL	1
V_5	4	3	4	5	NIL

Thank You