

Negative Binomial PMF:

γ is counting y failures given r successes. Total # of trials

$$n = y + r$$

$$f(y; p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

where p is the probability of success in each trial

Lemma:

$$\binom{y+r-1}{r-1} = \binom{y+r-2}{y}$$

Proof:

$$\begin{aligned}\binom{y+r-1}{r-1} &= \frac{(y+r-1)!}{(r-1)!((y+r-1)-r+1)!} = \frac{(y+r-1)!}{(r-2)!y!} \\ &= \frac{(y+r-1)!}{y!(r-1)!} = \frac{(y+r-2)!}{y!(y+r-2-y)!} \\ &= \binom{y+r-2}{y}\end{aligned}$$

□

Therefore

$$f(y; p, r) = \binom{y+r-1}{y} p^r (1-p)^y$$

$$\text{Mean: } \frac{r(1-p)}{p} \approx \mu, \quad \text{Variance: } \frac{r(1-p)}{p^2} = \text{Var}(\gamma)$$

$$\Rightarrow \mu p = r - rp$$

$$\Rightarrow p(r+\mu) = r$$

$$\Rightarrow p = r/r+\mu$$

$$f(y; \mu, r) = \binom{y+r-1}{y} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

$$\Gamma(n) = (n-1)!$$

if $n \in \mathbb{N}, n \geq 1$
if n is a real
the $\Gamma(\cdot)$

$$= \frac{\Gamma(y+r)}{y! \Gamma(r)} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

is well defined

Define $r := \frac{1}{\alpha}$,

$$f(y; \mu, r) = \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha\mu}{1+\alpha\mu} \right)^y$$

\Rightarrow The single data log-likelihood is

$$\begin{aligned} l_i^B &= y_i \log \left(\frac{\alpha\mu}{1+\alpha\mu} \right) - \frac{1}{\alpha} \log(1+\alpha\mu_i) + \log \Gamma(y_i + \frac{1}{\alpha}) \\ &\quad - \log \Gamma(y_i + 1) - \log \Gamma(\frac{1}{\alpha}) \end{aligned} \quad \dots \quad (1)$$

Zero-truncated NB.

The probability of a zero count is

$$f(0; \mu, r) = \frac{\Gamma(\frac{1}{\alpha})}{\Gamma(1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} = \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}}$$

The zero-truncated density is

$$\begin{aligned} f(y|y>0, \mu, r) &= \frac{f(y; \mu, r)}{1 - f(0; \mu, r)} \\ &= \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha\mu}{1+\alpha\mu} \right)^y \\ &\quad \frac{1}{1 - \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}}} \end{aligned}$$

This gives us the single data log-likelihood;

$$l_i^{ZT} = l_i^B - \log \left\{ 1 - \left(\frac{1}{1+\alpha\mu_i} \right)^{\frac{1}{\alpha}} \right\} \quad \dots \quad (2)$$

Where (2) is the ZTB single data log-likelihood as given in (1)

Zero Inflated Negative Binomial (Hurdle model in Tutz (2012))

We use a similar parameterization as in Hlood et al. (2017) SA I. written by a

$$g(y; \mu, r) = \begin{cases} 1-q & y=0 \\ q f(y|y>0, \mu, r) & \text{otherwise} \end{cases}$$

where q is the probability of a positive counts

Similar to Hlood et al. (2017), we adopt the unconstrained parameterization

$$\sigma = \log \mu, \eta = \log \hat{p} - \log(1-q)$$

This gives us the single-data log-likelihood (we drop the data index i)

$$l^{z^1} = \begin{cases} \log(1-q) & y=0 \\ \log q + \log f(y|y>0, \mu, r) & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{HB} - \log \left\{ 1 - (1+e^{-\eta})^{-\frac{1}{r}} \right\} & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

where $L^{AB} = L^{AB} - \log \{1 - ((1 + \alpha e^y)^{-\frac{1}{\alpha}})\}$ is the Z^{AB} log-likelihood.
and $L^{AB} = y \log \left(\frac{\alpha e^y}{1 + \alpha e^y} \right) - \frac{1}{\alpha} \log (1 + \alpha e^y) + \log \Gamma(y + \frac{1}{\alpha})$
 $- \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

$$\Rightarrow L^{Z^A} = y \log \left(\frac{\alpha e^y}{1 + \alpha e^y} \right) + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$$
 $+ \log \left\{ \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}} \right\} + \log \left\{ \frac{1}{1 - \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}}} \right\}$

looking at the last two terms basically is

$$\log \left\{ \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}} - 1 \right\} = - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$$
 $\Rightarrow L^{Z^A} = y \log \left\{ \frac{\alpha e^y}{1 + \alpha e^y} \right\} + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1)$
 $- \log \Gamma(\frac{1}{\alpha}) - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$
 $= y \log \alpha + y \gamma - y \log (1 + \alpha e^y) - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$
 $+ \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

Here as well, some care is required to evaluate L^{Z^A} without unnecessary overflow, since it is easy for $1 - e^{-\frac{m}{n}}$ and $(1 + \alpha e^y)^{\frac{1}{\alpha}} - 1$ to evaluate to zero in finite precision arithmetic. Hence the limiting results $\log(1 - e^{-\frac{m}{n}}) \rightarrow \log(e^m - e^{\frac{m}{n}} + \frac{m}{6}) \rightarrow y$ as $m \rightarrow \infty$.
Similar as $y \rightarrow -\infty$, $\log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \} \rightarrow \infty$ and as $\alpha \rightarrow 0$
 $\log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \} \rightarrow \log(e^y - 1) \xrightarrow{\text{provided } y > -\infty} e^y$

Implemented
taking α as
fixed.

Proof:

$$\textcircled{1} \quad \log(1 - e^{-\frac{m}{n}}) \rightarrow \log(e^m - e^{\frac{m}{n}} + \frac{m}{6}) \rightarrow y \quad n \rightarrow \infty$$

as $m \rightarrow \infty$, $e^{-\frac{m}{n}} \rightarrow 0$,
we know that $e^x \rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ as $x \rightarrow 0$

 $\Rightarrow e^{-\frac{m}{n}} \rightarrow 1 - e^{-\frac{m}{n}} + e^{\frac{m}{n}} - \frac{e^{\frac{m}{n}}}{6}$

$$= 0 \text{ as } \eta \rightarrow -\infty, \log(1 - e^{-e^\eta})$$

$$\rightarrow \log\left(1 - 1 + e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6}\right)$$

$$\text{as } \eta \rightarrow -\infty, e^{2\eta} \text{ and } e^{3\eta} \rightarrow 0 \text{ faster than } e^\eta$$

$$= \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$= 0 \quad \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$\rightarrow \log e^\eta \rightarrow \eta$$

② $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow r \text{ as } \alpha \rightarrow -\infty$

as $r \rightarrow -\infty$, $e^r \rightarrow 0 \Rightarrow \alpha e^r \rightarrow 0$ (fixed α)
 We know that $(1+x)^{\frac{1}{x}} \rightarrow 1 + \frac{1}{x}x$ as $x \rightarrow 0$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow 1 + \frac{1}{\alpha}(\alpha e^r) = 1 + e^r$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} - \log\{(1 + e^r) - 1\}$$

$$= \log(e^r) = r$$

as $r \rightarrow -\infty$

③ $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow e^r \text{ as } \alpha \rightarrow 0$ for fixed r

remember that $(1 + \alpha e^r)^{\frac{1}{\alpha}} = \left(1 + \frac{e^r}{r}\right)^r$,
 so as $\alpha \rightarrow 0$, $r \rightarrow \infty$

$$\text{and } \lim_{r \rightarrow \infty} \left(1 + \frac{e^r}{r}\right)^r = e^{e^r} \text{ for } r \in \mathbb{R}$$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow e^{e^r} \text{ as } \alpha \rightarrow 0$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow \log\{e^{e^r} - 1\} \text{ as } \alpha \rightarrow 0$$

also, for very large r , $e^{e^r} - 1$ is dominated by e^{e^r}

$$\Rightarrow \log\{e^{e^r} - 1\} \rightarrow \log e^{e^r} = e^r \quad \square$$

Implementation (Following Hood et al. (2017))

According to the framework proposed in Hood et al. (2017), we need the deviance and its derivatives to implement an extend GAM model (see section 3.3.1 of that paper);

Differentiating w.r.t. y_i , $\frac{\partial D_i}{\partial y_i}$, $\frac{\partial^2 D_i}{\partial y_i^2}$, $\frac{\partial^3 D_i}{\partial y_i^3}$, $\frac{\partial^4 D_i}{\partial y_i^4}$ (For \hat{p})

instead of m_i , $\frac{\partial^4 h_i}{\partial y_i^4}$, $\frac{\partial^5 D_i}{\partial m_i \partial y_i}$, $\frac{\partial^6 D_i}{\partial m_i^2 \partial y_i}$, $\frac{\partial^7 D_i}{\partial m_i^3 \partial y_i}$, $\frac{\partial^8 D_i}{\partial m_i^4 \partial y_i}$, $\frac{\partial^9 D_i}{\partial m_i^5 \partial y_i}$
as this is done in SAI. (For \hat{p} via full Newton).

As for the zip model, we first define the deviance as -2l for model estimation.

w.r.t. y for $y > 0$

$$\textcircled{1} \quad \frac{dL}{dy} = \frac{y}{e^y - 1}, \quad \textcircled{2} \quad \frac{d^2 L}{dy^2} = (1-e^y) \frac{dy}{dy} - \left(\frac{dy}{dy} \right)^2$$

exactly as for zip.

$$\textcircled{3} \quad \frac{d^3 L}{dy^3} = -e^y \frac{dy}{dy} + (1-e^y)^2 \frac{dy}{dy} - 3(1-e^y) \left(\frac{dy}{dy} \right)^2 + 2 \left(\frac{dy}{dy} \right)^3 \quad \text{and}$$

$$\textcircled{4} \quad \frac{d^4 L}{dy^4} = (3e^y - 4)e^y \frac{dy}{dy} + 4e^y \left(\frac{dy}{dy} \right)^2 + (1-e^y)^3 \frac{dy}{dy} - 7(1-e^y) \left(\frac{dy}{dy} \right)^2 + 12(1-e^y) \left(\frac{dy}{dy} \right)^3 - 6 \left(\frac{dy}{dy} \right)^4$$

See SAI of Hood et al. (2016) for dealing with these derivatives as $y \rightarrow \pm\infty$

w.r.t. γ for $y > 0$

$$\begin{aligned} \textcircled{5} \quad \frac{dy}{d\gamma} &= \frac{dL}{d\gamma} = \frac{d}{d\gamma} \left(y - y \log(1+\alpha e^\gamma) - \log \left\{ (1+\alpha e^\gamma)^{\frac{1}{\alpha}} - 1 \right\} \right) \\ &= y - \frac{y}{1+\alpha e^\gamma} - \frac{1-\alpha e^\gamma}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}} \cdot \frac{1}{\alpha} (1+\alpha e^\gamma)^{\frac{1}{\alpha}-1} \\ &= y - y \frac{e^\gamma}{1+\alpha e^\gamma} - \frac{e^\gamma (1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}} \end{aligned}$$

$$= y - \frac{y \alpha e^x}{1 + \alpha e^x} - \left(\frac{e^x}{1 + \alpha e^x} \right) \left(\frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1} \right)$$

Define $\beta := \frac{e^x}{1 + \alpha e^x}$, $\tau := \frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1}$

$$\Rightarrow \frac{dy}{dx} = y - y \alpha \beta - \beta \tau$$

$$\textcircled{2} \quad \frac{d^2 y}{dx^2} = \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

Proof: $\frac{d}{dx} \{ y - y \alpha \beta - \beta \tau \} = -y \alpha \frac{d\beta}{dx} - \frac{d\beta}{dx} \tau - \beta \frac{d\tau}{dx}$

But $\frac{d\beta}{dx} = \frac{(1 + \alpha e^x) e^x - e^x \cdot \alpha e^x}{(1 + \alpha e^x)^2}$

$$= \frac{e^x}{1 + \alpha e^x} - \frac{\alpha e^{2x}}{(1 + \alpha e^x)^2} = \beta - \alpha \beta^2$$

and $\frac{d\tau}{dx} = \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}$

$$= \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}{\left((1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right)^2}$$

$$= \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]} - \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}$$

$$= \beta \tau - \beta \tau^2$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y \alpha (\beta - \alpha \beta^2) - \beta \tau + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= y \alpha^2 \beta^2 - y \alpha \beta + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

□

$$\textcircled{3} \quad \frac{d^3L}{dt^3} = p \left\{ -8yx^3\beta^2 + 3yx^2\beta - 2x^2\beta^2z - yx + 3\alpha\beta z - 3\alpha\beta^2z^2 + 3\alpha\beta^2\tau - \tau + 3\beta\tau^2 - 3\beta\tau - 2\beta^2z^3 + 3\beta^2\tau^2 - \beta^2z^2y \right\}$$

Proof. Using a numerical linear algebra package, set ∞ . □

$$\textcircled{4} \quad \frac{d^4L}{dt^4} = p \left\{ 6yx^4\beta^3 - 12yx^3\beta^2 + 6x^2\beta^3z + 7yx^2\beta - 12x^2\beta^2z + 11x^2\beta^3z^2 - 11x^2\beta^3\tau - yx + 7\alpha\beta z - 18\alpha\beta^2z^2 + 18\alpha\beta^2\tau + 12\alpha\beta^3z^3 - 18\alpha\beta^3z^2 + 6\alpha\beta^3\tau - \tau + 7\beta\tau^2 - 7\beta\tau - 12\beta^2z^3 + 18\beta^2\tau^2 - 6\beta^2z + 6\beta^3\tau - 12\beta^3z^3 + 7\beta^3\tau^2 - \beta^3\tau \right\}$$

Proof. --



As with L^{22} , some care is required to ensure that the derivatives evaluate accurately without overflow even as wide a range of τ and κ as possible. As $\tau \rightarrow \infty$, $\beta = \frac{1}{e^{-\tau} + \kappa} \rightarrow \frac{1}{\kappa}$ for fixed κ , while

$$\tau = \frac{1}{1 - (1 + \kappa e^\tau)^{-1/\kappa}} \rightarrow 1, \quad \text{since } 1 + \kappa e^\tau \rightarrow \infty \Rightarrow (1 + \kappa e^\tau)^{-1/\kappa} \rightarrow 0$$

whereas as $\tau \rightarrow -\infty$, $\beta \rightarrow \frac{1}{e^{-\tau}} \rightarrow 0$ for fixed κ , while $e^\tau \rightarrow 0$
 $\Rightarrow (1 + \kappa e^\tau)^{-1/\kappa} \rightarrow 1 + \frac{1}{\kappa e^\tau} = 1 + e^\tau \rightarrow 0$ $\tau = \frac{(1 + \kappa e^\tau)^{1/\kappa}}{(1 + \kappa e^\tau)^{1/\kappa} - 1}$
first order Taylor expansion about $\tau = \kappa e^\tau = 0$

$$\rightarrow \frac{1 + e^\tau}{1 + e^\tau - 1} = \frac{1 + e^\tau}{e^\tau}$$

$$= e^{-\tau} + 1 \rightarrow e^{-\tau}$$

* How do we deal with products: $p_1, p_2, p_3, p_4, \dots$ etc. what does it converge to?

Since as $\tau \rightarrow -\infty$, $\beta \rightarrow e^\tau$ and $\tau \rightarrow e^{-\tau}$
 $\Rightarrow \beta \tau \rightarrow 1 \Rightarrow (\beta \tau)^\alpha \rightarrow 1 + \alpha e^\tau$
 (alternatively),

$$\lim_{\tau \rightarrow -\infty} \tau p = \lim_{\tau \rightarrow -\infty} \frac{\beta}{1 - (1 + \kappa e^\tau)^{-1/\kappa}}$$

$$\begin{aligned}
 (1^{\text{st}} \text{ Hopital}) &= \lim_{r \rightarrow -\infty} \frac{\beta - \alpha \beta^2}{\gamma_K K R^r (1 + K R^r)^{-\gamma_K - 1}} \\
 &= \lim_{r \rightarrow -\infty} \frac{\beta (1 - \alpha \beta)}{e^r (1 + \alpha e^r)^{-\gamma_K - 1}} \\
 \Rightarrow \lim_{r \rightarrow -\infty} &\frac{e^r (1 - \alpha \beta)}{(1 + \alpha e^r)^{-\gamma_K - 1}} \\
 &= \lim_{r \rightarrow -\infty} \frac{(1 - \alpha \beta)}{(1 + \alpha e^r)^{-\gamma_K}} = 1
 \end{aligned}$$

also ; $\beta^a t^b \rightarrow 0$ if $a > b$ as $r \rightarrow -\infty$
as $r \rightarrow -\infty$, $\beta \rightarrow \alpha^{-1}$ and $t \rightarrow 1 \Rightarrow \beta t \rightarrow \alpha^{-1}$

with these we get

$$\begin{aligned}
 \frac{\partial V}{\partial r} &\rightarrow y - 1 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^2 V}{\partial r^2} \rightarrow y - y - \frac{1}{\alpha} = -\frac{1}{\alpha} \text{ as } r \rightarrow -\infty \\
 \frac{\partial^2 V}{\partial r^2} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^3 V}{\partial r^3} \rightarrow y - y + \frac{3y}{\alpha^2} - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^2} \\
 &= \frac{1}{\alpha} - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^2} = 0 \\
 \frac{\partial^3 V}{\partial r^3} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^4 V}{\partial r^4} \rightarrow -2y + 3y - 2\frac{1}{\alpha} - y + \frac{3}{\alpha} \\
 &\quad - \frac{3}{\alpha^2} + \frac{3}{\alpha^2} - \frac{1}{\alpha} + \frac{3}{\alpha^2} \\
 &\quad - \frac{3}{\alpha^2} - \frac{2}{\alpha^3} \\
 &\quad + \frac{3}{\alpha^3} - \frac{1}{\alpha^3} \\
 &= \frac{1}{\alpha} - \frac{1}{\alpha} + \frac{3}{\alpha^2} - \frac{3}{\alpha^2} \\
 &\quad + 0 = 0 \text{ as } r \rightarrow -\infty
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^4 V}{\partial r^4} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^4 V}{\partial r^4} \rightarrow -2y + 3y - 2\frac{1}{\alpha} - y + \frac{3}{\alpha} \\
 &\quad - \frac{3}{\alpha^2} + \frac{3}{\alpha^2} - \frac{1}{\alpha} + \frac{3}{\alpha^2} \\
 &\quad - \frac{3}{\alpha^2} - \frac{2}{\alpha^3} + \frac{3}{\alpha^3} - \frac{1}{\alpha^3} \\
 &= 0 \text{ as } r \rightarrow -\infty
 \end{aligned}$$

For extended GAMs in which η is a Fn of γ and extra parameters θ e.g;

$$\eta = \theta_1 + e^{\theta_2} \gamma \quad (\text{as for the zip model in } \text{Hood et al. (2016)})$$

$$\frac{d\eta}{d\theta_1} = 1, \quad \frac{d\eta}{d\theta_2} = e^{\theta_2} \gamma, \quad \frac{d^2\eta}{d\theta_1 d\theta_2} = 0$$

$$\frac{d^2\eta}{d\theta_1^2} = 0, \quad \frac{d^2\eta}{d\theta_2^2} = e^{2\theta_2} \gamma$$

In this setting

$$\begin{aligned} \frac{dL}{d\gamma} &= \frac{\partial L}{\partial \gamma} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial \eta}{\partial \gamma}, \quad \frac{d^2L}{d\gamma^2} = \frac{\partial^2 L}{\partial \gamma^2} + \underbrace{\frac{\partial^2 L}{\partial \gamma \partial \eta} \cdot \frac{\partial \eta}{\partial \gamma}}_{=0} \\ &\quad + \underbrace{\frac{\partial^2 L}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial \gamma}\right)^2}_{=0} \\ &\quad + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \gamma^2} \\ &= \frac{\partial^2 L}{\partial \gamma^2} + \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \eta}{\partial \gamma}\right)^2 \\ &\quad + \frac{\partial L}{\partial \eta} \frac{\partial^2 \eta}{\partial \gamma^2} \end{aligned}$$

$$\begin{aligned} \frac{dL}{d\theta_i} &= \frac{\partial L}{\partial \eta} \cdot \frac{\partial \eta}{\partial \theta_i}, \quad \frac{d^2L}{d\gamma d\theta_i} = \underbrace{\frac{\partial^2 L}{\partial \gamma \partial \eta} \cdot \frac{\partial \eta}{\partial \theta_i}}_{=0} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} + \underbrace{\frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i}}_{-\frac{\partial \eta}{\partial \theta_i}} \\ &= \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial \eta}{\partial \gamma} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} \end{aligned}$$

$$\begin{aligned} \frac{d^3L}{d\theta_i^3} &= \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \left(\frac{\partial \eta}{\partial \gamma}\right)^2 + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^2 \eta}{\partial \theta_i^2} \cdot \frac{\partial \eta}{\partial \gamma} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i^2} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^3 \eta}{\partial \theta_i^2 \partial \gamma} \end{aligned}$$

$$= \frac{\partial^3 l}{\partial y^3} \frac{\partial y}{\partial \theta_i} \left(\frac{\partial y}{\partial x} \right)^2 + \frac{\partial^2 l}{\partial y^2} \left(2 \frac{\partial y}{\partial x} \cdot \frac{\partial y}{\partial \theta_i} + \frac{\partial y}{\partial \theta_i} \frac{\partial^2 y}{\partial x^2} \right) \\ + \frac{\partial l}{\partial y} - \frac{\partial^3 y}{\partial \theta^2 \partial \theta_i}$$

$$\frac{\partial^3 L}{\partial \gamma^3} = \frac{\partial^3 L}{\partial \gamma^3} + \frac{\dot{\gamma}^2}{\partial \gamma^3} \cdot \frac{\partial y}{\partial \gamma} \cdot \left(\frac{\partial y}{\partial \gamma} \right)^2 + \frac{\ddot{\gamma}^2}{\partial \gamma^2} \left[\frac{\partial^2 y}{\partial \gamma^2} \cdot \frac{\partial^2 y}{\partial \gamma^2} \right] + \frac{\partial^2 L}{\partial \gamma^2} \cdot \frac{\partial^2 y}{\partial \gamma^2} + \frac{\partial L}{\partial \gamma} \cdot \frac{\partial^3 y}{\partial \gamma^3}$$

$$= \frac{\partial^3 L}{\partial r^3} + \frac{\partial^2 L}{\partial y^2} \left(\frac{\partial y}{\partial r} \right)^3 + 3 \frac{\partial^2 L}{\partial y^2} \frac{\partial^2 y}{\partial r^2} \frac{\partial y}{\partial r} + \frac{\partial L}{\partial y} \cdot \frac{\partial^3 y}{\partial r^3}$$

$$\frac{d^k l}{d\theta_i d\theta_j} = \frac{\partial^2 l}{\partial \eta^2} \frac{\partial \eta}{\partial \theta_j} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial l}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}$$

$$\begin{aligned} \frac{\partial^3 L}{\partial r \partial \theta_i \partial \theta_j} &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_j} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial r \partial \theta_j} \right] \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial r} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r \partial \theta_i \partial \theta_j} \\ &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial r \partial \theta_j} \right. \\ &\quad \left. + \frac{\partial \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial r} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^3 \eta}{\partial r \partial \theta_i \partial \theta_j} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^4 L}{\partial r \partial T \partial \theta_i \partial \theta_j} &= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^3 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + 2 \frac{\partial \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j \partial r} \right] \\
 &+ \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_j} \left(2 \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i^2} \cdot \frac{\partial \eta}{\partial r} \right) \\
 &+ \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \partial^2 \eta}{\partial r \partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j \partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} \right) \\
 &+ \frac{\partial^2 \eta}{\partial r^2} \left(\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial r} \frac{\partial^2 \eta}{\partial \theta_j \partial r} \right)
 \end{aligned}$$

$$+ \frac{\partial^2 \gamma}{\partial y^2} \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial^3 \gamma}{\partial x^2 \partial z^2} + \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial^4 \gamma}{\partial x^2 \partial z^2 \partial y^2}$$

$$= \partial^4 L / \partial \eta^4 \partial^2 \eta / \partial \theta_i \partial \theta_i \left(\frac{\partial \eta}{\partial r} \right)^2$$

$$+ \frac{\partial^4 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + 2 \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_j \partial r} \right]$$

$$+ \frac{1}{2} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r \partial r} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial r^2}$$

$$+ \frac{\partial^4 L}{\partial \eta^2} \left[\frac{2 \partial^3 \eta}{\partial \theta_i \partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{2 \partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} + \frac{\partial^3 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} + \frac{\partial^3 \eta}{\partial r^2 \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j}$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i \partial \theta_j}$$

$$\frac{\partial^4 L}{\partial \theta^3 \partial \theta_i} = \frac{\partial^4 L}{\partial \theta^3 \partial \eta} - \frac{\partial^2 \eta}{\partial \theta_i} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^3 L}{\partial \eta^3} - 3 \left(\frac{\partial \eta}{\partial r} \right) \frac{\partial^2 \eta}{\partial \theta_i \partial r}$$

$\xrightarrow{=0}$

$$+ \frac{3 \partial^3 L}{\partial \eta^3} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + 3 \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial r} \right]$$

$$+ \frac{\partial^2 \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial r} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial r} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \theta_i} \frac{\partial^3 \eta}{\partial r^3} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r^2 \partial \theta_i}$$

$$= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + 3 \frac{\partial^3 L}{\partial \eta^3} \left[\left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial \theta_i \partial r} + \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 L}{\partial \eta^2} \left[\frac{3 \partial^2 \eta}{\partial \theta_i \partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial \theta_i \partial r^2} + \frac{\partial^3 \eta}{\partial r^3} \frac{\partial \eta}{\partial \theta_i} \right]$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^3 \partial \theta_i}$$

$$\frac{\partial^4 L}{\partial \theta^4} = \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial r} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^2 L}{\partial \eta^3} \cdot 3 \left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial r^2}$$

$$+ 3 \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial r} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial r^3} \right]$$

$$+ \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^4}$$

$$\begin{aligned}
&= \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial y^4} \left(\frac{\partial y}{\partial r} \right)^4 + 6 \frac{\partial^3 L}{\partial y^3} \left(\frac{\partial y}{\partial r} \right)^2 \frac{\partial^2 L}{\partial r^2} \\
&\quad + \frac{\partial^2 L}{\partial y^2} \left[3 \left(\frac{\partial^2 y}{\partial r^2} \right)^2 + 4 \frac{\partial y}{\partial r} \frac{\partial^3 y}{\partial r^3} \right] + \frac{\partial L}{\partial y} \frac{\partial^4 y}{\partial r^4}
\end{aligned}$$

Derivatives w.r.t. θ_0 .

Since $x = \frac{y}{r}$ and $r \in R_{>0} \Rightarrow x \in R_{>0}$. It is however easier to work with unrestricted parameter. Therefore, we define $\alpha := e^{\theta_0}$ with $\theta_0 \in \mathbb{R}$ unrestricted.

Recall that for $y > 0$,

$$\begin{aligned}
L^{2x} &= \log(1 - e^{-2x}) + y \log \alpha + yr - y \log(1 + \alpha e^r) - \log\{(1 + \alpha e^r)^{-1}\} \\
&\quad + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})
\end{aligned}$$

$$\begin{aligned}
(L^{\theta_0}) &= \log(1 - e^{-\theta_0}) + y \log e^{\theta_0} + yr - y \log(1 + e^{\theta_0} e^r) - \log\{(1 + e^{\theta_0} e^r)^{-1}\} \\
&\quad + \log \Gamma(y + e^{-\theta_0}) - \log \Gamma(y+1) - \log \Gamma(e^{-\theta_0})
\end{aligned}$$

Note that if $x = (1 + e^{\theta_0} e^r)^{\frac{1}{e^{-\theta_0}}}$, then
 $\log x = e^{-\theta_0} \log(1 + e^{\theta_0} e^r)$

$$\begin{aligned}
0 \quad \frac{dy}{d\theta_0} &= \frac{e^{-\theta_0}}{1 + e^{\theta_0} e^r} \cdot \frac{e^{\theta_0} e^r}{e^{-\theta_0}} + \log(1 + e^{\theta_0} e^r) \cdot e^{-\theta_0} \cdot -1
\end{aligned}$$

$$= \frac{e^r}{1 + e^{\theta_0} e^r} - e^{-\theta_0} \log(1 + e^{\theta_0} e^r)$$

$$\Rightarrow \frac{dy}{d\theta_0} = \left(1 + e^{\theta_0} e^r \right)^{\frac{1}{e^{-\theta_0}}} \left\{ \beta - e^{\theta_0} \log(1 + e^{\theta_0} e^r) \right\}$$

also, $\frac{d}{dz} \log \Gamma(z) =: \psi(z)$, where ψ is the digamma fn.

Now dropping the $2x$ superscript, we have,

$$\frac{dy}{d\theta_0} = 0 + y \frac{e^r}{e^{\theta_0}} - e^{\theta_0} + 0 - y \frac{e^{\theta_0} e^r}{1 + e^{\theta_0} e^r} \cdot \frac{e^{\theta_0} e^r}{e^{-\theta_0}} - \frac{1}{(1 + e^{\theta_0} e^r)^{\frac{1}{e^{-\theta_0}}}} - 1$$

$$\begin{aligned}
 & \left. \left(\frac{\partial}{\partial \theta_0} \right) e^{-\theta_0} \right\} \beta - e^{-\theta_0} \log(1 + e^{\theta_0} e^r) \Big] \\
 & - \varphi(y + e^{-\theta_0}) e^{-\theta_0} + \varphi(e^{-\theta_0}) e^{-\theta_0} \\
 = & y - y \beta e^{\theta_0} - \tau \left\{ \beta - e^{-\theta_0} \log(1 + e^{\theta_0} e^r) \right\} \\
 & - e^{-\theta_0} \varphi(y + e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})
 \end{aligned}$$

To compute $\frac{d^2 l}{d\theta_0^2}$, we need $\frac{d\beta}{d\theta_0}$, and $\frac{dy}{d\theta_0}$,

$$\frac{d\beta}{d\theta_0} = \frac{d}{d\theta_0} \frac{\mu r}{1 + e^{\theta_0} e^r} = \frac{1 + e^{\theta_0} e^r \cdot 0 - e^r \cdot e^r e^{\theta_0}}{(1 + e^{\theta_0} e^r)^2}$$

$$= - \frac{e^{2r}}{(1 + e^{\theta_0} e^r)^2} e^{\theta_0} = - \beta^2 e^{\theta_0}$$

Let leave the rest to the computer; see ..;

$$\begin{aligned}
 \frac{d^2 l}{d\theta_0^2} = & -y \beta e^{\theta_0} + y \beta^2 e^{2\theta_0} + \tau^2 \left(e^{-\theta_0} \log(1 + e^{\theta_0} e^r) - \beta \right)^2 \\
 & - \tau \left(e^{-\theta_0} \log(1 + e^{\theta_0} e^r) - \beta \right)^2 \\
 & + \tau \left(\beta + \beta^2 e^{\theta_0} - e^{-\theta_0} \log(1 + e^{\theta_0} e^r) \right) \\
 & + e^{-\theta_0} \varphi(y + e^{-\theta_0}) - e^{-\theta_0} \varphi(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) \\
 & - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0})
 \end{aligned}$$

For $\varphi^{(1)}$, see
Polygamma
Fn on Wikipedia.

* How do we deal with products $\tau e^{-\theta_0} \log(1 + e^{\theta_0} e^r)$, ...
 since as $r \rightarrow -\infty$, $\tau \rightarrow +\infty$ and $\log(1 + e^{\theta_0} e^r) \rightarrow \log(1) = 0$
 we end up with the product $+\infty \cdot 0$, which is indeterminate.
 But we have hope if $\tau \rightarrow +\infty$ slower than the terms of interact with $\rightarrow 0$, in which case the interaction $\rightarrow 0$ as $r \rightarrow -\infty$.
 Taking θ_0 as fixed, we have

$$(i) \lim_{r \rightarrow -\infty} \tau \log(1 + e^{\theta_0} e^r) = \lim_{r \rightarrow -\infty} \frac{\log(1 + e^{\theta_0} e^r)}{1 - (1 + e^{\theta_0} e^r)^{-e^{-\theta_0}}}$$

$$\text{(L'Hopital)} = \lim_{r \rightarrow -\infty} \frac{e^{\theta_0} e^r}{(1 + e^{\theta_0} e^r)(0 + e^{-\theta_0} (1 + e^{\theta_0} e^r)^{-e^{-\theta_0}} \cdot e^{\theta_0} e^r)}$$

$$\begin{aligned}
 &= \lim_{r \rightarrow -\infty} \frac{e^{\theta_0 r}}{e^r (1 + e^{\theta_0 r})^{-e^{-\theta_0}}} \\
 &= \lim_{r \rightarrow -\infty} \frac{e^{\theta_0}}{(1 + e^{\theta_0 r})^{-e^{-\theta_0}}} = e^{\theta_0}
 \end{aligned}$$

$$\Rightarrow (c \log(1 + e^{\theta_0} e^r))^a \rightarrow e^{a\theta_0} \quad \forall a \in \mathbb{R}$$

also, $c^a \log(1 + e^{\theta_0} e^r)^b \rightarrow 0 \quad \text{if } b > a \text{ as } r \rightarrow -\infty, a, b \in \mathbb{R}$

$$(ii) \lim_{r \rightarrow -\infty} c \left(e^{-\theta_0} \log(1 + e^{\theta_0} e^r) - \beta \right) \stackrel{(i)}{=} \frac{e^{-\theta_0} - e^{-\theta_0} - 1}{1 - 1} \xrightarrow{b \rightarrow 1, \text{ as } r \rightarrow -\infty} 0$$

$$\text{Define } K = e^{-\theta_0} \log(1 + e^{\theta_0} e^r) - \beta$$

$$\Rightarrow \lim_{r \rightarrow -\infty} cK = 0 \quad / \quad \Rightarrow \lim_{r \rightarrow -\infty} (ck)^a = 0 \quad \forall a \in \mathbb{R}$$

$$\Rightarrow \lim_{r \rightarrow -\infty} c^a K^b = 0 \quad \forall a, b \in \mathbb{R} \quad \text{if } b > a$$

$$(iii) \lim_{r \rightarrow -\infty} c \left(\beta + \beta^2 e^{\theta_0} - e^{-\theta_0} \log(1 + e^{\theta_0} e^r) \right)$$

$$= 1 + 0 - e^{-\theta_0} e^{\theta_0} = 1 - 1 = 0$$

for fixed θ_0

For $r \rightarrow -\infty$, $c \rightarrow 1$, $\beta \rightarrow e^{-\theta_0}$, $\log(1 + e^{\theta_0} e^r) \approx \log(e^{\theta_0} e^r)$
since $e^{\theta_0} e^r$ will dominate 1. and $\log(e^{\theta_0} e^r) \rightarrow \theta_0 + r$.

We'll use this even though we don't have indeterminacy since
the fit implementation also uses it. See how $\log(e^r - 1)$ is approximated
as e^r as $r \rightarrow \infty$ for the zip model likelihood here 😎.

With these, we get, as $r \rightarrow -\infty$

$$\frac{dy}{d\theta_0} \rightarrow y - e^{-\theta_0} \varphi(y + e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})$$

$$\frac{d^2y}{d\theta_0^2} \rightarrow e^{-\theta_0} \varphi(y + e^{-\theta_0}) - e^{-\theta_0} \varphi(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0})$$

when $r \rightarrow -\infty$

$$\frac{dy}{d\theta_0} \rightarrow y - y e^{-\theta_0} e^{\theta_0} - (-e^{-\theta_0} (\theta_0 + r) + e^{-\theta_0}) \\ - e^{-\theta_0} \varphi(y e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})$$

$$= e^{-\theta_0} \left\{ \theta_0 + r - 1 - \varphi(y e^{-\theta_0}) + \varphi(e^{-\theta_0}) \right\}$$

$$\frac{d^2y}{d\theta_0^2} \rightarrow e^{-\theta_0} + e^{-2\theta_0} e^{\theta_0} - e^{-\theta_0} (\theta_0 + r) + e^{-\theta_0} \varphi(y e^{-\theta_0}) \\ - e^{-\theta_0} \varphi'(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y e^{-\theta_0}) - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0})$$

$$= e^{-\theta_0} \left\{ 2 - \theta_0 - r + \varphi(y e^{-\theta_0}) - \varphi(e^{-\theta_0}) \right. \\ \left. + e^{-\theta_0} \varphi^{(1)}(y e^{-\theta_0}) - e^{-\theta_0} \varphi^{(1)}(e^{-\theta_0}) \right\}$$