Negative Binomial PMF:

Y is counting y failures given r successes. Total # of trials $n = y+r$

$$f(y; p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

where p is the probability of success in each trial

Lemma: $\binom{y+r-1}{r-1} = \binom{y+r-1}{y}$

proof:

$$\binom{y+r-1}{r-1} = \frac{(y+r-1)!}{(r-1)!((y+r-1)-r-1)!} = \frac{(y+r-1)!}{(r-1)!\,y!}$$

$$= \frac{(y+r-1)!}{y!\,(r-1)!} = \frac{(y+r-1)!}{y!\,(y+r-1-y)!}$$

$$= \binom{y+r-1}{y}$$

$\square$

Therefore

$$f(y; p, r) = \binom{y+r-1}{y} p^r (1-p)^y$$

Mean: $\frac{r(1-p)}{p} =: \mu$, **Variance:** $\frac{r(1-p)}{p^2} = Var(Y)$

$\Rightarrow \mu p = r - rp$

$\Rightarrow p(r+\mu) = r$

$\Rightarrow p = r/_{r+\mu}$

$$f(y; \mu, r) = \binom{y+r-1}{y} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

$\Gamma(n) = (n-1)!$ if $n \in \mathbb{N}, n \geq 1$

if n is a real # the $\Gamma(\cdot)$

$$= \frac{\Gamma(y+r)}{y!\,\Gamma(r)} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

is well defined

Define $r := 1/\alpha$,

$$f(y;\mu,r) = \frac{\Gamma(y+1/\alpha)}{\Gamma(y+1)\,\Gamma(1/\alpha)} \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha} \left(\frac{\alpha\mu}{1+\alpha\mu}\right)^{y}$$

$\Rightarrow$ The single data log-likelihood is

$$l_i^{NB} = y_i \log\left(\frac{\alpha\mu_i}{1+\alpha\mu_i}\right) - 1/\alpha \log(1+\alpha\mu_i) + \log\Gamma(y_i + 1/\alpha)$$
$$- \log\Gamma(y_i+1) - \log\Gamma(1/\alpha) \quad \cdots \; (1)$$

Zero-truncated NB.

The probability of a zero count is

$$f(0;\mu,r) = \frac{\Gamma(1/\alpha)}{\Gamma(1)\,\Gamma(1/\alpha)}\left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha} = \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha}$$

The zero-truncated density is

$$f(y|y>0,\mu,r) = \frac{f(y;\mu,r)}{1-f(0,\mu,r)}$$

$$= \frac{\frac{\Gamma(y+1/\alpha)}{\Gamma(y+1)\,\Gamma(1/\alpha)}\left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha}\left(\frac{\alpha\mu}{1+\alpha\mu}\right)^{y}}{1-\left(1/(1+\alpha\mu)\right)^{1/\alpha}}$$

This gives us the single data log-likelihood;

$$l_i^{ZT} = l_i^{NB} - \log\left\{1-(1+\alpha\mu_i)^{-1/\alpha}\right\} \quad \cdots \; (2)$$

Where (2) is the NB single data log-likelihood as given in (1)

Zero Inflated Negative Binomial (Hurdle model in Tutz (2012))

We use a similar parameterization as in Wood et al. (2017) SA I.
erated
y a

$$g(y;\mu,r) = \begin{cases} 1-q & y=0 \\ q\, f(y \mid y>0,\mu,r) & \text{otherwise} \end{cases}$$

where q is the probability of a positive counts

Similar to Wood et al. (2017), we adopt the unconstrained parameterization

$$\gamma = \log \mu \quad , \quad \eta = \log\{-\log(1-q)\} .$$

This gives us the single data log-likelihood (we drop the data index i)

$$l^{z1} = \begin{cases} \log(1-q) & y=0 \\ \log q + \log f(y \mid y>0,\mu,r) & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{z1} & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{NB} - \log\left\{1-(1+\alpha e^{\gamma})^{-1/\alpha}\right\} & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{z1} & \text{otherwise} \end{cases}$$

where $l^{ZI} = l^{NB} - \log\{1-(1+\alpha e^{\gamma})^{-1/\alpha}\}$ is the ZINB log-likelihood.

and $l^{NB} = y\log\left(\alpha e^{\gamma}/_{1+\alpha e^{\gamma}}\right) - \frac{1}{\alpha}\log(1+\alpha e^{\gamma}) + \log\Gamma(y+1/\alpha)$

$$-\log\Gamma(y+1) - \log\Gamma(1/\alpha)$$

$$\Rightarrow l^{ZI} = y\log\left(\alpha e^{\gamma}/_{1+\alpha e^{\gamma}}\right) + \log\Gamma(y+1/\alpha) - \log\Gamma(y+1) - \log\Gamma(1/\alpha)$$
$$+\log\left\{\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}}\right\} + \log\left\{\frac{1}{1-\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}}}\right\}$$

looking at the last two terms basically is

$$\log\left\{\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}-1}\right\} = -\log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\}$$

$$\Rightarrow l^{ZI} = y\log\left\{\alpha e^{\gamma}/_{1+\alpha e^{\gamma}}\right\} + \log\Gamma(y+1/\alpha) - \log\Gamma(y+1)$$
$$-\log\Gamma(1/\alpha) - \log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\}$$

$$= y\log\alpha + y\gamma - y\log(1+\alpha e^{\gamma}) - \log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\}$$
$$+\log\Gamma(y+1/\alpha) - \log\Gamma(y+1) - \log\Gamma(1/\alpha)$$

Here as well, some care is required to evaluate l^{ZI} without unnecessary overflow, since it is easy for $1-e^{-e^{\eta}}$ and $(1+\alpha e^{\gamma})^{1/\alpha}-1$ to evaluate to zero in finite precision arithmetic. Hence the limiting results $\log(1-e^{-e^{\eta}}) \to \log(e^{\eta}-e^{2\eta/2}+e^{3\eta}/6) \to \eta$ as $\eta \to -\infty$

Similar as $\gamma \to -\infty$, $\log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to \gamma$ and as $\alpha \to 0$

$\log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to \log(e^{e^{\gamma}}-1) \to e^{\gamma}$

provided $\gamma \not\to -\infty$

Proof:

① $\log(1-e^{-e^{\eta}}) \to \log\left(e^{\eta}-e^{2\eta}/2+e^{3\eta}/6\right) \to \eta \quad n \to -\infty$

as $\eta \to -\infty$, $-e^{\eta} \to 0$,

we know that $e^{x} \to 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ as $x \to 0$

$$\Rightarrow e^{-e^{\eta}} \to 1-e^{\eta}+e^{2\eta}/2-e^{3\eta}/6$$

$\Rightarrow$ as $\eta \to -\infty$, $\log(1-e^{-e^{\eta}})$

$$\to \log\left(1 - 1 + e^{\eta} - \frac{e^{2\eta}}{2} + e^{3\eta}/6\right)$$

$$= \log\left(e^{\eta} - \frac{e^{2\eta}}{2} + e^{3\eta}/6\right)$$

as $\eta \to -\infty$ $e^{2\eta}$ and $e^{3\eta} \to 0$ faster than e^{η}

$$\Rightarrow \log\left(e^{\eta} - e^{2\eta}/2 + \frac{e^{3\eta}}{6}\right)$$

$$\to \log e^{\eta} \to \eta$$

② $\log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\} \to \gamma$ as $\gamma \to -\infty$

as $\gamma \to -\infty$, $e^{\gamma} \to 0$ $\Rightarrow \alpha e^{\gamma} \to 0$ (fixed α)

We know that $(1+x)^{1/\alpha} \to 1 + \frac{1}{\alpha} x$ as $x \to 0$

$$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \to 1 + \frac{1}{\alpha}(\alpha e^{\gamma}) = 1 + e^{\gamma}$$

$$\Rightarrow \log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\} \to \log\{(1+e^{\gamma}) - 1\}$$

$$= \log(e^{\gamma}) = \gamma$$

as $\gamma \to -\infty$

$e^x := \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$

③ $\log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\} \to e^{\gamma}$ as $\alpha \to 0$ for fixed γ

remember that $(1+\alpha e^{\gamma})^{1/\alpha} = \left(1 + \frac{e^{\gamma}}{r}\right)^{r}$,

so as $\alpha \to 0$, $r \to \infty$

and $\lim_{r\to\infty} \left(1 + \frac{e^{\gamma}}{r}\right)^{r} = e^{e^{\gamma}}$ for $r \in \mathbb{N}_0$

$$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \to e^{e^{\gamma}} \text{ as } \alpha \to 0$$

$\Rightarrow \log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\} \to \log\{e^{e^{\gamma}} - 1\}$ as $\alpha \to 0$

also for very large γ, $e^{e^{\gamma}} - 1$ is dominated by $e^{e^{\gamma}}$

$$\Rightarrow \log\{e^{e^{\gamma}} - 1\} \to \log e^{e^{\gamma}} = e^{\gamma} \quad \square$$

Implementatn (Following Wood et al. (2017))

According to the framework proposed in Wood et al. (2017), we need the deviance an its derivatives to implement an extend GAM model (see sectn 3.3.1 of that paper);

Differentiating w.r.t. η_i instead of μ_i as this is done in SAI.

1- $\frac{\partial D_i}{\partial \eta_i}$, $\frac{\partial^2 D_i}{\partial \eta_i^2}$, $\frac{dh_i}{d\mu_i}$, $\frac{d^2 h_i}{d\mu_i^2}$ (For $\hat{\vec{\beta}}$)

2- $d^4 h_i / d\eta^4$, $\frac{\partial^2 D_i}{\partial\theta_i \partial\theta_j}$, $\frac{\partial^3 D_i}{\partial\eta_i \partial\theta_j \partial\theta_k}$, $\frac{\partial^4 D_i}{\partial\eta_i^4}$, $\frac{\partial^4 D_i}{\partial\mu_i^3 \partial\theta_j}$, $\frac{\partial^4 D}{\partial\mu_i^2 \partial\theta_j \partial\theta_k}$

(For $\hat{\vec{\rho}}$ via full Newton).

As for the ziP model, we first define the deviance as $-2l$ for model estimatn.

w.r.t. η for $y>0$

Exactly as for zip.

① $dl/d\eta = \frac{e^\eta}{e^{e^\eta}-1}$, ② $\frac{d^2 l}{d\eta^2} = (1-e^\eta)\,\partial l/\partial\eta - \left(\frac{\partial l}{\partial\eta}\right)^2$

③ $d^3 l/d\eta^3 = -e^\eta \partial l/\partial\eta + (1-e^\eta)^2 \partial l/\partial\eta - 3(1-e^\eta)\left(\partial l/\partial\eta\right)^2 + 2\left(\frac{\partial l}{\partial\eta}\right)^3$ and

④ $$d^4 l/d\eta^4 = (3e^\eta - 4)e^\eta \partial l/\partial\eta + 4e^\eta\left(\partial l/\partial\eta\right)^2 + (1-e^\eta)^3\frac{\partial l}{\partial\eta} - 7(1-e^\eta)^2\left(\partial l/\partial\eta\right)^2 + 12(1-e^\eta)\left(\partial l/\partial\eta\right)^3 - 6\left(\frac{\partial l}{\partial\eta}\right)^4$$

See SAI of Wood et al. (2016) for dealing with these derivatives as $\eta \to \pm\infty$

w.r.t. γ for $y>0$

① $$dl/d\gamma = dl^{zt}/d\gamma = \frac{d}{d\gamma}\left(y\gamma - y\log(1+\alpha e^\gamma) - \log\{(1+\alpha e^\gamma)^{1/\alpha} - 1\}\right)$$
$$= y - \frac{y}{1+\alpha e^\gamma}\cdot\alpha e^\gamma - \frac{1\cdot\alpha e^\gamma \cdot 1/\alpha\,(1+\alpha e^\gamma)^{1/\alpha - 1}}{(1+\alpha e^\gamma)^{1/\alpha}-1}$$
$$= y - y\alpha\frac{e^\gamma}{1+\alpha e^\gamma} - \frac{e^\gamma(1+\alpha e^\gamma)^{1/\alpha - 1}}{(1+\alpha e^\gamma)^{1/\alpha}-1}$$

$$= y - \frac{y\,\alpha e^{\gamma}}{1+\alpha e^{\gamma}} - \left(\frac{e^{\gamma}}{1+\alpha e^{\gamma}}\right)\left(\frac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e)^{1/\alpha} - 1}\right)$$

Define $\beta := \dfrac{e^{\gamma}}{1+\alpha e^{\gamma}}$, $\tau := \dfrac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})^{1/\alpha} - 1}$

$$\Rightarrow \quad d\ell/d\gamma = y - y\alpha\beta - \beta\tau$$

2 $$\frac{d^2\ell}{d\gamma^2} = \beta\left\{ y\alpha^2\beta - y\alpha + \alpha\beta\tau - \tau + \beta\tau^2 - \beta\tau \right\}$$

Proof: $$\frac{d}{d\gamma}\left\{ y - y\alpha\beta - \beta\tau \right\} = -y\alpha\, d\beta/d\gamma - \frac{d\beta}{d\gamma}\tau - \beta\frac{d\tau}{d\gamma}$$

But $$d\beta/d\gamma = \frac{(1+\alpha e^{\gamma})e^{\gamma} - e^{\gamma}\cdot\alpha e^{\gamma}}{(1+\alpha e^{\gamma})^2}$$

$$= \frac{e^{\gamma}}{1+\alpha e^{\gamma}} - \frac{\alpha e^{2\gamma}}{(1+\alpha e^{\gamma})^2} = \beta - \alpha\beta^2$$

and $$d\tau/d\gamma = \frac{\left\{\left[(1+\alpha e^{\gamma})^{1/\alpha} - 1\right]\left[\alpha e^{\gamma}\cdot \tfrac{1}{\alpha}\cdot(1+\alpha e^{\gamma})^{1/\alpha - 1}\right] - \left[(1+\alpha e^{\gamma})^{1/\alpha}\right]\left[\alpha e^{\gamma}\cdot\tfrac{1}{\alpha}(1+\alpha e^{\gamma})^{1/\alpha-1}\right]\right\}}{\left((1+\alpha e^{\gamma})^{1/\alpha} - 1\right)^2}$$

$$= \frac{e^{\gamma}(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})\left[(1+\alpha e^{\gamma})^{1/\alpha} - 1\right]} - \frac{e^{\gamma}(1+\alpha e^{\gamma})^{2/\alpha}}{(1+\alpha e^{\gamma})\left[(1+\alpha e^{\gamma})^{1/\alpha} - 1\right]^2}$$

$$= \beta\tau - \beta\tau^2$$

$$\Rightarrow \frac{d^2\ell}{d\gamma^2} = -y\alpha(\beta - \alpha\beta^2) - \beta\tau + \alpha\beta^2\tau - \beta^2\tau + \beta^2\tau^2$$

$$= y\alpha^2\beta^2 - y\alpha\beta + \alpha\beta^2\tau - \beta\tau + \beta^2\tau^2 - \beta^2\tau$$

$$= \beta\left\{ y\alpha^2\beta - y\alpha + \alpha\beta\tau - \tau + \beta\tau^2 - \beta\tau \right\}$$

□

(3) $d^3\ell/d\gamma^3 = \beta\{ -2y\alpha^3\beta^2 + 3y\alpha^2\beta - 2\alpha^2\beta^2\tau - y\alpha + 3\alpha\beta\tau$
$- 3\alpha\beta^2\tau^2 + 3\alpha\beta^2\tau - \tau + 3\beta\tau^2 - 3\beta\tau - 2\beta^2\tau^3$
$+ 3\beta^2\tau^2 - \beta^2\tau^2 \}$

Proof. Using a numerical linear algebra package, see 👓. $\square$

(4) $d^4\ell/d\gamma^4 = \beta\{ 6y\alpha^4\beta^3 - 12y\alpha^3\beta^2 + 6\alpha^2\beta^3\tau + 7y\alpha^2\beta - 12\alpha^2\beta^2\tau + 11\alpha^2\beta^3\tau^2 - 11\alpha^2\beta^3\tau$
$- y\alpha + 7\alpha\beta\tau - 18\alpha\beta^2\tau^2 + 18\alpha\beta^2\tau + 12\alpha\beta^3\tau^3 - 18\alpha\beta^3\tau^2 + 6\alpha\beta^3\tau$
$- \tau + 7\beta\tau^2 - 7\beta\tau - 12\beta^2\tau^3 + 18\beta^2\tau^2 - 6\beta^2\tau + 6\beta^3\tau^4$
$- 12\beta^3\tau^3 + 7\beta^3\tau^2 - \beta^3\tau \}$

proof. 👓

$\square$

As with ℓ^{21}, some care is required to ensure that the derivatives evaluate accurately without overflow over as wide a range of γ and α as possible. As $\gamma \to \infty$, $\beta = \frac{1}{e^{-\gamma} + \alpha} \to 1/\alpha$ for fixed α, while

$\tau = \frac{1}{1 - (1+\alpha e^{\gamma})^{-1/\alpha}} \to 1$, since $1 + \alpha e^{\gamma} \to \infty \Rightarrow (1+\alpha e^{\gamma})^{-1/\alpha} \to 0$ for fixed α

whereas as $\gamma \to -\infty$, $\beta \to 1/e^{-\gamma} \to 0$ for fixed α, while $e^{\gamma} \to 0$
$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \to 1 + \frac{1}{\alpha}\alpha e^{\gamma} = 1 + e^{\gamma} \Rightarrow \tau = \frac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})^{1/\alpha} - 1}$

first order Taylor expansion about $x = \alpha e^{\gamma} = 0$

$\to \frac{1+e^{\gamma}}{1+e^{\gamma}-1} = \frac{1+e^{\gamma}}{e^{\gamma}}$

$* = e^{-\gamma} + 1 \to e^{-\gamma}$

* How do we deal with products $\beta\tau$, $\beta^2\tau^2$, $\beta^2\tau$ etc. what does it converge to?

Since as $\gamma \to -\infty$, $\beta \to e^{\gamma}$ and $\tau \to e^{-\gamma}$
$\Rightarrow \beta\tau \to 1 \Rightarrow (\beta\tau)^a \to 1 \quad \forall a \in \mathbb{R}$

also $\Rightarrow \beta^a\alpha^b \to 0$ if $a > b$ as $\gamma \to -\infty$

as $\gamma \to \infty$, $\beta \to \alpha^{-1}$ and $\tau \to 1 \Rightarrow \beta\tau \to \alpha^{-1}$

with these we get

$\partial c/\partial \gamma \to y-1$ as $\gamma \to -\infty$, $\partial c/\partial \gamma \to y - y - 1/\alpha = -1/\alpha$ as $\gamma \to \infty$

$\partial^2 c/\partial \gamma^2 \to 0$ as $\gamma \to -\infty$, $\partial^2 c/\partial \gamma^2 \to y - y + \alpha/\alpha^2 - 1/\alpha + 1/\alpha^2 - \frac{1}{\alpha^2}$

$= 1/\alpha - 1/\alpha + 1/\alpha^2 - 1/\alpha^2 = 0$

as $\gamma \to \infty$

$\partial^3 c/\partial \gamma^3 \to 0$ as $\gamma \to -\infty$, $\partial^3 c/\partial \gamma^3 \to -2y + 3y - 2/\alpha - y + 3/\alpha$

$- 3/\alpha^2 + 3/\alpha^2 - 1/\alpha + \frac{3}{\alpha^2}$

$- 3/\alpha^2 - 2/\alpha^3$

$+ 3/\alpha^3 - 1/\alpha^3$

$= 1/\alpha - 1/\alpha + 3/\alpha^2 - 3/\alpha^2$

$+ 0 = 0$ as $\gamma \to \infty$

$\partial^4 c/\partial \gamma^3 \to 0$ as $\gamma \to -\infty$, $\partial^4 c/\partial \gamma^4 \to \cancel{-2y} + \cancel{3y} - \cancel{2/\alpha} - \cancel{y} + \cancel{3/\alpha}$

$- \cancel{3/\alpha^2} + \cancel{3/\alpha^2} - \cancel{1/\alpha} + \cancel{3/\alpha^2}$

$- \cancel{3/\alpha^2} - \cancel{2/\alpha^3} + \cancel{3/\alpha^3} - \cancel{1/\alpha^3}$

$= 0$ as $\gamma \to \infty$

For extended GAMs in which η is a Fn of γ and extra parameters θ e.g.;

$$\eta = \theta_1 + e^{\theta_2}\gamma$$ (as for the ziP model in Wood et al. (2016)).

$$d\eta/d\theta_1 = 1, \quad d\eta/d\theta_2 = e^{\theta_2}\gamma, \quad d^2\eta/d\theta_1 d\theta_2 = 0$$

$$d^2\eta/d\theta_1^2 = 0, \quad d^2\eta/d\theta_2^2 = e^{\theta_2}\gamma$$

In this setting

$$dl/d\gamma = \partial l/\partial\gamma + \partial l/\partial\eta \cdot \partial\eta/\partial\gamma, \quad \frac{d^2 l}{d\gamma^2} = \frac{\partial^2 l}{\partial\gamma^2} + \overbrace{\frac{\partial^2 l}{\partial\gamma\partial\eta} \cdot \frac{\partial\eta}{\partial\gamma}}^{=0}$$

$$+ \frac{\partial^2 l}{\partial\eta\partial\gamma}\frac{\partial\eta}{\partial\gamma}\Big\} = 0$$

$$+ \frac{\partial^2 l}{\partial\eta^2}\cdot\left(\frac{\partial\eta}{\partial\gamma}\right)^2$$

$$+ \frac{\partial l}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma^2}$$

$$= \frac{\partial^2 l}{\partial\gamma^2} + \frac{\partial^2 l}{\partial\eta^2}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial l}{\partial\eta}\frac{\partial^2\eta}{\partial\gamma^2}$$

$$dl/d\theta_i = \partial l/\partial\eta \cdot \partial\eta/d\theta_i, \quad \frac{d^2 l}{d\gamma d\theta_i} = \underbrace{\frac{\partial^2 l}{\partial\gamma\partial\eta}}_{=0}\cdot\frac{\partial\eta}{d\theta_i} + \frac{\partial l}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma\partial\theta_i} + \frac{\partial^2 l}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial\theta_i}\cdot\partial\eta/\partial\gamma$$

$$= \partial^2 l/\partial\eta^2 \cdot \frac{\partial\eta}{\partial\theta_i}\cdot\frac{\partial\eta}{\partial\gamma} + \frac{\partial l}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma\partial\theta_i}$$

$$\frac{d^3 l}{d\gamma^2 d\theta_i} = \frac{\partial^3 l}{\partial\eta^3}\cdot\frac{\partial\eta}{\partial\theta_i}\cdot\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^2 l}{\partial\eta^2}\, 2\,\frac{\partial\eta}{\partial\gamma}\cdot\frac{\partial\eta}{\partial\gamma\partial\theta_i} + \frac{\partial^2 l}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial\theta_i}\frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ \partial l/\partial\eta \cdot \partial^3\eta/\partial\gamma^2\partial\theta_i$$

$$= \frac{\partial^3 l}{\partial\eta^3}\,\frac{\partial\eta}{\partial\theta_i}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^2 l}{\partial\eta^2}\left(2\,\frac{\partial\eta}{\partial\gamma}\cdot\frac{\partial\eta}{\partial\gamma\partial\theta_i} + \frac{\partial\eta}{\partial\theta_i}\,\frac{\partial^2\eta}{\partial\gamma^2}\right)$$

$$+ \frac{\partial l}{\partial\eta}\cdot\frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i}$$

$$\frac{d^3 l}{d\gamma^3} = \frac{\partial^3 l}{\partial\gamma^3} + \frac{\partial^3 l}{\partial\eta^3}\cdot\frac{\partial\eta}{\partial\gamma}\cdot\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^2 l}{\partial\eta^2}\left[2\,\frac{\partial\eta}{\partial\gamma}\cdot\frac{\partial^2\eta}{\partial\gamma^2}\right]$$

$$+ \frac{\partial^2 l}{\partial\eta^2}\cdot\frac{\partial^2\eta}{\partial\gamma^2} + \frac{\partial l}{\partial\eta}\cdot\frac{\partial^3\eta}{\partial\gamma^3}$$

$$= \partial^3 l/\partial\gamma^3 + \frac{\partial^3 l}{\partial\eta^3}\left(\frac{\partial\eta}{\partial\gamma}\right)^3 + 3\,\frac{\partial^2 l}{\partial\eta^2}\,\frac{\partial^2\eta}{\partial\gamma^2}\,\frac{\partial\eta}{\partial\gamma} + \frac{\partial l}{\partial\eta}\cdot\frac{\partial^3\eta}{\partial\gamma^3}$$

$$d^2 l/d\theta_i d\theta_j = \frac{\partial^2 l}{\partial\eta^2}\,\partial\eta/\partial\theta_j\,\partial\eta/\partial\theta_i + \frac{\partial l}{\partial\eta}\cdot\partial^2\eta/\partial\theta_i\partial\theta_j$$

$$\frac{d^3 l}{d\gamma\, d\theta_i\, d\theta_j} = \frac{\partial^3 l}{\partial\eta^3}\,\partial\eta/\partial\theta_j\,\partial\eta/\partial\theta_i\,\partial\eta/\partial\gamma + \frac{\partial^2 l}{\partial\eta^2}\left[\frac{\partial^2\eta}{\partial\theta_i\partial\theta_j}\,\frac{\partial\eta}{\partial\gamma} + \frac{\partial\eta}{\partial\theta_i}\,\frac{\partial^2\eta}{\partial\gamma\partial\theta_j}\right]$$

$$+ \frac{\partial^2 l}{\partial\eta^2}\,\frac{\partial\eta}{\partial\theta_j}\,\frac{\partial^2\eta}{\partial\theta_i\partial\gamma} + \frac{\partial l}{\partial\eta}\,\frac{\partial^3\eta}{\partial\gamma\partial\theta_i\partial\theta_j}$$

$$= \partial^3 l/\partial\eta^3\,\partial\eta/\partial\theta_i\,\partial\eta/\partial\theta_j\,\partial\eta/\partial\gamma + \partial^2 l/\partial\eta^2\left[\frac{\partial^2\eta}{\partial\theta_i\partial\theta_j}\,\frac{\partial\eta}{\partial\gamma} + \frac{\partial\eta}{\partial\theta_i}\,\frac{\partial^2\eta}{\partial\gamma\partial\theta_j}\right.$$

$$\left.+ \partial\eta/\partial\theta_j\,\frac{\partial^2\eta}{\partial\theta_i\partial\gamma}\right] + \partial l/\partial\eta\;\partial^3\eta/\partial\gamma\partial\theta_i\partial\theta_j$$

$$d^4 l/d\gamma d\gamma d\theta_i d\theta_j = \frac{\partial^4 l}{\partial\eta^4}\,\frac{\partial\eta}{\partial\theta_j}\,\frac{\partial\eta}{\partial\theta_i}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^3 l}{\partial\eta^3}\left[\frac{\partial^2\eta}{\partial\theta_i\partial\theta_j}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + 2\,\frac{\partial\eta}{\partial\theta_i}\,\frac{\partial\eta}{\partial\gamma}\,\frac{\partial^2\eta}{\partial\gamma\partial\theta_j}\right]$$

$$+ \frac{\partial^3 l}{\partial\eta^3}\,\frac{\partial\eta}{\partial\theta_j}\left(2\,\partial^2\eta/\partial\gamma\partial\theta_i\;\partial\eta/\partial\gamma + \frac{\partial^2\eta}{\partial\gamma^2}\cdot\frac{\partial\eta}{\partial\theta_i}\right)$$

$$+ \frac{\partial^2 l}{\partial\eta^2}\left(2\,\frac{\partial^3\eta}{\partial\gamma\partial\theta_i\partial\theta_j}\,\frac{\partial\eta}{\partial\gamma} + \frac{2\,\partial^2\eta}{\partial\gamma\partial\theta_i}\,\frac{\partial^2\eta}{\partial\gamma\partial\theta_j} + \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_j}\,\frac{\partial\eta}{\partial\theta_i}\right.$$

$$\left.+ \frac{\partial^2\eta}{\partial\gamma^2}\,\frac{\partial^2\eta}{\partial\theta_i\partial\theta_j}\right)$$

$$+ \partial^2 l/\partial\eta^2\;\partial\eta/\partial\theta_j\;\partial^3\eta/\partial\gamma^2\partial\theta_i + \partial l/\partial\eta\;\frac{\partial^4\eta}{\partial\gamma^2\partial\theta_i\partial\theta_j}$$

$$= \partial^4 l/\partial\eta^4 \; \partial\eta/\partial\theta_j \; \partial\eta/\partial\theta_i \left(\frac{\partial\eta}{\partial\gamma}\right)^2$$

$$+ \partial^3 l/\partial\eta^3 \Big[\partial^2\eta/\partial\theta_i\partial\theta_j \left(\frac{\partial\eta}{\partial\gamma}\right)^2 + 2\, \partial\eta/\partial\theta_i \; \partial\eta/\partial\gamma \; \frac{\partial^2\eta}{\partial\theta_j\partial\gamma}$$

$$+ 2\,\partial\eta/\partial\theta_j \; \partial^2\eta/\partial\gamma\partial\theta_i \; \partial\eta/\partial\gamma + \partial\eta/\partial\theta_i \; \frac{\partial\eta}{\partial\theta_j} \frac{\partial^2\eta}{\partial\gamma^2} \Big)$$

$$+ \partial^2 l/\partial\eta^2 \Big[\frac{2\,\partial^3\eta}{\partial\gamma\partial\theta_i\partial\theta_j} \; \partial\eta/\partial\gamma + \frac{2\,\partial^2\eta}{\partial\gamma\partial\theta_i} \frac{\partial^2\eta}{\partial\gamma\partial\theta_j} + \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_j} \frac{\partial\eta}{\partial\theta_i}$$

$$+ \partial^2\eta/\partial\gamma^2 \; \frac{\partial^2\eta}{\partial\theta_i\partial\theta_j} + \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} \; \partial\eta/\partial\theta_j \Big]$$

$$+ \partial l/\partial\eta \; \partial^4\eta/\partial\gamma^2\partial\theta_i\partial\theta_j$$

$$d^4 l/d\gamma^3 d\theta_i = \underbrace{\frac{\partial^4 l}{\partial^3\gamma\,\partial\eta}}_{=0} \cdot \partial\eta/\partial\theta_i + \frac{\partial^4 l}{\partial\eta^4} \frac{\partial\eta}{\partial\theta_i} \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + \frac{\partial^3 l}{\partial\eta^3} \cdot 3\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma\partial\theta_i}$$

$$+ \frac{3\,\partial^3 l}{\partial\eta^3} \; \partial\eta/\partial\theta_i \; \partial\eta/\partial\gamma \; \partial^2\eta/\partial\gamma^2 + \frac{3\,\partial^2 l}{\partial\eta^2} \Big[\frac{\partial^2\eta}{\partial\gamma\partial\theta_i} \frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ \partial\eta/\partial\gamma \; \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} \Big] + \frac{\partial^2 l}{\partial\eta^2} \frac{\partial\eta}{\partial\theta_i} \frac{\partial^3\eta}{\partial\gamma^3} + \frac{\partial l}{\partial\eta} \frac{\partial^4\eta}{\partial\gamma^3\partial\theta_i}$$

$$= \frac{\partial^4 l}{\partial\eta^4} \frac{\partial\eta}{\partial\theta_i} \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + 3\,\frac{\partial^3 l}{\partial\eta^3} \Big[\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma\partial\theta_i} + \frac{\partial\eta}{\partial\theta_i} \frac{\partial\eta}{\partial\gamma} \frac{\partial^2\eta}{\partial\gamma^2} \Big]$$

$$+ \partial^2 l/\partial\eta^2 \Big[3\,\partial^2\eta/\partial\gamma\partial\theta_i \; \frac{\partial^2\eta}{\partial\gamma^2} + 3\,\frac{\partial\eta}{\partial\gamma} \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} + \frac{\partial^3\eta}{\partial\gamma^3} \frac{\partial\eta}{\partial\theta_i} \Big]$$

$$+ \partial l/\partial\eta \; \frac{\partial^4\eta}{\partial\gamma^3\partial\theta_i}$$

$$d^4 l/d\gamma^4 = \partial^4 l/\partial\gamma^4 + \frac{\partial^4 l}{\partial\eta^4} \; \partial\eta/\partial\gamma \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + \frac{\partial^3 l}{\partial\eta^3} \cdot 3\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ 3\,\frac{\partial^3 l}{\partial\eta^3} \cdot \partial\eta/\partial\gamma \; \frac{\partial\eta}{\partial\gamma} \frac{\partial^2\eta}{\partial\gamma^2} + \frac{3\,\partial^2 l}{\partial\eta^2} \Big[\partial^2\eta/\partial\gamma^2 \; \frac{\partial\eta^2}{\partial\gamma^2} +$$

$$\partial\eta/\partial\gamma \; \partial^3\eta/\partial\gamma^3 \Big] + \partial^2 l/\partial\eta^2 \; \partial\eta/\partial\gamma \; \frac{\partial^3\eta}{\partial\gamma^3} + \frac{\partial l}{\partial\eta} \frac{\partial^4\eta}{\partial\gamma^4}$$

$$= \frac{\partial^4 l}{\partial \gamma^4} + \frac{\partial^4 l}{\partial \eta^4}\left(\frac{\partial \eta}{\partial \gamma}\right)^4 + 6\frac{\partial^3 l}{\partial \eta^3}\left(\frac{\partial \eta}{\partial \gamma}\right)^2 \frac{\partial^2 \eta}{\partial \gamma^2}$$

$$+ \partial^2 l/\partial \eta^2 \left[3\left(\partial^2 \eta/\partial \gamma^2\right)^2 + 4\,\partial \eta/\partial \gamma\, \partial^3 \eta/\partial \gamma^3\right) + \frac{\partial l}{\partial \eta}\frac{\partial^4 \eta}{\partial \gamma^4}$$

Derivatives w.r.t. α.

Recall that for $y > 0$,

$$l^{(2)} = \log(1-e^{-e^\eta}) + y\log\alpha + y\gamma - y\log(1+\alpha e^\gamma)$$
$$- \log\{(1+\alpha e^\gamma)^{1/\alpha} - 1\} + \log\Gamma(y+1/\alpha) - \log\Gamma(y+1)$$
$$- \log\Gamma(1/\alpha)$$

Note that if $x = (1+\alpha e^\gamma)^{1/\alpha}$

$$\Rightarrow \log x = 1/\alpha \log(1+\alpha e^\gamma)$$

$$\Rightarrow 1/x\, dx/d\alpha = -1/\alpha^2 \log(1+\alpha e^\gamma) + 1/\alpha\left(\frac{1}{1+\alpha e^\gamma}\cdot e^\gamma\right)$$

$$\Rightarrow 1/x\, dx/d\alpha = \frac{e^\gamma}{\alpha(1+\alpha e^\gamma)} - 1/\alpha^2 \log(1+\alpha e^\gamma)$$

$$\Rightarrow dx/d\alpha = (1+\alpha e^\gamma)^{1/\alpha}\left\{\frac{e^\gamma}{\alpha(1+\alpha e^\gamma)} - \frac{1}{\alpha^2}\log(1+\alpha e^\gamma)\right\}$$

also $\frac{d}{dz}\log\Gamma(z) =: \varphi(z)$, where φ is the digamma Fn.

η, γ don't depend on α

Now $$\frac{dl}{d\alpha} = 0 + y/\alpha + 0 - \frac{y e^\gamma}{1+\alpha e^\gamma} - \frac{(1+\alpha e^\gamma)^{1/\alpha}}{[(1+\alpha e^\gamma)^{1/\alpha} - 1]}$$
$$\left\{\frac{e^\gamma}{\alpha(1+\alpha e^\gamma)} - 1/\alpha^2 \log(1+\alpha e^\gamma)\right\}$$
$$- \frac{1}{\alpha^2}\varphi(y+1/\alpha) + 1/\alpha^2\, \varphi(1/\alpha)$$
$$= y/\alpha - y\beta - \tau\left\{\beta/\alpha - 1/\alpha^2 \log(1+\alpha e^\gamma)\right\}$$
$$- 1/\alpha^2\, \varphi(y+1/\alpha) + 1/\alpha^2\, \varphi(1/\alpha).$$

as $\gamma \to -\infty$ $\log(1+\alpha e^\gamma) \to 0$

as $\gamma \to \infty$ $\log(1+\alpha e^\gamma) \to \log(\alpha e^\gamma) \to \log\alpha + \gamma \to \gamma$

$$d\beta/d\alpha = \frac{(1+\alpha e^\gamma)0 - e^\gamma e^\gamma}{(1+\alpha e^\gamma)^2} = \frac{-e^{2\gamma}}{(1+\alpha e^\gamma)^2} = -\beta^2$$

without manually computing $d\tau/d\alpha$, we write

$$d^2\mathcal{L}/d\alpha^2 = y\beta^2 + \frac{\tau}{\alpha}\left\{\beta^2 + \frac{2}{\alpha}\beta - \frac{2}{\alpha^2}\log(1+\alpha e^{\gamma})\right\}$$

$$- y/\alpha^2 + \frac{\tau^2}{\alpha^2}\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$- \frac{\tau}{\alpha^2}\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2 - \frac{2}{\alpha^3}\psi(1/\alpha)$$

$$+ 2/\alpha^3\,\psi(y+1/\alpha) - 1/\alpha^4\,\psi'(1/\alpha) + 1/\alpha^4\,\psi'(y+1/\alpha)$$

(See 👓)

Since $\partial\eta/\partial\alpha = 0$, these are already the total derivatives for the extended GAM implementation. What we further need as per section 3.3.1 of Wood et al. (2016) for full Newton are:

$d^2\mathcal{L}/d\theta_1 d\alpha$, $d^2\mathcal{L}/d\theta_2 d\alpha$, $\frac{d^3\mathcal{L}}{d\gamma\, d\theta_1\, d\alpha}$, $\frac{d^3\mathcal{L}}{d\gamma\, d\theta_2\, d\alpha}$, $\frac{d^3\mathcal{L}}{d\gamma\, d\alpha^2}$, $\frac{d^4\mathcal{L}}{d\gamma^3\, d\alpha}$, $\frac{d^4\mathcal{L}}{d\gamma^2\, d\alpha^2}$

$d^4\mathcal{L}/d\gamma^3 d\theta_1 d\alpha$, $d^4\mathcal{L}/d\gamma^3 d\theta_2\, d\alpha$

(1) $$d^2\mathcal{L}/d\theta_1 d\alpha = \frac{d}{d\alpha}\left[\frac{d\mathcal{L}}{d\eta}\cdot d\eta/d\theta_1\right] = 0$$

(2) Similarly $d^2\mathcal{L}/d\theta_2\, d\alpha = 0$,

(3) $$d^3\mathcal{L}/d\gamma\, d\theta_1\, d\alpha = 0$$

(4) $$d^3\mathcal{L}/d\gamma\, d\theta_2\, d\alpha = 0$$

(5) $$d^3\mathcal{L}/d\gamma\, d\alpha^2 = \beta\Big\{-2\alpha y\beta^2 + 2y\beta - 2\beta^2\tau - \frac{\tau^2}{\alpha}\left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$+ \tau/\alpha\left(\beta^2 + 2/\alpha\,\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$- 2\frac{\beta\tau^2}{\alpha}\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+ \frac{2}{\alpha}\beta\tau\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$- \frac{2}{\alpha^2} \tau^3 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)^2$$

$$+ \frac{3\tau^2}{\alpha^2} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)^2$$

$$- \frac{\tau}{\alpha^2} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)^2 \Bigg\}$$

(See 👓)

⑥ $$\frac{d^4 l}{d\gamma^3 d\alpha} = \beta \Bigg\{ 6\alpha^3 y \beta^3 - 12\alpha^2 y \beta^2 + 6\alpha^2 \beta^3 \tau + 7\alpha y \beta$$

$$+ 2\alpha\beta^2\tau^2 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right) - 2\alpha\beta^2\tau \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- 10\alpha\beta^2\tau + 9\alpha\beta^3\tau^2 - 9\alpha\beta^3\tau - y$$

$$- 3\beta\tau^2 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right) + 3\beta\tau \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$+ 4\beta\tau + 6\beta^2\tau^3 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- 9\beta^2\tau^2 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- 9\beta^2\tau^2 + 3\beta^2\tau \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$+ 9\beta^2\tau + 6\beta^3\tau^3 - 9\beta^3\tau^2 + 3\beta^3\tau$$

$$+ \frac{\tau^2}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right) - \frac{\tau}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- \frac{6}{\alpha} \beta \tau^3 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$+ \frac{9\beta\tau^2}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- \frac{3\beta\tau}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$+ \frac{6\beta^2\tau^4}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- 12\beta^2\tau^3 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$+ \frac{7}{\alpha} \beta^2\tau^2 \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

$$- \frac{\beta^2\tau}{\alpha} \left(\beta - \frac{\log(1+\alpha e^\gamma)}{\alpha} \right)$$

(See 👓)

(7) $$\frac{d^4 C}{d\gamma^2 d\alpha^2} = \beta \Big\{ 6\alpha^2 y \beta^3 - 10\alpha y \beta^2 + 6\alpha\beta^3\tau + 4 y \beta$$

$$+ \beta\alpha^2 \left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$- \beta\tau \left(\beta^2 + 2\beta/\alpha - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$+ 4\beta^2\tau^2 \left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$- 4\beta^2\tau \left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$- 6\beta^2\tau + 6\beta^3\tau^2 - 6\beta^3\tau$$

$$- \frac{\tau^2}{\alpha}\left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$+ \frac{\tau}{\alpha}\left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$+ \frac{2\beta}{\alpha}\tau^3\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+ \frac{2}{\alpha}\beta\tau^3\left(\beta^2 + 2\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$- \frac{3}{\alpha}\beta\tau^2\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$- \frac{4}{\alpha}\beta\tau^2\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$- 3\beta\tau^2\left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha^2}\right)$$

$$+ \frac{\beta\tau}{\alpha}\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$+ \frac{4}{\alpha}\beta\tau\left(\beta - \frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+ \frac{\beta}{\alpha}\tau\left(\beta^2 + \frac{2}{\alpha}\beta - \frac{2\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+\frac{8\beta^2\tau^3}{\alpha}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$-\frac{12}{\alpha}\beta^2\tau^2\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+\frac{4\beta^2}{\alpha}\tau\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$-2\frac{\tau^3}{\alpha^2}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+\frac{3\tau^2}{\alpha^2}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$-\frac{\tau}{\alpha^2}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$+\frac{6\beta\tau^4}{\alpha^2}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$-\frac{12\beta\tau^3}{\alpha^2}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$+\frac{7}{\alpha^2}\beta\tau^2\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)^2$$

$$-\frac{\beta}{\alpha^2}\tau\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)\Bigg\}$$

(See 👓)

For quasi-Newton, we need:

① $$\partial^2 L/\partial\gamma\partial\alpha = \beta\left\{y\alpha\beta-y+\beta\tau+\frac{\tau^2}{\alpha}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)-\frac{\tau}{\alpha}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)\right\}$$

② $$\partial^3 L/\partial\gamma^2\partial\alpha = \beta\left\{-2y\alpha^2\beta^2+3y\alpha\beta-2\alpha\beta^2\tau-y-\beta\tau^2\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)\right.$$

$$+\beta\tau\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+2\beta\tau-2\beta^2\tau^2+2\beta^2\tau$$

$$+\frac{\tau^2}{\alpha}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$-\tau/\alpha\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$-\frac{2\beta\tau^3}{\alpha}\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$+\frac{3\beta}{\alpha}\tau^2\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)$$

$$-\frac{\beta}{\alpha}\tau\left(\beta-\frac{\log(1+\alpha e^{\gamma})}{\alpha}\right)\Bigg\} \qquad \square$$

(See 👓)

Todo interactions of τ and $\log(1+\alpha e^{\gamma})$

$$\lim_{\gamma\to-\infty}\tau\cdot\log(1+\alpha e^{\gamma})$$