

Negative Binomial PMF:

γ is counting y failures given r successes. Total # of trials

$$n = y + r$$

$$f(y; p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

where p is the probability of success in each trial

Lemma:

$$\binom{y+r-1}{r-1} = \binom{y+r-2}{y}$$

Proof:

$$\begin{aligned}\binom{y+r-1}{r-1} &= \frac{(y+r-1)!}{(r-1)!((y+r-1)-r+1)!} = \frac{(y+r-1)!}{(r-2)!y!} \\ &= \frac{(y+r-1)!}{y!(r-1)!} = \frac{(y+r-2)!}{y!(y+r-2-y)!} \\ &= \binom{y+r-2}{y}\end{aligned}$$

□

Therefore

$$f(y; p, r) = \binom{y+r-1}{y} p^r (1-p)^y$$

$$\text{Mean: } \frac{r(1-p)}{p} \approx \mu, \quad \text{Variance: } \frac{r(1-p)}{p^2} = \text{Var}(\gamma)$$

$$\Rightarrow \mu p = r - rp$$

$$\Rightarrow p(r+\mu) = r$$

$$\Rightarrow p = r/r+\mu$$

$$f(y; \mu, r) = \binom{y+r-1}{y} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

$$\Gamma(n) = (n-1)!$$

if $n \in \mathbb{N}, n \geq 1$
if n is a real
the $\Gamma(\cdot)$

$$= \frac{\Gamma(y+r)}{y! \Gamma(r)} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

is well defined

Define $r := \frac{1}{\alpha}$,

$$f(y; \mu, r) = \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha\mu}{1+\alpha\mu} \right)^y$$

\Rightarrow The single data log-likelihood is

$$\begin{aligned} l_i^B &= y_i \log \left(\frac{\alpha\mu}{1+\alpha\mu} \right) - \frac{1}{\alpha} \log(1+\alpha\mu_i) + \log \Gamma(y_i + \frac{1}{\alpha}) \\ &\quad - \log \Gamma(y_i + 1) - \log \Gamma(\frac{1}{\alpha}) \end{aligned} \quad \dots \quad (1)$$

Zero-truncated NB.

The probability of a zero count is

$$f(0; \mu, r) = \frac{\Gamma(\frac{1}{\alpha})}{\Gamma(1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} = \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}}$$

The zero-truncated density is

$$\begin{aligned} f(y|y>0, \mu, r) &= \frac{f(y; \mu, r)}{1 - f(0; \mu, r)} \\ &= \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha\mu}{1+\alpha\mu} \right)^y \\ &\quad / \left(1 - \left(\frac{1}{1+\alpha\mu} \right)^{\frac{1}{\alpha}} \right) \end{aligned}$$

This gives us the single data log-likelihood;

$$l_i^{ZT} = l_i^B - \log \left\{ 1 - \left(\frac{1}{1+\alpha\mu_i} \right)^{\frac{1}{\alpha}} \right\} \quad \dots \quad (2)$$

Where (2) is the ZTB single data log-likelihood as given in (1)

Zero Inflated Negative Binomial (Hurdle model in Tutz (2012))

We use a similar parameterization as in Hlood et al. (2017) SA I. written by a

$$g(y; \mu, r) = \begin{cases} 1-q & y=0 \\ q f(y|y>0, \mu, r) & \text{otherwise} \end{cases}$$

where q is the probability of a positive counts

Similar to Hlood et al. (2017), we adopt the unconstrained parameterization

$$\sigma = \log \mu, \eta = \log \hat{p} - \log(1-q)$$

This gives us the single-data log-likelihood (we drop the data index i)

$$l^{z^1} = \begin{cases} \log(1-q) & y=0 \\ \log q + \log f(y|y>0, \mu, r) & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{HB} - \log \left\{ 1 - (1+e^{-\eta})^{-\frac{1}{r}} \right\} & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

where $L^{AB} = L^{AB} - \log \{1 - ((1 + \alpha e^x)^{-\frac{1}{\alpha}})\}$ is the Z^{AB} log-likelihood.
and $L^{AB} = y \log \left(\frac{\alpha e^x}{1 + \alpha e^x} \right) - \frac{1}{\alpha} \log (1 + \alpha e^x) + \log \Gamma(y + \frac{1}{\alpha})$
 $- \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

$$\Rightarrow L^{Z^A} = y \log \left(\frac{\alpha e^x}{1 + \alpha e^x} \right) + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$$
 $+ \log \left\{ \frac{1}{(1 + \alpha e^x)^{\frac{1}{\alpha}}} \right\} + \log \left\{ \frac{1}{1 - \frac{1}{(1 + \alpha e^x)^{\frac{1}{\alpha}}}} \right\}$

looking at the last two terms basically is

$$\log \left\{ \frac{1}{(1 + \alpha e^x)^{\frac{1}{\alpha}}} + 1 \right\} = - \log \{ (1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \}$$
 $\Rightarrow L^{Z^A} = y \log \left\{ \frac{\alpha e^x}{1 + \alpha e^x} \right\} + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1)$
 $- \log \Gamma(\frac{1}{\alpha}) - \log \{ (1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \}$
 $= y \log \alpha + yx - y \log (1 + \alpha e^x) - \log \{ (1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \}$
 $+ \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

Here as well, some care is required to evaluate L^{Z^A} without unnecessary overflow, since it is easy for $1 - e^{-\frac{1}{\alpha}}$ and $(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1$ to evaluate to zero in finite precision arithmetic. Hence the limiting results $\log(1 - e^{-\frac{1}{\alpha}}) \rightarrow \log(e^n - e^{\frac{1}{\alpha}n} + e^{\frac{1}{\alpha}n}/6) \rightarrow y$ as $n \rightarrow \infty$.
Similar as $x \rightarrow -\infty$, $\log \{ (1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \} \rightarrow x$ and as $x \rightarrow \infty$
 $\log \{ (1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \} \rightarrow \log(e^x - 1) \xrightarrow{x \rightarrow \infty} e^x$ provided $x > -\infty$

Proof:

$$\textcircled{1} \quad \log(1 - e^{-\frac{1}{\alpha}}) \rightarrow \log(e^n - e^{\frac{1}{\alpha}n} + e^{\frac{1}{\alpha}n}/6) \rightarrow y \quad n \rightarrow \infty$$

as $n \rightarrow \infty$, $e^{-\frac{1}{\alpha}} \rightarrow 0$,
we know that $e^n \rightarrow 1 + n + \frac{n^2}{2} + \frac{n^3}{6}$ as $n \rightarrow \infty$

 $\Rightarrow e^{-\frac{1}{\alpha}} \rightarrow 1 - e^{-\frac{1}{\alpha}} + e^{\frac{1}{\alpha}n} - e^{\frac{1}{\alpha}n}/6$

$$= 0 \text{ as } \eta \rightarrow -\infty, \log(1 - e^{-e^\eta})$$

$$\rightarrow \log\left(1 - 1 + e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6}\right)$$

$$\text{as } \eta \rightarrow -\infty, e^{2\eta} \text{ and } e^{3\eta} \rightarrow 0 \text{ faster than } e^\eta$$

$$= \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$= 0 \quad \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$\rightarrow \log e^\eta \rightarrow \eta$$

② $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow r \text{ as } \alpha \rightarrow -\infty$

as $r \rightarrow -\infty$, $e^r \rightarrow 0 \Rightarrow \alpha e^r \rightarrow 0$ (fixed α)
 We know that $(1+x)^{\frac{1}{x}} \rightarrow 1 + \frac{1}{x}x$ as $x \rightarrow 0$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow 1 + \frac{1}{\alpha}(\alpha e^r) = 1 + e^r$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} - \log\{(1 + e^r) - 1\}$$

$$= \log(e^r) = r$$

as $r \rightarrow -\infty$

③ $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow e^r \text{ as } \alpha \rightarrow 0$ for fixed r

remember that $(1 + \alpha e^r)^{\frac{1}{\alpha}} = \left(1 + \frac{e^r}{r}\right)^r$,
 so as $\alpha \rightarrow 0$, $r \rightarrow \infty$

$$\text{and } \lim_{r \rightarrow \infty} \left(1 + \frac{e^r}{r}\right)^r = e^{e^r} \text{ for } r \in \mathbb{R}$$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow e^{e^r} \text{ as } \alpha \rightarrow 0$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow \log\{e^{e^r} - 1\} \text{ as } \alpha \rightarrow 0$$

also, for very large r , $e^{e^r} - 1$ is dominated by e^{e^r}

$$\Rightarrow \log\{e^{e^r} - 1\} \rightarrow \log e^{e^r} = e^r \quad \square$$

Implementation (Following Hood et al. (2017))

According to the framework proposed in Hood et al. (2017), we need the deviance and its derivatives to implement an extend GAM model (see section 3.3.1 of that paper);

Differentiating w.r.t. y_i , $\frac{\partial D_i}{\partial y_i}$, $\frac{\partial^2 D_i}{\partial y_i^2}$, $\frac{\partial^3 D_i}{\partial y_i^3}$, $\frac{\partial^4 D_i}{\partial y_i^4}$ (For \hat{p})

instead of m_i , $\frac{\partial^4 h_i}{\partial y_i^4}$, $\frac{\partial^5 D_i}{\partial m_i \partial y_i}$, $\frac{\partial^6 D_i}{\partial m_i^2 \partial y_i}$, $\frac{\partial^7 D_i}{\partial m_i^3 \partial y_i}$, $\frac{\partial^8 D_i}{\partial m_i^4 \partial y_i}$ as this is done in SAI. (For \hat{p} via full Newton).

As for the zip model, we first define the deviance as -2l for model estimation.

w.r.t. y for $y > 0$

$$\textcircled{1} \quad \frac{dL}{dy} = \frac{y}{e^y - 1}, \quad \textcircled{2} \quad \frac{d^2 L}{dy^2} = (1-e^y) \frac{dy}{dy} - \left(\frac{dy}{dy} \right)^2$$

exactly as for zip.

$$\textcircled{3} \quad \frac{d^3 L}{dy^3} = -e^y \frac{dy}{dy} + (1-e^y)^2 \frac{dy}{dy} - 3(1-e^y) \left(\frac{dy}{dy} \right)^2 + 2 \left(\frac{dy}{dy} \right)^3 \quad \text{and}$$

$$\textcircled{4} \quad \frac{d^4 L}{dy^4} = (3e^y - 4)e^y \frac{dy}{dy} + 4e^y \left(\frac{dy}{dy} \right)^2 + (1-e^y)^3 \frac{dy}{dy} - 7(1-e^y) \left(\frac{dy}{dy} \right)^2 + 12(1-e^y) \left(\frac{dy}{dy} \right)^3 - 6 \left(\frac{dy}{dy} \right)^4$$

See SAI of Hood et al. (2016) for dealing with these derivatives as $y \rightarrow \pm\infty$

w.r.t. γ for $y > 0$

$$\begin{aligned} \textcircled{5} \quad \frac{dy}{d\gamma} &= \frac{dL}{d\gamma} = \frac{d}{d\gamma} \left(y - y \log(1+\alpha e^\gamma) - \log \left\{ (1+\alpha e^\gamma)^{\frac{1}{\alpha}} - 1 \right\} \right) \\ &= y - \frac{y}{1+\alpha e^\gamma} - \frac{1-\alpha e^\gamma}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}} \cdot \frac{1}{\alpha} (1+\alpha e^\gamma)^{\frac{1}{\alpha}-1} \\ &= y - y \frac{e^\gamma}{1+\alpha e^\gamma} - \frac{e^\gamma (1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}} \end{aligned}$$

$$= y - \frac{y \alpha e^x}{1 + \alpha e^x} - \left(\frac{e^x}{1 + \alpha e^x} \right) \left(\frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1} \right)$$

Define $\beta := \frac{e^x}{1 + \alpha e^x}$, $\tau := \frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1}$

$$\Rightarrow \frac{dy}{dx} = y - y \alpha \beta - \beta \tau$$

$$\textcircled{2} \quad \frac{d^2 y}{dx^2} = \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

Proof: $\frac{d}{dx} \{ y - y \alpha \beta - \beta \tau \} = -y \alpha \frac{d\beta}{dx} - \frac{d\beta}{dx} \tau - \beta \frac{d\tau}{dx}$

But $\frac{d\beta}{dx} = \frac{(1 + \alpha e^x) e^x - e^x \cdot \alpha e^x}{(1 + \alpha e^x)^2}$

$$= \frac{e^x}{1 + \alpha e^x} - \frac{\alpha e^{2x}}{(1 + \alpha e^x)^2} = \beta - \alpha \beta^2$$

and $\frac{d\tau}{dx} = \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}$

$$= \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}{\left((1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right)^2}$$

$$= \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]} - \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}$$

$$= \beta \tau - \beta \tau^2$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y \alpha (\beta - \alpha \beta^2) - \beta \tau + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= y \alpha^2 \beta^2 - y \alpha \beta + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

□

$$\textcircled{3} \quad \frac{d^3L}{dt^3} = p \left\{ -8yx^3\beta^2 + 3yx^2\beta - 2x^2\beta^2z - yx + 3\alpha\beta z - 3\alpha\beta^2z^2 + 3\alpha\beta^2t - t + 3\beta t^2 - 3\beta z - 2\beta^2z^3 + 3\beta^2t^2 - \beta^2z^2y \right\}$$

Proof. Using a numerical linear algebra package, set ∞ . □

$$\textcircled{4} \quad \frac{d^4L}{dt^4} = p \left\{ 6yx^4\beta^3 - 12yx^3\beta^2 + 6x^2\beta^3z + 7yx^2\beta - 12x^2\beta^2z + 11x^2\beta^3z^2 - 11x^2\beta^3t - yx + 7\alpha\beta z - 18\alpha\beta^2z^2 + 18\alpha\beta^2t + 12\alpha\beta^3z^3 - 18\alpha\beta^3t^2 + 6\alpha\beta^3z^4 - z + 7\beta t^2 - 7\beta z - 12\beta^2z^3 + 18\beta^2t^2 - 6\beta^2z + 6\beta^3t^4 - 12\beta^3z^3 + 7\beta^3t^2 - \beta^3z^2 \right\}$$

Proof. ∞ □

As with L^2 , some care is required to ensure that the derivatives evaluate accurately without overflow even as with a range of r and κ as possible. As $t \rightarrow \infty$, $\beta = \frac{1}{e^{-r} + \kappa} \rightarrow \frac{1}{\kappa}$ for fixed κ , while

$$t = \frac{1}{1 - (1 + \kappa e^r)^{-1/\kappa}} \rightarrow 1, \quad \text{since } 1 + \kappa e^r \rightarrow \infty \Rightarrow (1 + \kappa e^r)^{-1/\kappa} \rightarrow 0$$

for fixed κ

whereas as $r \rightarrow -\infty$, $\beta \rightarrow 0$ for fixed κ , while $e^r \rightarrow 0$
 $\Rightarrow (1 + \kappa e^r)^{-1/\kappa} \rightarrow 1 + \frac{1}{\kappa e^r} = 1 + e^r \rightarrow 0$, $t = \frac{(1 + \kappa e^r)^{-1/\kappa}}{(1 + \kappa e^r)^{1/\kappa}}$
first order Taylor expansion about $r = \kappa e^r = 0$

$$\begin{aligned} &\rightarrow \frac{1 + e^r}{1 + e^r - 1} = \frac{1 + e^r}{e^r} \\ &= e^{-r} + 1 \rightarrow e^{-r} \\ &\approx 0(e^{-r}) \end{aligned}$$

* How do we deal with products: $pz, p\bar{z}, p^2z$ etc. what does it converge to

$$\begin{aligned} p\bar{z} &= \frac{e^r}{1 + \kappa e^r} \cdot \frac{1}{1 - (1 + \kappa e^r)^{-1/\kappa}} \xrightarrow{r \rightarrow -\infty} \frac{1}{e^{-r} + \kappa} (e^{-r} + 1) \\ &= \frac{e^{-r} + 1}{e^{-r} + \kappa} = \frac{e^{-r}}{e^{-r} + \kappa} + \frac{1}{e^{-r} + \kappa} \\ &= \frac{1}{1 + e^r} + \frac{1}{e^{-r} + \kappa} \\ &\rightarrow 1 + 0 = 1 \end{aligned}$$

For extended GAMs in which η is a Fn of γ and extra parameters θ e.g;

$$\eta = \theta_1 + e^{\theta_2} \gamma \quad (\text{as for the zip model in } \text{Hood et al. (2016)})$$

$$\frac{d\eta}{d\theta_1} = 1, \quad \frac{d\eta}{d\theta_2} = e^{\theta_2} \gamma, \quad \frac{d^2\eta}{d\theta_1 d\theta_2} = 0$$

$$\frac{d^2\eta}{d\theta_1^2} = 0, \quad \frac{d^2\eta}{d\theta_2^2} = e^{2\theta_2} \gamma$$

In this setting

$$\begin{aligned} \frac{dL}{d\gamma} &= \frac{\partial L}{\partial \gamma} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial \eta}{\partial \gamma}, \quad \frac{d^2L}{d\gamma^2} = \frac{\partial^2 L}{\partial \gamma^2} + \underbrace{\frac{\partial^2 L}{\partial \gamma \partial \eta} \cdot \frac{\partial \eta}{\partial \gamma}}_{=0} \\ &\quad + \underbrace{\frac{\partial^2 L}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial \gamma}\right)^2}_{=0} \\ &\quad + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \gamma^2} \\ &= \frac{\partial^2 L}{\partial \gamma^2} + \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \eta}{\partial \gamma}\right)^2 \\ &\quad + \frac{\partial L}{\partial \eta} \frac{\partial^2 \eta}{\partial \gamma^2} \end{aligned}$$

$$\begin{aligned} \frac{dL}{d\theta_i} &= \frac{\partial L}{\partial \eta} \cdot \frac{\partial \eta}{\partial \theta_i}, \quad \frac{d^2L}{d\gamma d\theta_i} = \underbrace{\frac{\partial^2 L}{\partial \gamma \partial \eta} \cdot \frac{\partial \eta}{\partial \theta_i}}_{=0} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} + \underbrace{\frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i}}_{-\frac{\partial \eta}{\partial \theta_i}} \\ &= \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial \eta}{\partial \gamma} + \frac{\partial L}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 L}{d\theta_i^2} &= \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \left(\frac{\partial \eta}{\partial \gamma}\right)^2 + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^2 \eta}{\partial \theta_i^2} \cdot \frac{\partial \eta}{\partial \gamma} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} \\ &\quad + \frac{\partial^4 L}{\partial \eta^4} \cdot \frac{\partial^3 \eta}{\partial \theta_i^2 \partial \gamma} \end{aligned}$$

$$= \frac{\partial^3 L}{\partial \gamma^3} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial \gamma} \right)^2 + \frac{\partial^2 L}{\partial \eta^2} \left(2 \frac{\partial \eta}{\partial \gamma} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \gamma^2} \right) \\ + \frac{\partial L}{\partial \gamma} - \frac{\partial^3 \eta}{\partial \theta_i^2}$$

$$\frac{\partial^3 L}{\partial \gamma^3} = \frac{\partial^3 L}{\partial \gamma^3} + \frac{\dot{\gamma}^2}{\partial \gamma^3} \cdot \frac{\partial y}{\partial \gamma} \cdot \left(\frac{\partial y}{\partial \gamma} \right)^2 + \frac{\ddot{\gamma}^2}{\partial \gamma^2} \left[\frac{\partial^2 y}{\partial \gamma^2} \cdot \frac{\partial^2 y}{\partial \gamma^2} \right] \\ + \frac{\partial^2 L}{\partial \gamma^2} \cdot \frac{\partial \dot{y}}{\partial \gamma^2} + \frac{\partial L}{\partial \gamma} \cdot \frac{\partial \dot{y}}{\partial \gamma^3}$$

$$= \frac{\partial^3 L}{\partial r^3} + \frac{\partial^2 L}{\partial y^2} \left(\frac{\partial y}{\partial r} \right)^3 + 3 \frac{\partial^2 L}{\partial y^2} \frac{\partial^2 y}{\partial r^2} \frac{\partial y}{\partial r} + \frac{\partial L}{\partial y} \cdot \frac{\partial^3 y}{\partial r^3}$$

$$\frac{d^k l}{d\theta_i d\theta_j} = \frac{\partial^2 l}{\partial \eta^2} \frac{\partial \eta}{\partial \theta_j} \frac{\partial \eta}{\partial \theta_i} + \frac{\partial l}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}$$

$$\begin{aligned} \frac{\partial^3 L}{\partial r \partial \theta_i \partial \theta_j} &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_j} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial r \partial \theta_j} \right] \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial r} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r \partial \theta_i \partial \theta_j} \\ &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial r \partial \theta_j} \right. \\ &\quad \left. + \frac{\partial \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial r} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^3 \eta}{\partial r \partial \theta_i \partial \theta_j} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^4 L}{\partial r \partial T \partial \theta_i \partial \theta_j} &= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^3 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + 2 \frac{\partial \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j \partial r} \right] \\
 &+ \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_j} \left(2 \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i^2} \cdot \frac{\partial \eta}{\partial r} \right) \\
 &+ \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \partial^2 \eta}{\partial r \partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j \partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} \right) \\
 &+ \frac{\partial^2 \eta}{\partial r^2} \left(\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \theta_i \partial r} \frac{\partial^2 \eta}{\partial \theta_j \partial r} + \frac{\partial^2 \eta}{\partial \theta_j \partial r} \frac{\partial \eta}{\partial r} \right)
 \end{aligned}$$

$$+ \frac{\partial^2 \gamma}{\partial y^2} \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial^3 \gamma}{\partial x^2 \partial z^2} + \frac{\partial^2 \gamma}{\partial z^2} \frac{\partial^3 \gamma}{\partial x^2 \partial z^2 \partial y^2}$$

$$= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^2$$

$$+ \frac{\partial^4 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + 2 \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \right]$$

$$+ \frac{1}{2} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial r^2}$$

$$+ \frac{\partial^4 L}{\partial \eta^2} \left[\frac{2 \partial^3 \eta}{\partial \theta_i \partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{2 \partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} + \frac{\partial^3 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} + \frac{\partial^3 \eta}{\partial r^2 \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j}$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i \partial \theta_j}$$

$$\frac{\partial^4 L}{\partial \theta_i^3} = \frac{\partial^4 L}{\partial \theta_i^3 \partial \eta} - \frac{\partial^2 \eta}{\partial \theta_i} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^3 L}{\partial \eta^3} - 3 \left(\frac{\partial \eta}{\partial r} \right) \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i}$$

$\xrightarrow{=0}$

$$+ \frac{3 \partial^3 L}{\partial \eta^3} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} \right.$$

$$\left. + \frac{\partial^2 \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^3 \eta}{\partial r^3} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i}$$

$$= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + 3 \frac{\partial^3 L}{\partial \eta^3} \left[\left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} + \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial \theta_i \partial \theta_i} + \frac{\partial^3 \eta}{\partial r^3} \frac{\partial \eta}{\partial \theta_i} \right]$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i}$$

$$\frac{\partial^4 L}{\partial r^4} = \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial r} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^2 L}{\partial \eta^2} \cdot 3 \left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial r^2}$$

$$+ 3 \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial r} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial r^2} + \right.$$

$$\left. \frac{\partial^2 \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^3} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial r^2} + \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2}$$

$$\begin{aligned}
 &= \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial y^4} \left(\frac{\partial y}{\partial r} \right)^4 + 6 \frac{\partial^3 L}{\partial y^3} \left(\frac{\partial y}{\partial r} \right)^2 \frac{\partial^2 L}{\partial r^2} \\
 &\quad + \frac{\partial^2 L}{\partial y^2} \left[3 \left(\frac{\partial^2 y}{\partial r^2} \right)^2 + 4 \frac{\partial y}{\partial r} \frac{\partial^3 y}{\partial r^3} \right] + \frac{\partial L}{\partial y} \frac{\partial^4 y}{\partial r^4}
 \end{aligned}$$

To do: Derivative w.r.t. α .

Recall that for $y > 0$,

$$\begin{aligned}
 L^2 &= \log(1 - e^{-\alpha y}) + y \log \alpha + y r - y \log(1 + \alpha e^r) \\
 &\quad - \log \left\{ (1 + \alpha e^r)^{\frac{y}{\alpha}} - 1 \right\} + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y + 1)
 \end{aligned}$$

Also that if $x = (1 + \alpha e^r)^{\frac{y}{\alpha}}$,

$$\begin{aligned}
 &\Rightarrow \log x = \frac{y}{\alpha} \log(1 + \alpha e^r) \\
 &\Rightarrow \frac{1}{x} \frac{dx}{dx} = -\frac{1}{\alpha^2} \log(1 + \alpha e^r) + \frac{1}{\alpha} \left(\frac{1}{1 + \alpha e^r} \cdot e^r \right)
 \end{aligned}$$

$$\Rightarrow \frac{1}{x} \frac{dy}{dx} = \frac{e^r}{\alpha(1 + \alpha e^r)} - \frac{1}{\alpha^2} \log(1 + \alpha e^r)$$

$$\Rightarrow \frac{dy}{dx} = (1 + \alpha e^r)^{\frac{y}{\alpha}} \left\{ \frac{e^r}{\alpha(1 + \alpha e^r)} - \frac{1}{\alpha^2} \log(1 + \alpha e^r) \right\}$$

also $\frac{d}{y} \log \Gamma(y) := \varphi(y)$, where φ is the digamma fn.

Now $\frac{dy}{dx} = 0 + y_{\alpha} + 0 - \frac{y e^r}{1 + \alpha e^r} - \frac{1}{[(1 + \alpha e^r)^{\frac{y}{\alpha}} - 1]} (1 + \alpha e^r)^{\frac{y}{\alpha}}$

$$\begin{aligned}
 &\quad \left\{ \frac{e^r}{\alpha(1 + \alpha e^r)} - \frac{1}{\alpha^2} \log(1 + \alpha e^r) \right\} \\
 &\quad - \frac{1}{\alpha^2} \varphi(y + \frac{1}{\alpha}) + \frac{1}{\alpha^2} \varphi(\frac{1}{\alpha}) \\
 &= y_{\alpha} - y_{\beta} - 2 \left\{ \beta_{\alpha} - \frac{1}{\alpha^2} \log(1 + \alpha e^r) \right\} \\
 &\quad - \frac{1}{\alpha^2} \varphi(y + \frac{1}{\alpha}) + \frac{1}{\alpha^2} \varphi(\frac{1}{\alpha}).
 \end{aligned}$$

$$\frac{dp}{d\alpha} = \frac{(1 + \alpha e^r) 0 - e^r e^r}{(1 + \alpha e^r)^2} = \frac{-e^{2r}}{(1 + \alpha e^r)^2} = -p^2$$

without manually computing $\frac{d^2L}{d\alpha^2}$, we write

$$\begin{aligned} \frac{d^2L}{d\alpha^2} = & y\beta^2 + \frac{\tau}{\kappa} \left\{ \beta^2 + \frac{2}{\kappa}\beta - \frac{2}{\alpha^2} \log(1+\alpha e^\tau) \right\} \\ & - \frac{y}{\alpha^2} + \frac{\tau^2}{\kappa^2} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2 \\ & - \frac{2}{\alpha^2} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2 - \frac{2}{\alpha^3} \varphi(\frac{y}{\kappa}) \\ & + \frac{2}{\kappa^3} \varphi(y + \frac{1}{\kappa}) - \frac{1}{\kappa^4} \varphi'(\frac{y}{\kappa}) + \frac{1}{\kappa^4} \varphi'(y + \frac{1}{\kappa}) \end{aligned}$$

(See 

Since $\frac{\partial y}{\partial \kappa} = 0$, these are already the total derivatives for the extended GAM implementation. What we further need as per section 3.3.1 of Wood et al (2016) for full reflux are:

$$\frac{d^2L}{d\theta_1 d\alpha}, \frac{d^2L}{d\theta_2 d\alpha}, \frac{d^2L}{d\theta d\alpha}, \frac{d^2L}{d\theta d\theta_1 d\alpha}, \frac{d^2L}{d\theta d\theta_2 d\alpha}, \frac{d^2L}{d\theta^3 d\alpha}, \frac{d^2L}{d\theta^2 d\alpha^2}$$

$$\frac{d^2L}{d\theta^3 d\theta_1 d\alpha}, \frac{d^2L}{d\theta^3 d\theta_2 d\alpha}$$

$$\textcircled{1} \quad \frac{d^2L}{d\theta_1 d\alpha} = \frac{d}{d\alpha} \left[\frac{d\theta_1}{dy} \cdot \frac{dy}{d\theta_1} \right] = 0$$

$$\textcircled{2} \quad \text{Similarly } \frac{d^2L}{d\theta_2 d\alpha} = 0$$

$$\textcircled{3} \quad \frac{d^2L}{d\theta d\theta_1 d\alpha} = 0$$

$$\textcircled{4} \quad \frac{d^2L}{d\theta d\theta_2 d\alpha} = 0$$

$$\begin{aligned} \textcircled{5} \quad \frac{d^2L}{d\theta d\alpha^2} = & \beta^2 - 2\kappa y\beta^2 + 2y\beta - 2\beta^2\tau - \frac{2^2}{\kappa} \left(\beta^2 + \frac{2}{\kappa}\beta - \frac{2\log(1+\alpha e^\tau)}{\alpha^2} \right) \\ & + \frac{2}{\kappa} \left(\beta^2 + \frac{2}{\kappa}\beta - \frac{2\log(1+\alpha e^\tau)}{\alpha^2} \right) \\ & - 2 \frac{\beta\tau^2}{\kappa} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right) \\ & + \frac{2}{\kappa} \beta\tau \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{2}{\alpha^2} z^3 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right)^2 \\
 & + \frac{3}{\alpha^2} z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right)^2 \\
 & - \frac{2}{\alpha^2} \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right)^2 \}
 \end{aligned}$$

(See )

$$\begin{aligned}
 \textcircled{6} \quad \frac{d^4y}{dr^3 d\alpha} = \beta \Bigg\{ & 6\alpha^3 y \beta^3 - [2\alpha^2 y \beta^2 + 6\alpha^2 \beta^3] z f + 2\alpha y \beta \\
 & + 2\alpha \beta^2 z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) - 2\alpha \beta^2 z \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - 6\alpha \beta^2 z f + g \alpha \beta^3 z^2 - g \alpha \beta^3 z - y \\
 & - 3\beta z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) + 3\beta z \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & + 4\beta z + 6\beta^2 z^3 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - g \beta^2 z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - g \beta^2 z^2 + 3\beta^2 z \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & + g \beta^2 z + 6\beta^3 z^3 - g \beta^3 z^2 + 3\beta^3 z \\
 & + \frac{z^2}{\alpha} \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) - \frac{z}{\alpha} \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - \frac{6}{\alpha} \beta z^3 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & + g \beta z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - \frac{3}{\alpha} \beta z \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & + 6\beta^2 z^4 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - 12\beta^2 z^3 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & + \frac{7}{\alpha} \beta^2 z^2 \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right) \\
 & - \frac{\beta^2 z}{\alpha} \left(\beta - \log \frac{(1+\alpha e^z)}{\alpha} \right)
 \end{aligned}$$

(See )

$$\begin{aligned}
 \textcircled{4} \quad \frac{d^4 C}{dx^2 d\alpha^2} &= \beta \left\{ 6x^2 y \beta^3 - 6xy\beta^2 + 6x\beta^3 \tau + 4y\beta \right. \\
 &\quad + \beta x^2 \left(\beta^2 + \frac{2}{\alpha} \beta - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right) \\
 &\quad - \beta \tau \left(\beta^2 + 2\beta/\alpha - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right) \\
 &\quad + 4\beta^2 \tau^2 \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right) \\
 &\quad - 4\beta^2 \tau \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right) \\
 &\quad - 6\beta^2 \tau + 6\beta^3 \tau^2 - 6\beta^3 \tau \\
 &\quad - \frac{x^2}{\alpha} \left(\beta^2 + \frac{2}{\alpha} \beta - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right) \\
 &\quad + \frac{2\beta \tau^3}{\alpha} \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right) \\
 &\quad + \frac{2\beta \tau^3}{\alpha} \left(\beta^2 + 2\beta - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right) \\
 &\quad - \frac{3\beta \tau^2}{\alpha} \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right)^2 \\
 &\quad - \frac{4\beta \tau^2}{\alpha} \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right) \\
 &\quad - \frac{3\beta \tau^2}{\alpha} \left(\beta^2 + \frac{2}{\alpha} \beta - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right) \\
 &\quad + \frac{\beta \tau}{\alpha} \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right)^2 \\
 &\quad + \frac{4\beta \tau}{\alpha} \left(\beta - \frac{\log(1+\alpha x^2)}{\alpha} \right) \\
 &\quad + \frac{\beta \tau}{\alpha} \left(\beta^2 + \frac{2}{\alpha} \beta - 2 \frac{\log(1+\alpha x^2)}{\alpha^2} \right)
 \end{aligned}$$

$$+ \frac{8\beta^2\tau^3}{\kappa} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)$$

$$- \frac{12}{\kappa} \beta^2 \tau^2 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)$$

$$+ 4 \frac{\beta^2}{\kappa} \tau \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)$$

$$- 2 \frac{\tau^3}{\kappa^2} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)$$

$$+ \frac{3}{\kappa^2} \tau^2 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2$$

$$- \frac{\tau}{\kappa^2} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2$$

$$+ \frac{6\beta}{\kappa^2} \tau^4 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2$$

$$- \frac{12\beta}{\kappa^2} \tau^3 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2$$

$$+ \frac{7}{\kappa^2} \beta \tau^2 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2$$

$$- \frac{\beta}{\kappa^2} \tau \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right)^2 \Big]$$

(See )

For quasi-Newton, we need:

$$\textcircled{1} \quad \frac{\partial^2 L}{\partial y \partial \alpha} = \beta \left\{ y \times \beta^{-y} + \beta \tau + \frac{\tau^2}{\kappa} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right) \right. \\ \left. - \frac{\tau}{\kappa} \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right) \right\}$$

$$\textcircled{2} \quad \frac{\partial^2 L}{\partial \tau^2 \partial \alpha} = \beta \left\{ -2y \times \beta^{-y} + 3y \times \beta - 2\kappa \beta^2 \tau - y \right. \\ \left. - \beta \tau^2 \left(\beta - \frac{\log(1+\alpha e^\tau)}{\kappa} \right) \right\}$$

$$+ \beta \tau \left(\beta - \frac{\log(1+\alpha e^r)}{\alpha} \right)$$

$$+ 2\beta \tau - 2\beta^2 \tau^2 + 2\beta^2 \tau$$

$$+ \frac{\tau^2}{\alpha} \left(\beta - \frac{\log(1+\alpha e^r)}{\alpha} \right)$$

$$- \frac{\tau}{\alpha} \left(\beta - \frac{\log(1+\alpha e^r)}{\alpha} \right)$$

$$- \frac{2\beta}{\alpha} \tau^2 \left(\beta - \frac{\log(1+\alpha e^r)}{\alpha} \right)$$

$$+ \frac{3\beta}{\alpha} \tau^2 \left(\beta - \frac{\log(1+\alpha e^r)}{\alpha} \right)$$

$$- \frac{\beta}{\alpha} \tau \left(\beta - \log \frac{(1+\alpha e^r)}{\alpha} \right) \}$$

(See )

□