Negative Binomial PMF:

Y is counting y failures given r successes. Total # of trials $n = y + r$

$$f(y; p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

where p is the probability of success in each trial

Lemma: $$\binom{y+r-1}{r-1} = \binom{y+r-1}{y}$$

proof:

$$\binom{y+r-1}{r-1} = \frac{(y+r-1)!}{(r-1)!((y+r-1)-r-1)!} = \frac{(y+r-1)!}{(r-1)!\, y!}$$

$$= \frac{(y+r-1)!}{y!\,(r-1)!} = \frac{(y+r-1)!}{y!\,(y+r-1-y)!}$$

$$= \binom{y+r-1}{y}$$

$\square$

$\alpha(\tilde{c} - c)$

$\overline{\alpha c} - \underbrace{\alpha c}_{\text{Der. resid}}$

- s-ll

Therefore

$$f(y; p, r) = \binom{y+r-1}{y} p^r (1-p)^y$$

Mean: $\dfrac{r(1-p)}{p} =: \mu$, **Variance:** $\dfrac{r(1-p)}{p^2} = Var(Y)$

$\Rightarrow \quad \mu p = r - rp$

$\Rightarrow \quad p(r+\mu) = r$

$\Rightarrow \quad p = r/_{r+\mu} \Rightarrow Var(Y) = \mu/_{r/_{r+\mu}} = \mu/_r (r+\mu) = \mu + \frac{\mu^2}{r}$

$$f(y; \mu, r) = \binom{y+r-1}{y} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

$$= \frac{\Gamma(y+r)}{y!\,\Gamma(r)} \left(\frac{r}{r+\mu}\right)^r \left(\frac{\mu}{r+\mu}\right)^y$$

$\Gamma(n) = (n-1)!$
if $n \in \mathbb{N}, n \geq 1$
if n is a real # the $\Gamma(\cdot)$

is well defined

Define $r := 1/\alpha$, $\Rightarrow p = \frac{1}{\alpha}\left(\frac{1}{1/\alpha} + \mu\right) = 1/(1+\alpha\mu)$

$$f(y;\mu,r) = \frac{\Gamma(y+1/\alpha)}{\Gamma(y+1)\,\Gamma(1/\alpha)} \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha} \left(\frac{\alpha\mu}{1+\alpha\mu}\right)^{y}$$

$\Rightarrow$ The single data log-likelihood is

$$l_i^{NB} = y_i \log\left(\frac{\alpha\mu_i}{1+\alpha\mu_i}\right) - 1/\alpha \log(1+\alpha\mu_i) + \log\Gamma(y_i + 1/\alpha)$$
$$- \log\Gamma(y_i+1) - \log\Gamma(1/\alpha) \qquad \cdots (1)$$

Zero-truncated NB.

The probability of a zero count is

$$f(0;\mu,r) = \frac{\Gamma(1/\alpha)}{\Gamma(1)\,\Gamma(1/\alpha)} \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha} = \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha}$$

The zero-truncated density is

$$f(y \mid y>0,\mu,r) = \frac{f(y;\mu,r)}{1-f(0,\mu,r)}$$

$$= \frac{\frac{\Gamma(y+1/\alpha)}{\Gamma(y+1)\,\Gamma(1/\alpha)} \left(\frac{1}{1+\alpha\mu}\right)^{1/\alpha} \left(\frac{\alpha\mu}{1+\alpha\mu}\right)^{y}}{1-\left(1/(1+\alpha\mu)\right)^{1/\alpha}}$$

This gives us the single data log-likelihood;

$$l_i^{ZT} = l_i^{NB} - \log\left\{1-(1+\alpha\mu_i)^{-1/\alpha}\right\} \qquad \cdots (2)$$

Where (2) is the NB single data log-likelihood as given in (1)

Zero Inflated Negative Binomial (Hurdle model in Tutz (2012))

erated

We use a similar parameterization as in Hood et al. (2017) SA I. y a

$$
g(y; \mu, r) = \begin{cases} 1-q & y=0 \\ q\, f(y \mid y>0, \mu, r) & \text{otherwise} \end{cases}
$$

where q is the probability of a positive counts

Similar to Hood et al. (2017), we adopt the unconstrained parameterization

$$
\gamma = \log \mu, \quad \eta = \log\{-\log(1-q)\}.
$$

This gives us the single-data log-likelihood (we drop the data index i)

$$
l^{ZI} = \begin{cases} \log(1-q) & y=0 \\ \log q + \log f(y \mid y>0, \mu, r) & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{ZT} & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{NB} - \log\left\{1-\left(1+\alpha e^{\gamma}\right)^{-1/\alpha}\right\} & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} -e^{\eta} & y=0 \\ \log\left(1-e^{-e^{\eta}}\right) + l^{ZT} & \text{otherwise} \end{cases}
$$

where $l^{ZT} = l^{AB} - \log\{1 - (1+\alpha e^{\gamma})^{-1/\alpha}\}$ is the ZTAB log-likelihood.
and $l^{AB} = y \log\left(\alpha e^{\gamma} / 1+\alpha e^{\gamma}\right) - \frac{1}{\alpha}\log(1+\alpha e^{\gamma}) + \log\Gamma(y + 1/\alpha)$

$$- \log\Gamma(y+1) - \log\Gamma(1/\alpha)$$

$$\Rightarrow l^{ZT} = y\log\left(\alpha e^{\gamma} / 1+\alpha e^{\gamma}\right) + \log\Gamma(y+1/\alpha) - \log\Gamma(y+1) - \log\Gamma(1/\alpha)$$
$$+ \log\left\{\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}}\right\} + \log\left\{\frac{1}{1-\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}}}\right\}$$

looking at the last two terms basically is

$$\log\left\{\frac{1}{(1+\alpha e^{\gamma})^{1/\alpha}} - 1\right\} = -\log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\}$$

$$\Rightarrow l^{ZT} = y\log\left\{\alpha e^{\gamma} / 1+\alpha e^{\gamma}\right\} + \log\Gamma(y+1/\alpha) - \log\Gamma(y+1)$$
$$- \log\Gamma(1/\alpha) - \log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\}$$

$$= y\log\alpha + y\gamma - y\log(1+\alpha e^{\gamma}) - \log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\}$$
$$+ \log\Gamma(y+1/\alpha) - \log\Gamma(y+1) - \log\Gamma(1/\alpha)$$

Here as well, some care is required to evaluate l^{ZT} without unnecessary overflow, since it is easy for $1-e^{-e^{\eta}}$ and $(1+\alpha e^{\gamma})^{1/\alpha} - 1$ to evaluate to zero in finite precision arithmetic. Hence the limiting results $\log(1-e^{-e^{\eta}}) \to \log(e^{\eta} - e^{2\eta} + e^{3\eta}/6) \to \eta$ as $\eta \to -\infty$. Similar as $\gamma \to -\infty$, $\log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\}^{2} \to \gamma$ and as $\alpha \to 0$ $\log\{(1+\alpha e^{\gamma})^{1/\alpha} - 1\} \to \log(e^{e^{\gamma}} - 1) \to e^{\gamma}$
provided $\gamma \not\to -\infty$

Implemented taking α as fixed.

Proof:

(1) $\log(1-e^{-e^{\eta}}) \to \log\left(e^{\eta} - e^{2\eta}/2 + e^{3\eta}/6\right) \to \eta$ $n \to -\infty$

as $\eta \to -\infty$, $e^{\eta} \to 0$,

we know that $e^{x} \to 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ as $x \to 0$

$$\Rightarrow e^{-e^{\eta}} \to 1 - e^{\eta} + e^{2\eta}/2 - e^{3\eta}/6$$

$\Rightarrow$ as $\eta \to -\infty$, $\log(1-e^{-e^{\eta}})$

$\to \log\left(1-1+e^{\eta}-\frac{e^{2\eta}}{2}+\frac{e^{3\eta}}{6}\right)$

$= \log\left(e^{\eta}-\frac{e^{2\eta}}{2}+\frac{e^{3\eta}}{6}\right)$

as $\eta \to -\infty$ $e^{2\eta}$ and $e^{3\eta} \to 0$ faster than e^{η}

$\Rightarrow \log\left(e^{\eta}-e^{2\eta}/2+\frac{e^{3\eta}}{6}\right)$

$\to \log e^{\eta} \to \eta$

② $\log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to \gamma$ as $\gamma \to -\infty$

as $\gamma \to -\infty$, $e^{\gamma} \to 0 \Rightarrow \alpha e^{\gamma} \to 0$ (fixed α)

We know that $(1+x)^{1/\alpha} \to 1+\frac{1}{\alpha}x$ as $x \to 0$

$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \to 1+\frac{1}{\alpha}(\alpha e^{\gamma}) = 1+e^{\gamma}$

$\Rightarrow \log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to \log\{(1+e^{\gamma})-1\}$

$= \log(e^{\gamma}) = \gamma$

as $\gamma \to -\infty$

$e^{x} := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$

③ $\log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to e^{\gamma}$ as $\alpha \to 0$ for fixed γ

remember that $(1+\alpha e^{\gamma})^{1/\alpha} = \left(1+\frac{e^{\gamma}}{r}\right)^{r}$,

so as $\alpha \to 0$, $r \to \infty$

and $\lim_{r\to\infty}\left(1+\frac{e^{\gamma}}{r}\right)^{r} = e^{e^{\gamma}}$ for $r \in \mathbb{N}_0$

$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \to e^{e^{\gamma}}$ as $\alpha \to 0$

$\Rightarrow \log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\} \to \log\{e^{e^{\gamma}}-1\}$ as $\alpha \to 0$

also for very large γ, $e^{e^{\gamma}}-1$ is dominated by $e^{e^{\gamma}}$

$\Rightarrow \log\{e^{e^{\gamma}}-1\} \to \log e^{e^{\gamma}} = e^{\gamma}$ $\square$

On the expectation of y in the ZINB.

Recall the model's density

$$g(y;\mu,r) = \begin{cases} 1-q & y=0 \\ q\, f(y\mid y>0, \mu, r) & y>0 \end{cases}$$

where $f(y\mid y>0,\mu,r) = \dfrac{f(y;\mu,r)}{1-f(0;\mu,r)}$

Therefore, the model adopts the form of hurdle models discussed in Tuty, section 7.8.

Let $v := \dfrac{P(y>0)}{1-f(0;\mu,r)} \overset{\text{ZINB}}{=} q\tau$, since in ZINB $P(y>0) = q$ and $f(0;\mu,r) = \dfrac{1}{(1+\alpha\mu)^{1/\alpha}}$, $\mu = e^{\gamma}$

Therefore, the mean is given by

$$\mathbb{E}[y] = \sum_{s=1}^{\infty} s f(s;\mu,r)\, v = \sum_{i=1}^{\infty} s f(s;\mu,r) \cdot \overbrace{\frac{\mathbb{P}(y>0)}{1-f(0;\mu,r)}}^{\text{doesn't depend on the value of } s}$$

$$= \mathbb{P}(y>0) \sum_{s=1}^{\infty} s \frac{f(s;\mu,r)}{1-f(0;\mu,r)}$$

$$= \mathbb{P}(y>0)\, \mathbb{E}[y \mid y>0, \mu, r]$$

(y_2 the underlying NB r.v.) or $= \dfrac{P(y>0)}{1-f(0;\mu,r)}\, \mathbb{E}[y_2]$ ← doesn't depend on s.

For the ZINB model, which as $f(s;\mu,r)$ is the PMF of a NB distribution with mean μ;

$$\Rightarrow \quad \mathbb{E}[y] = v\mu = q\tau\mu = q\tau e^{\gamma} \quad (\mu = e^{\gamma}).$$

where $\tau = \dfrac{1}{1-(1+\alpha\mu)^{-1/\alpha}} = \dfrac{1}{1-(1+\alpha e^{\gamma})^{-1/\alpha}}$

also $\lim_{r \to -\infty} \tau_\ell^r = \lim_{r \to -\infty} e^r / \tfrac{1}{\tau}$

(L'Hopital)

$$= \lim_{r \to -\infty} \frac{e^r}{\frac{1}{\alpha} \alpha e^r (1 + \alpha e^r)^{-1/\alpha - 1}} \quad \text{(for fixed } \alpha\text{)}$$

$$= \lim_{r \to -\infty} \frac{1}{(1 + \alpha e^r)^{-1/\alpha - 1}} = 1$$

$\Rightarrow \mathbb{E}[Y] \to q$ as $r \to -\infty$

as $r \to \infty$, $\tau \to 1$ for fixed α

$\Rightarrow \mathbb{E}[y] \to q e^r = q\mu$.

On the variance of y in the ZINB.

$$Var(y) = \sum_{s=1}^{\infty} s^2 f(s;\mu,r)v - \left(\sum_{r=1}^{\infty} s f(s;\mu,r) v\right)^2$$

$$= P(y>0) \sum_{s=1}^{\infty} s^2 \frac{f(s;\mu,r)}{1-f(0;\mu,r)} - \left(P(y>0)\, \mathbb{E}[y \mid y>0,\mu,r]\right)^2$$

$$= P(y>0)\left(var(y \mid y>0,\mu,r) + \mathbb{E}[y \mid y>0,\mu,r]^2\right)$$
$$- P(y>0)^2\, \mathbb{E}[y \mid y>0,\mu,r]^2$$

$$= P(y>0)\, var(y \mid y>0,\mu,r) + P(y>0)\left(1-P(y>0)\right)\mathbb{E}[y \mid y>0,\mu,r]^2$$

Which in the ZINB case gives, for $f(y;\mu,r)$ the underlying NB, y_2, with $var(y_2;\mu,r) = \frac{\mu}{r}(r+\mu) = \mu + \frac{\mu^2}{r}$, we have,
$\mathbb{E}[y_2^2;\mu,r] = \mu + \mu^2/r + \mu^2$

Since $r = 1/\alpha \Rightarrow \mathbb{E}[y_2^2;\mu,r] = \mu + \mu^2 + \alpha\mu^2$ and $var(y_2;\mu,r) = \mu + \alpha\mu^2$

$$\Rightarrow Var(y) = \underbrace{\frac{P(y>0)}{1-f(0;\mu,r)}}_{=v}\left\{\overbrace{var(y_2;\mu,r) + \mathbb{E}[y_2;\mu,r]}^{=\mathbb{E}[y_2^2;\mu,r]}\right\} - \left(\frac{P(y>0)}{1-f(0;\mu,r)}\right)^2\mu^2$$

$$= v(\mu + \mu^2 + \alpha\mu^2) - v^2\mu^2$$
$$= v\mu(1 + \mu + \alpha\mu - v\mu)$$

$$\Rightarrow \frac{Var(y)}{\mathbb{E}[y]} = 1 + \mu + \alpha\mu - v\mu$$
$$= 1 + \mu(1 + \alpha - v)$$, assume non trivial μ

Over dispersion if $1 + \alpha - v > 0 \Rightarrow -v > -1 - \alpha$
$\Rightarrow v < 1 + \alpha$; but we know
that $v > 0$ from its definition (Provided $P(y>0) \neq 0$)

$\Rightarrow$ overdispersion if $0 < v < 1+\alpha$

For underdispersion, we have that

$$1 + \mu(1+\alpha-v) < 1 \quad \text{and} \quad \text{b/c} \quad var(y) > 0, \; 1+\mu(1+\alpha-v) > 0$$

$$\Rightarrow \quad 1+\alpha-v < 0 \quad \text{and} \quad 1+\alpha-v > -1/\mu$$

$$\Rightarrow \quad -v < -1-\alpha \quad \text{and} \quad -v > -1/\mu - 1 - \alpha$$

$$\Rightarrow \quad v > 1+\alpha \quad \text{and} \quad v < 1/\mu + 1 + \alpha$$

$\Rightarrow$ underdispersion, provided,

$$1+\alpha < v < \frac{1}{\mu} + 1 + \alpha$$

Thus the model is indeed flexible and allows for both under- and over-dispersion.

Finally if $q = 1/\tau = \dfrac{(1+\alpha\mu)^{1/\alpha} - 1}{(1+\alpha\mu)^{1/\alpha}}$ then we recover the NB model.

Proof:

$$g(y;\mu,r) = \begin{cases} 1-q & y=0 \\ \dfrac{q}{1-f(0;\mu,r)} \cdot f(y;\mu,r) & y>0 \end{cases}$$

but $1 - f(0;\mu,r) = \dfrac{(1+\alpha\mu)^{1/\alpha} - 1}{(1+\alpha\mu)^{1/\alpha}} = q$ and $1-q = 1/(1+\alpha\mu)^{1/\alpha} = f(0;\mu,r)$

$$\Rightarrow g(y;\mu,r) = \begin{cases} f(0;\mu,r) & y=0 \\ f(y;\mu,r) & y>0 \end{cases} = f(y;\mu,r)$$

The NB density $\square$.

Implementatn (Following Wood et al. (2017))

According to the framework proposed in Wood et al. (2017), we need the deviance an its derivatives to implement an extend GAM model (see sectn 3.3.1 of that paper);

Differentiating w.r.t. η_i instead of μ_i as this is done in SAI.

1- $\frac{\partial D_i}{\partial \eta_i}$, $\frac{\partial^2 D_i}{\partial \eta_i^2}$, $\frac{dh_i}{d\mu_i}$, $\frac{d^2 h_i}{d\mu_i^2}$ (For $\vec{\hat{\beta}}$)

2- $d^4 h_i / d\eta^4$, $\frac{\partial^2 D_i}{\partial \theta_i \partial \theta_j}$, $\frac{\partial^3 D_i}{\partial \eta_i \partial \theta_j \partial \theta_k}$, $\frac{\partial^4 D_i}{\partial \eta_i^4}$, $\frac{\partial^4 D_i}{\partial \eta_i^3 \partial \theta_j}$, $\frac{\partial^4 D}{\partial \eta_i^2 \partial \theta_j \partial \theta_k}$

(For $\vec{\hat{\rho}}$ via full Newton).

As for the zip model, we first define the deviance as $-2l$ for model estimatn.

w.r.t. η for $y>0$

Exactly as for zip.

① $$dl/d\eta = \frac{e^{\eta}}{e^{e^{\eta}}-1}$$, ② $$\frac{d^2 l}{d\eta^2} = (1-e^{\eta})\, \partial l/\partial\eta - \left(\frac{\partial l}{\partial \eta}\right)^2$$

③ $$d^3 l/d\eta^3 = -e^{\eta}\,\partial l/\partial\eta + (1-e^{\eta})^2\,\partial l/\partial\eta - 3(1-e^{\eta})\left(\partial l/\partial\eta\right)^2 + 2\left(\frac{\partial l}{\partial\eta}\right)^3$$ and

④ $$d^4 l/d\eta^4 = (3e^{\eta}-4)\,e^{\eta}\,\partial l/\partial\eta + 4e^{\eta}\left(\partial l/\partial\eta\right)^2 + (1-e^{\eta})^3\frac{\partial l}{\partial\eta} - 7(1-e^{\eta})^2\left(\partial l/\partial\eta\right)^2 + 12(1-e^{\eta})\left(\partial l/\partial\eta\right)^3 - 6\left(\frac{\partial l}{\partial\eta}\right)^4$$

See SAL of Wood et al. (2016) for dealing with these derivatives as $\eta \to \pm\infty$

w.r.t. γ for $y>0$

① $$dl/d\gamma = dl^{zt}/d\gamma = \frac{d}{d\gamma}\left(y\gamma - y\log(1+\alpha e^{\gamma}) - \log\{(1+\alpha e^{\gamma})^{1/\alpha}-1\}\right)$$

$$= y - \frac{y}{1+\alpha e^{\gamma}}\cdot \alpha e^{\gamma} - \frac{1\cdot \alpha e^{\gamma}\cdot 1/\alpha\,(1+\alpha e^{\gamma})^{1/\alpha-1}}{(1+\alpha e^{\gamma})^{1/\alpha}-1}$$

$$= y - y\alpha\frac{e^{\gamma}}{1+\alpha e^{\gamma}} - \frac{e^{\gamma}(1+\alpha e^{\gamma})^{1/\alpha-1}}{(1+\alpha e^{\gamma})^{1/\alpha}-1}$$

$$= y - \frac{y\,\alpha e^{\gamma}}{1+\alpha e^{\gamma}} - \left(\frac{e^{\gamma}}{1+\alpha e^{\gamma}}\right)\left(\frac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e)^{1/\alpha}-1}\right)$$

Define $\beta := \dfrac{e^{\gamma}}{1+\alpha e^{\gamma}}$, $\tau := \dfrac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})^{1/\alpha}-1}$

$$\Rightarrow \; d\ell/d\gamma = y - y\alpha\beta - \beta\tau$$

② $$d^2\ell/d\gamma^2 = \beta\left\{y\alpha^2\beta - y\alpha + \alpha\beta\tau - \tau + \beta\tau^2 - \beta\tau\right\}$$

Proof: $$\frac{d}{d\gamma}\left\{y - y\alpha\beta - \beta\tau\right\} = -y\alpha\, d\beta/d\gamma - \frac{d\beta}{d\gamma}\tau - \beta\frac{d\tau}{d\gamma}$$

But $$d\beta/d\gamma = \frac{(1+\alpha e^{\gamma})e^{\gamma} - e^{\gamma}\cdot\alpha e^{\gamma}}{(1+\alpha e^{\gamma})^2}$$

$$= \frac{e^{\gamma}}{1+\alpha e^{\gamma}} - \frac{\alpha e^{2\gamma}}{(1+\alpha e^{\gamma})^2} = \beta - \alpha\beta^2$$

and $$d\tau/d\gamma = \frac{\left\{\left[(1+\alpha e^{\gamma})^{1/\alpha}-1\right]\left[\alpha e^{\gamma}\cdot \frac{1}{\alpha}\cdot(1+\alpha e^{\gamma})^{1/\alpha-1}\right] - \left[(1+\alpha e^{\gamma})^{1/\alpha}\right]\left[\alpha e^{\gamma}\cdot\frac{1}{\alpha}(1+\alpha e^{\gamma})^{1/\alpha-1}\right]\right\}}{\left((1+\alpha e^{\gamma})^{1/\alpha}-1\right)^2}$$

$$= \frac{e^{\gamma}(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})\left[(1+\alpha e^{\gamma})^{1/\alpha}-1\right]} - \frac{e^{\gamma}(1+\alpha e^{\gamma})^{2/\alpha}}{(1+\alpha e^{\gamma})\left[(1+\alpha e^{\gamma})^{1/\alpha}-1\right]^2}$$

$$= \beta\tau - \beta\tau^2$$

$$\Rightarrow \frac{d^2\ell}{d\gamma^2} = -y\alpha(\beta - \alpha\beta^2) - \beta\tau + \alpha\beta^2\tau - \beta^2\tau + \beta^2\tau^2$$

$$= y\alpha^2\beta^2 - y\alpha\beta + \alpha\beta^2\tau - \beta\tau + \beta^2\tau^2 - \beta^2\tau$$

$$= \beta\left\{y\alpha^2\beta - y\alpha + \alpha\beta\tau - \tau + \beta\tau^2 - \beta\tau\right\}$$

$\square$

We also need

$$\mathbb{E}\left[d^2 l / d\gamma^2 \right] = \mathbb{E}_Y \, \mathbb{E}\left[d^2 l / d\gamma^2 \mid Y \right]$$

$$= \mathbb{E}_Y \left\{ \mathbb{E}\Big[\underbrace{d^2 l / d\gamma^2}_{=0 \text{ for } y=0} \mid Y = 0 \Big] \mathbb{P}(Y=0) + \mathbb{E}\left[\frac{d^2 l}{d\gamma^2} \mid Y > 0 \right] \mathbb{P}(Y>0) \right\}$$

$$= \mathbb{E}_Y \left\{ \mathbb{E}\left[\frac{d^2 l}{d\gamma^2} \mid Y > 0 \right] \underbrace{\mathbb{P}(Y>0)}_{=q} \right\}$$

$$= \mathbb{E}_Y \left\{ q \left(y\alpha^2\beta^2 - y\alpha\beta + \alpha\beta^2\tau - \tau\beta + \beta^2\tau^2 - \beta^2\tau \right) \right\}$$

$$= q \left(\mathbb{E}[y]\alpha^2\beta^2 - \mathbb{E}[y]\alpha\beta + \alpha\beta^2\tau - \tau\beta + \beta^2\tau^2 - \beta^2\tau \right)$$

where $\mathbb{E}[y] = q\tau e^{\gamma}$, as derived on page 6 🕶️.

③ $d^3\ell/d\gamma^3 = \beta\{-2y\alpha^3\beta^2 + 3y\alpha^2\beta - 2\alpha^2\beta^2\tau - y\alpha + 3\alpha\beta\tau$
$- 3\alpha\beta^2\tau^2 + 3\alpha\beta^2\tau - \tau + 3\beta\tau^2 - 3\beta\tau - 2\beta^2\tau^3$
$+ 3\beta^2\tau^2 - \beta^2\tau\}$

Proof. Using a numerical linear algebra package, see 👓.
$\square$

④ $d^4\ell/d\gamma^4 = \beta\{6y\alpha^4\beta^3 - 12y\alpha^3\beta^2 + 6\alpha^3\beta^3\tau + 7y\alpha^2\beta - 12\alpha^2\beta^2\tau + 11\alpha^2\beta^3\tau^2 - 11\alpha^2\beta^3\tau$
$- y\alpha + 7\alpha\beta\tau - 18\alpha\beta^2\tau^2 + 18\alpha\beta^2\tau + 12\alpha\beta^3\tau^3 - 18\alpha\beta^3\tau^2 + 6\alpha\beta^3\tau$
$- \tau + 7\beta\tau^2 - 7\beta\tau - 12\beta^2\tau^3 + 18\beta^2\tau^2 - 6\beta^2\tau + 6\beta^3\tau^4$
$- 12\beta^3\tau^3 + 7\beta^3\tau^2 - \beta^3\tau\}$

proof. 👓

$\square$

As with ℓ^{21}, some care is required to ensure that the derivatives evaluate accurately without overflow over as wide a range of γ and α as possible. As $\gamma \to \infty$, $\beta = \frac{1}{e^{-\gamma}+\alpha} \to 1/\alpha$ for fixed α, while

$\tau = \frac{1}{1-(1+\alpha e^{\gamma})^{-1/\alpha}} \to 1$, since $1+\alpha e^{\gamma} \to \infty \Rightarrow (1+\alpha e^{\gamma})^{-1/\alpha} \to 0$ for fixed α

whereas as $\gamma \to -\infty$, $\beta \to 1/e^{-\gamma} \to 0$ for fixed α, while $e^{\gamma} \to 0$

$\Rightarrow (1+\alpha e^{\gamma})^{1/\alpha} \underset{\text{first order Taylor expansion about } x = \alpha e^{\gamma} = 0}{\to} 1 + \frac{1}{\alpha}\alpha e^{\gamma} = 1 + e^{\gamma} \Rightarrow \tau = \frac{(1+\alpha e^{\gamma})^{1/\alpha}}{(1+\alpha e^{\gamma})^{1/\alpha}-1}$

$\to \frac{1+e^{\gamma}}{1+e^{\gamma}-1} = \frac{1+e^{\gamma}}{e^{\gamma}}$

$* = e^{-\gamma} + 1 \to e^{-\gamma}$

* How do we deal with products $\beta\tau$, $\beta^2\tau^2$, $\beta^2\tau$ etc. what does it converge to?

Since as $\gamma \to -\infty$, $\beta \to e^{\gamma}$ and $\tau \to e^{-\gamma}$

$\Rightarrow \beta\tau \to 1 \Rightarrow (\beta\tau)^a \to 1 \quad \forall a \in \mathbb{R}$

(alternatively),

$\lim_{\gamma\to-\infty} \tau\beta = \lim_{\gamma\to-\infty} \beta / 1-(1+\alpha e^{\gamma})^{-1/\alpha}$

$$(\text{l'Hopital}) = \lim_{\gamma\to-\infty} \frac{\beta-\alpha\beta^2}{\frac{1}{\alpha}\alpha e^\gamma (1+\alpha e^\gamma)^{-1/\alpha-1}}$$

$$= \lim_{\gamma\to-\infty} \frac{\beta(1-\alpha\beta)}{e^\gamma (1+\alpha e^\gamma)^{-1/\alpha-1}}$$

$$= \lim_{\gamma\to-\infty} \frac{\frac{e^\gamma}{1+\alpha e^\gamma}(1-\alpha\beta)}{e^\gamma(1+\alpha e^\gamma)^{-1/\alpha-1}}$$

$$= \lim_{\gamma\to-\infty} \frac{1-\alpha\beta}{(1+\alpha e^\gamma)^{-1/\alpha}} = 1$$

also ; $\beta^a \tau^b \to 0$ if $a > b$ as $\gamma\to-\infty$

as $\gamma\to\infty$, $\beta\to\alpha^{-1}$ and $\tau\to 1$ $\Rightarrow$ $\beta\tau\to\alpha^{-1}$

with these we get

$\partial l/\partial\gamma \to y-1$ as $\gamma\to-\infty$, $\partial l/\partial\gamma \to y-y-1/\alpha = -1/\alpha$ as $\gamma\to\infty$

$\partial^2 l/\partial\gamma^2 \to 0$ as $\gamma\to-\infty$, $\partial^2 l/\partial\gamma^2 \to y-y+\alpha/\alpha^2 - 1/\alpha + 1/\alpha^2 - 1/\alpha^2$

$= 1/\alpha - 1/\alpha + 1/\alpha^2 - 1/\alpha^2 = 0$

as $\gamma\to\infty$

$\partial^3 l/\partial\gamma^3 \to 0$ as $\gamma\to-\infty$, $\partial^3 l/\partial\gamma^3 \to -2y+3y-2/\alpha-y+3/\alpha$

$- 3/\alpha^2 + 3/\alpha^2 - 1/\alpha + \frac{3}{\alpha^2}$

$- 3/\alpha^2 - 2/\alpha^3$

$+ 3/\alpha^3 - 1/\alpha^3$

$= 1/\alpha - 1/\alpha + 3/\alpha^2 - 3/\alpha^2$

$+ 0 = 0$ as $\gamma\to\infty$

$\partial^4 l/\partial\gamma^4 \to 0$ as $\gamma\to-\infty$, $\partial^4 l/\partial\gamma^4 \to 0$ as $\gamma\to\infty$

For extended GAMs in which η is a fn of γ and extra parameters θ e.g.,

$$\eta = \theta_1 + (b + e^{\theta_2})\gamma$$ (as for the ziP model in Wood et al (2016))

$$d\eta/d\theta_1 = 1, \quad d\eta/d\theta_2 = e^{\theta_2}\gamma, \quad d\eta/d\gamma = b + e^{\theta_2}, \quad d^2\eta/d\gamma^2 = 0$$

$$d^2\eta/d\theta_1^2 = d^2\eta/d\theta_1 d\theta_2 = \frac{d^2\eta}{d\gamma\, d\theta_1} = 0, \quad d^2\eta/d\theta_2^2 = e^{\theta_2}\gamma, \quad \frac{d^2\eta}{d\gamma\, d\theta_2} = e^{\theta_2}$$

$$d^3\eta/d\gamma d\theta_1^2 = 0, \quad \frac{d^3\eta}{d\gamma\, d\theta_1\, d\theta_2} = 0, \quad \frac{d^3\eta}{d\gamma\, d\theta_2^2} = e^{\theta_2},$$

$$d^3\eta/d\gamma^2 d\theta_1 = d^3\eta/d\gamma^2 d\theta_2 = d^3\eta/d\gamma^3 = 0$$

$$d^4\eta/d\gamma^4 = d^4\eta/d\gamma^3 d\theta_1 = d^4\eta/d\gamma^3 d\theta_2 = d^4\eta/d\gamma^2 d\theta_1^2 = \frac{d^4\eta}{d\gamma^2 d\theta_2^2}$$

$$= \frac{d^4\eta}{d\gamma^2 d\theta_1\, d\theta_2} = 0$$

Total derivatives.

I We focus on θ_i, for $i \in \{1,2\}$. derivatives involving θ_0 are considered in sect. 2

$$d\ell/d\gamma = \partial\ell/\partial\gamma + \partial\ell/\partial\eta \cdot \partial\eta/\partial\gamma, \quad \frac{d^2\ell}{d\gamma^2} = \frac{\partial^2\ell}{\partial\gamma^2} + \frac{\partial^2\ell}{\partial\gamma\partial\eta}\cdot\frac{\partial\eta}{\partial\gamma}$$

$$+ \frac{\partial^2\ell}{\partial\eta\partial\gamma}\frac{\partial\eta}{\partial\gamma} = 0$$

$$+ \frac{\partial^2\ell}{\partial\eta^2}\cdot\left(\frac{\partial\eta}{\partial\gamma}\right)^2$$

$$+ \frac{\partial\ell}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma^2}$$

$$= \frac{\partial^2\ell}{\partial\gamma^2} + \frac{\partial^2\ell}{\partial\eta^2}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial\ell}{\partial\eta}\frac{\partial^2\eta}{\partial\gamma^2}$$

$$d\ell/d\theta_i = \partial\ell/\partial\eta \cdot \partial\eta/d\theta_i, \quad \frac{d^2\ell}{d\gamma d\theta_i} = \underbrace{\frac{\partial^2\ell}{\partial\gamma\partial\eta}}_{=0}\cdot\frac{\partial\eta}{d\theta_i} + \frac{\partial\ell}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma\partial\theta_i} + \frac{\partial^2\ell}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial\theta_i}\cdot\partial\eta/\partial\gamma$$

$$= \partial^2\ell/\partial\eta^2 \cdot \frac{\partial\eta}{\partial\theta_i}\cdot\frac{\partial\eta}{\partial\gamma} + \frac{\partial\ell}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial\gamma\partial\theta_i}$$

$$\frac{d^3\ell}{d\gamma^2 d\theta_i} = \frac{\partial^3\ell}{\partial\eta^3}\cdot\frac{\partial\eta}{\partial\theta_i}\cdot\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^2\ell}{\partial\eta^2}\, 2\,\frac{\partial\eta}{\partial\gamma}\cdot\frac{\partial\eta}{\partial\gamma\partial\theta_i} + \frac{\partial^2\ell}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial\theta_i}\cdot\frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ \partial\ell/\partial\eta \cdot \partial^3\eta/\partial\gamma^2\partial\theta_i$$

$$= \frac{\partial^3\ell}{\partial\eta^3}\frac{\partial\eta}{\partial\theta_i}\left(\frac{\partial\eta}{\partial\gamma}\right)^2 + \frac{\partial^2\ell}{\partial\eta^2}\left(2\frac{\partial\eta}{\partial\gamma}\cdot\frac{\partial\eta}{\partial\gamma\partial\theta_i} + \frac{\partial\eta}{\partial\theta_i}\frac{\partial^2\eta}{\partial\gamma^2}\right)$$

$$+ \partial\ell/\partial\eta \cdot \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i}$$

$$\frac{d^3 l}{d\gamma^3} = \frac{\partial^3 l}{\partial \gamma^3} + \frac{\partial^3 l}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial \gamma} \cdot \left(\frac{\partial \eta}{\partial \gamma}\right)^2 + \frac{\partial^2 l}{\partial \eta^2}\left[2\frac{\partial \eta}{\partial \gamma} \cdot \frac{\partial^2 \eta}{\partial \gamma^2}\right]$$

$$+ \frac{\partial^2 l}{\partial \eta^2} \cdot \frac{\partial^2 \eta}{\partial \gamma^2} + \frac{\partial l}{\partial \eta} \cdot \frac{\partial^3 \eta}{\partial \gamma^3}$$

$$= \partial^3 l/\partial \gamma^3 + \frac{\partial^3 l}{\partial \eta^3}\left(\frac{\partial \eta}{\partial \gamma}\right)^3 + 3\frac{\partial^2 l}{\partial \eta^2}\frac{\partial^2 \eta}{\partial \gamma^2}\frac{\partial \eta}{\partial \gamma} + \frac{\partial l}{\partial \eta} \cdot \frac{\partial^3 \eta}{\partial \gamma^3}$$

$$d^2 l/d\theta_i d\theta_j = \frac{\partial^2 l}{\partial \eta^2}\, \partial \eta/\partial \theta_j \cdot \partial \eta/\partial \theta_i + \frac{\partial l}{\partial \eta} \cdot \partial^2 \eta/\partial \theta_i \partial \theta_j$$

$$\frac{d^3 l}{d\gamma\, d\theta_i\, d\theta_j} = \frac{\partial^3 l}{\partial \eta^3}\, \partial \eta/\partial \theta_j\, \partial \eta/\partial \theta_i\, \partial \eta/\partial \gamma + \frac{\partial^2 l}{\partial \eta^2}\left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}\frac{\partial \eta}{\partial \gamma} + \frac{\partial \eta}{\partial \theta_i}\frac{\partial^2 \eta}{\partial \gamma \partial \theta_j}\right]$$

$$+ \frac{\partial^2 l}{\partial \eta^2}\frac{\partial \eta}{\partial \theta_j}\frac{\partial^2 \eta}{\partial \theta_i \partial \gamma} + \frac{\partial l}{\partial \eta}\frac{\partial^3 \eta}{\partial \gamma \partial \theta_i \partial \theta_j}$$

$$= \partial^3 l/\partial \eta^3\; \partial \eta/\partial \theta_i\; \partial \eta/\partial \theta_j\; \partial \eta/\partial \gamma + \partial^2 l/\partial \eta^2\left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}\frac{\partial \eta}{\partial \gamma} + \frac{\partial \eta}{\partial \theta_i}\frac{\partial^2 \eta}{\partial \gamma \partial \theta_j}\right.$$

$$\left.+ \partial \eta/\partial \theta_j\, \frac{\partial^2 \eta}{\partial \theta_i \partial \gamma}\right] + \partial l/\partial \eta\; \partial^3 \eta/\partial \gamma \partial \theta_i \partial \theta_j$$

$$d^4 l/d\gamma\, d\gamma\, d\theta_i\, d\theta_j = \frac{\partial^4 l}{\partial \eta^4}\frac{\partial \eta}{\partial \theta_j}\frac{\partial \eta}{\partial \theta_i}\left(\frac{\partial \eta}{\partial \gamma}\right)^2 + \frac{\partial^3 l}{\partial \eta^3}\left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}\left(\frac{\partial \eta}{\partial \gamma}\right)^2 + 2\frac{\partial \eta}{\partial \theta_i}\frac{\partial \eta}{\partial \gamma}\frac{\partial^2 \eta}{\partial \gamma \partial \theta_j}\right]$$

$$+ \frac{\partial^3 l}{\partial \eta^3}\frac{\partial \eta}{\partial \theta_j}\left(2\,\partial^2 \eta/\partial \gamma \partial \theta_i\; \partial \eta/\partial \gamma + \frac{\partial^2 \eta}{\partial \gamma^2} \cdot \frac{\partial \eta}{\partial \theta_i}\right)$$

$$+ \frac{\partial^2 l}{\partial \eta^2}\left(2\frac{\partial^3 \eta}{\partial \gamma \partial \theta_i \partial \theta_j}\frac{\partial \eta}{\partial \gamma} + \frac{2\partial^2 \eta}{\partial \gamma \partial \theta_i}\frac{\partial^2 \eta}{\partial \gamma \partial \theta_j} + \frac{\partial^3 \eta}{\partial \gamma^2 \partial \theta_j}\frac{\partial \eta}{\partial \theta_i}\right.$$

$$\left.+ \frac{\partial^2 \eta}{\partial \gamma^2}\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j}\right)$$

$$+ \partial^2 l/\partial \eta^2\; \partial \eta/\partial \theta_j\; \partial^3 \eta/\partial \gamma^2 \partial \theta_i + \partial l/\partial \eta\; \frac{\partial^4 \eta}{\partial \gamma^2 \partial \theta_i \partial \theta_j}$$

$$= \partial^4 l/\partial\eta^4 \; \partial\eta/\partial\theta_j \; \partial\eta/\partial\theta_i \left(\frac{\partial\eta}{\partial\gamma}\right)^2$$

$$+ \partial^3 l/\partial\eta^3 \Big[\partial^2\eta/\partial\theta_i\partial\theta_j \left(\frac{\partial\eta}{\partial\gamma}\right)^2 + 2\,\partial\eta/\partial\theta_i \; \partial\eta/\partial\gamma \; \frac{\partial^2\eta}{\partial\theta_j\partial\gamma}$$

$$+ 2\,\partial\eta/\partial\theta_j \; \partial^2\eta/\partial\gamma\partial\theta_i \; \partial\eta/\partial\gamma + \partial\eta/\partial\theta_i \; \frac{\partial\eta}{\partial\theta_j} \; \frac{\partial^2\eta}{\partial\gamma^2}\Big)$$

$$+ \partial^2 l/\partial\eta^2 \Big[\frac{2\,\partial^3\eta}{\partial\gamma\partial\theta_i\partial\theta_j} \; \partial\eta/\partial\gamma + \frac{2\,\partial^2\eta}{\partial\gamma\partial\theta_i} \; \frac{\partial^2\eta}{\partial\gamma\partial\theta_j} + \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_j} \; \frac{\partial\eta}{\partial\theta_i}$$

$$+ \partial^2\eta/\partial\gamma^2 \; \frac{\partial^2\eta}{\partial\theta_i\partial\theta_j} + \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} \; \partial\eta/\partial\theta_j \Big]$$

$$+ \partial l/\partial\eta \; \partial^4\eta/\partial\gamma^2\partial\theta_i\partial\theta_j$$

$$d^4 l/d\gamma^3 d\theta_i = \underbrace{\frac{\partial^4 l}{\partial^3\gamma\partial\eta}}_{=0} \cdot \partial\eta/\partial\theta_i + \frac{\partial^4 l}{\partial\eta^4} \; \frac{\partial\eta}{\partial\theta_i} \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + \frac{\partial^3 l}{\partial\eta^3} \cdot 3\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma\partial\theta_i}$$

$$+ 3\frac{\partial^3 l}{\partial\eta^3} \; \partial\eta/\partial\theta_i \; \partial\eta/\partial\gamma \; \partial^2\eta/\partial\gamma^2 + 3\frac{\partial^2 l}{\partial\eta^2} \Big[\frac{\partial^2\eta}{\partial\gamma\partial\theta_i} \; \frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ \partial\eta/\partial\gamma \; \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} \Big] + \frac{\partial^2 l}{\partial\eta^2} \; \frac{\partial\eta}{\partial\theta_i} \; \frac{\partial^3\eta}{\partial\gamma^3} + \frac{\partial l}{\partial\eta} \; \frac{\partial^4\eta}{\partial\gamma^3\partial\theta_i}$$

$$= \frac{\partial^4 l}{\partial\eta^4} \; \frac{\partial\eta}{\partial\theta_i} \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + 3\,\frac{\partial^3 l}{\partial\eta^3} \Big[\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma\partial\theta_i} + \frac{\partial\eta}{\partial\theta_i} \; \frac{\partial\eta}{\partial\gamma} \; \frac{\partial^2\eta}{\partial\gamma^2} \Big]$$

$$+ \partial^2 l/\partial\eta^2 \Big[3\,\partial^2\eta/\partial\gamma\partial\theta_i \; \frac{\partial^2\eta}{\partial\gamma^2} + 3\,\frac{\partial\eta}{\partial\gamma} \; \frac{\partial^3\eta}{\partial\gamma^2\partial\theta_i} + \frac{\partial^3\eta}{\partial\gamma^3} \; \frac{\partial\eta}{\partial\theta_i} \Big]$$

$$+ \partial l/\partial\eta \; \frac{\partial^4\eta}{\partial\gamma^3\partial\theta_i}$$

$$d^4 l/d\gamma^4 = \partial^4 l/\partial\gamma^4 + \frac{\partial^4 l}{\partial\eta^4} \; \partial\eta/\partial\gamma \left(\frac{\partial\eta}{\partial\gamma}\right)^3 + \frac{\partial^3 l}{\partial\eta^3} \cdot 3\left(\frac{\partial\eta}{\partial\gamma}\right)^2 \frac{\partial^2\eta}{\partial\gamma^2}$$

$$+ 3\,\frac{\partial^3 l}{\partial\eta^3} \cdot \partial\eta/\partial\gamma \; \frac{\partial\eta}{\partial\gamma} \; \frac{\partial^2\eta}{\partial\gamma^2} + 3\frac{\partial^2 l}{\partial\eta^2} \Big[\partial^2\eta/\partial\gamma^2 \; \frac{\partial\eta^2}{\partial\gamma^2} +$$

$$\partial\eta/\partial\gamma \; \partial^3\eta/\partial\gamma^3 \Big] + \partial^2 l/\partial\eta^2 \; \partial\eta/\partial\gamma \; \frac{\partial^3\eta}{\partial\gamma^3} + \frac{\partial l}{\partial\eta} \; \frac{\partial^4\eta}{\partial\gamma^4}$$

$$= \frac{\partial^4 l}{\partial \gamma^4} + \frac{\partial^4 l}{\partial \eta^4}\left(\frac{\partial \eta}{\partial \gamma}\right)^4 + 6\,\frac{\partial^3 l}{\partial \eta^3}\left(\frac{\partial \eta}{\partial \gamma}\right)^2 \frac{\partial^2 \eta}{\partial \gamma^2}$$

$$+ \partial^2 l/\partial \eta^2 \left[3\left(\partial^2 \eta/\partial \gamma^2\right)^2 + 4\,\partial \eta/\partial \gamma \; \partial^3 \eta/\partial \gamma^3 \right) + \frac{\partial l}{\partial \eta}\,\frac{\partial^4 \eta}{\partial \gamma^4}$$

II Derivatives involving θ_0

Since $\alpha = 1/r$ and $r \in \mathbb{R}_{>0} \Rightarrow \alpha \in \mathbb{R}_{>0}$. It is however easier to work with unrestricted parameter. Therefore, we define $\alpha := e^{\theta_0}$ with $\theta_0 \in \mathbb{R}$ unrestricted.

Recall that for $y>0$,

$$l^{ZI} = \log\left(1 - e^{-e^{\eta}}\right) + y\log\alpha + y\gamma - y\log\left(1+\alpha e^{\gamma}\right) - \log\left\{\left(1+\alpha e^{\gamma}\right)^{1/\alpha} - 1\right\}$$
$$+ \log\Gamma\left(y + 1/\alpha\right) - \log\Gamma(y+1) - \log\Gamma\left(1/\alpha\right)$$

$$(\alpha = e^{\theta_0}) \quad = \log\left(1 - e^{-e^{\eta}}\right) + y\log e^{\theta_0} + y\gamma - y\log\left(1+e^{\theta_0}e^{\gamma}\right) - \log\left\{\left(1+e^{\theta_0}e^{\gamma}\right)^{e^{-\theta_0}} - 1\right\}$$
$$+ \log\Gamma\left(y + e^{-\theta_0}\right) - \log\Gamma(y+1) - \log\Gamma\left(e^{-\theta_0}\right)$$

Note that if $x = \left(1+e^{\theta_0}e^{\gamma}\right)^{e^{-\theta_0}}$, then
$$\log x = e^{-\theta_0}\log\left(1+e^{\theta_0}e^{\gamma}\right)$$

$$\Rightarrow \frac{1}{x}\,dx/d\theta_0 = \frac{e^{-\theta_0}}{1+e^{\theta_0}e^{\gamma}} \cdot e^{\theta_0}e^{\gamma} + \log\left(1+e^{\theta_0}e^{\gamma}\right)\cdot e^{-\theta_0}\cdot -1$$

$$= \frac{e^{\gamma}}{1+e^{\theta_0}e^{\gamma}} - e^{-\theta_0}\log\left(1+e^{\theta_0}e^{\gamma}\right)$$

$$\Rightarrow dx/d\theta_0 = \left(1+e^{\theta_0}e^{\gamma}\right)^{e^{-\theta_0}}\left\{\beta - e^{-\theta_0}\log\left(1+e^{\theta_0}e^{\gamma}\right)\right\}$$

also, $d/dz \log\Gamma(z) =: \psi(z)$, where ψ is the digamma Fn.

Now dropping the ZI superscript, we have,

$$dl/d\theta_0 = 0 + y/e^{\theta_0}\cdot e^{\theta_0} + 0 - \frac{y}{1+e^{\theta_0}e^{\gamma}}\cdot e^{\theta_0}e^{\gamma} - \frac{1}{\left(1+e^{\theta_0}e^{\gamma}\right)^{e^{-\theta_0}} - 1}$$

$$(1+e^{\theta_0}e^{\gamma})^{e^{-\theta_0}} \left\{ \beta - e^{-\theta_0}\log(1+e^{\theta_0}e^{\gamma}) \right\}$$

$$-\varphi(y+e^{-\theta_0})e^{-\theta_0} + \varphi(e^{-\theta_0})e^{-\theta_0}$$

Define $\quad \omega := e^{-\theta_0}\log(1+e^{\theta_0}e^{\gamma}) - \beta$

$$= y - y\beta e^{\theta_0} + \tau\omega - e^{-\theta_0}\varphi(y+e^{-\theta_0}) + e^{-\theta_0}\varphi(e^{-\theta_0})$$

To compute $d^2 l/d\theta_0^2$, we need $d\beta/d\theta_0$, and $d\tau/d\theta_0$

$$d\beta/d\theta_0 = \frac{d}{d\theta_0}\frac{e^{\gamma}}{1+e^{\theta_0}e^{\gamma}} = \frac{1+e^{\theta_0}e^{\gamma}\cdot 0 - e^{\gamma}\cdot e^{\gamma}e^{\theta_0}}{(1+e^{\theta_0}e^{\gamma})^2}$$

$$= -\frac{e^{2\gamma}}{(1+e^{\theta_0}e^{\gamma})^2}e^{\theta_0} = -\beta^2 e^{\theta_0}$$

We leave the rest to the computer; see 👓;

For $\varphi^{(1)}$, see Polygamma Fn on Wikipedia.

$$d^2l/d\theta_0^2 = -y\beta e^{\theta_0} + y\beta^2 e^{2\theta_0} + \tau^2\omega^2 - \tau\omega^2 + \tau(\beta^2 e^{\theta_0} - \omega)$$
$$+ e^{-\theta_0}\varphi(y+e^{-\theta_0}) - e^{-\theta_0}\varphi(e^{-\theta_0}) + e^{-2\theta_0}\varphi^{(1)}(y+e^{-\theta_0})$$
$$- e^{-2\theta_0}\varphi^{(1)}(e^{-\theta_0})$$

* How do we deal with products $\tau\omega$ ...

Since as $\gamma \to -\infty$, $\tau \to +\infty$ and $\log(1+e^{\theta_0}e^{\gamma}) \to \log(1) = 0$ we end up with the product $+\infty\cdot 0$, which is indeterminate. But we have hope if $\tau \to +\infty$ slower than the terms it interacts with $\to 0$, in which case the interaction $\to 0$ as $\gamma \to -\infty$.

Taking θ_0 as fixed, we have

(i) $$\lim_{\gamma\to-\infty} \tau\log(1+e^{\theta_0}e^{\gamma}) = \lim_{\gamma\to-\infty} \frac{\log(1+e^{\theta_0}e^{\gamma})}{1-(1+e^{\theta_0}e^{\gamma})^{-e^{-\theta_0}}}$$

(L'Hopital) $$= \lim_{\gamma\to-\infty} \frac{e^{\theta_0}e^{\gamma}}{(1+e^{\theta_0}e^{\gamma})\left(0+e^{-\theta_0}(1+e^{\theta_0}e^{\gamma})^{-e^{-\theta_0}-1}\cdot e^{\theta_0}e^{\gamma}\right)}$$

$$= \lim_{\gamma\to-\infty} \frac{e^{\theta_0}e^{\gamma}}{e^{\gamma}(1+e^{\theta_0}e^{\gamma})^{-e^{-\theta_0}}}$$

$$= \lim_{\gamma \to -\infty} \frac{e^{\theta_0}}{(1+e^{\theta_0}e^{\gamma})^{-e^{-\theta_0}}} = e^{\theta_0}$$

$$\Rightarrow \left(\tau \log(1+e^{\theta_0}e^{\gamma})\right)^a \to e^{a\theta_0} \quad \forall a \in \mathbb{R}$$

also, $\tau^a \log(1+e^{\theta_0}e^{\gamma})^b \to 0$ if $b > a$ as $\gamma \to -\infty$, $a, b \in \mathbb{R}$

$\leftarrow \beta\tau \to 1$, as $\gamma \to -\infty$

(ii) $$\lim_{\gamma \to -\infty} \tau\left(e^{-\theta_0}\log(1+e^{\theta_0}e^{\gamma}) - \beta\right) \overset{(i)}{=} e^{-\theta_0} \cdot e^{\theta_0} - 1 = 1 - 1 = 0$$

$$\Rightarrow \lim_{\gamma \to -\infty} \tau\omega = 0, \quad \Rightarrow \lim_{\gamma \to -\infty} (\tau\omega)^a = 0 \quad \forall a \in \mathbb{R}$$

$$\Rightarrow \lim_{\gamma \to -\infty} \tau^a \omega^b = 0 \quad \forall a, b \in \mathbb{R} \text{ if } b > a$$

(iii) $$\lim_{\gamma \to -\infty} \tau\left(\beta + \beta^2 e^{\theta_0} - e^{-\theta_0}\log(1+e^{\theta_0}e^{\gamma})\right) = \lim_{\gamma \to -\infty} \tau(\beta^2 e^{\theta_0} - \omega)$$
$$= \lim_{\gamma \to -\infty} (\beta^2 \tau e^{\theta_0} - \tau\omega) = 0$$

for fixed θ_0

For $\gamma \to +\infty$, $\tau \to 1$, $\beta \to e^{-\theta_0}$, $\log(1+e^{\theta_0}e^{\gamma}) \approx \log(e^{\theta_0}e^{\gamma})$ since $e^{\theta_0}e^{\gamma}$ will dominate 1. and $\log(e^{\theta_0}e^{\gamma}) \to \theta_0 + \gamma$.
We'll use this even though we don't have indeterminacy since the zip implementation also uses it. See how $\log(e^{e^{\gamma}} - 1)$ is approximated as e^{γ} as $\gamma \to \infty$ for the zip model likelihood here 😎.

With these, we get, as $\gamma \to -\infty$

$$dl/d\theta_0 \to y - e^{-\theta_0}\varphi(y + e^{-\theta_0}) + e^{-\theta_0}\varphi(e^{-\theta_0})$$

$$d^2l/d\theta_0^2 \to e^{-\theta_0}\varphi(y+e^{-\theta_0}) - e^{-\theta_0}\varphi(e^{-\theta_0}) + e^{-2\theta_0}\varphi^{(1)}(y+e^{-\theta_0}) - e^{-2\theta_0}\varphi^{(1)}(e^{-\theta_0})$$

while as $\gamma \to +\infty$

$$dl/d\theta_0 \to y - y e^{-\theta_0} e^{\theta_0} - \left(-e^{-\theta_0}(\theta_0 + \gamma) + e^{-\theta_0}\right)$$
$$- e^{-\theta_0}\varphi(y + e^{-\theta_0}) + e^{-\theta_0}\varphi(e^{-\theta_0})$$

$$= e^{-\theta_0}\left\{\theta_0 + \gamma - 1 - \varphi(y + e^{-\theta_0}) + \varphi(e^{-\theta_0})\right\}$$

$$d^2l/d\theta_0^2 \to e^{-\theta_0} + e^{-2\theta_0}e^{\theta_0} - e^{-\theta_0}(\theta_0 + \gamma) + e^{-\theta_0}\varphi(y + e^{-\theta_0})$$
$$- e^{-\theta_0}\varphi(e^{-\theta_0}) + e^{-2\theta_0}\varphi^{(1)}(y + e^{-\theta_0}) - e^{-2\theta_0}\varphi^{(1)}(e^{-\theta_0})$$

$$= e^{-\theta_0}\left\{2 - \theta_0 - \gamma + \varphi(y + e^{-\theta_0}) - \varphi(e^{-\theta_0}) + e^{-\theta_0}\varphi^{(1)}(y + e^{-\theta_0}) - e^{-\theta_0}\varphi^{(1)}(e^{-\theta_0})\right\}$$

$$d^2l/d\theta_0 d\theta_1 = d^2l/d\theta_0 d\theta_2 = 0$$

$$d^2l/d\gamma d\theta_0 = -y\beta e^{\theta_0} + y\beta^2 e^{2\theta_0} - \beta\tau^2\omega + \beta\tau\omega + e^{\theta_0}\beta^2\tau$$

$$d^3l/d\gamma^2 d\theta_0 = -y\beta e^{\theta_0} + 3y\beta^2 e^{2\theta_0} - 2y\beta^3 e^{3\theta_0} - \beta\tau^2\omega + \beta\tau\omega + 2\beta^2\tau^3\omega + e^{\theta_0}\beta^2\tau^2\omega$$
$$- 3\beta^2\tau^2\omega - e^{\theta_0}\beta^2\tau\omega + \beta^2\tau\omega + 2e^{\theta_0}\beta^2\tau - 2e^{\theta_0}\beta^3\tau^2$$
$$- 2e^{2\theta_0}\beta^3\tau + 2e^{\theta_0}\beta^3\tau$$

$$d^3l/d\gamma d\theta_0 d\theta_1 = d^3l/d\gamma d\theta_0 d\theta_2 = 0$$

$$d^3l/d\gamma d\theta_0^2 = -y\beta e^{\theta_0} + 3y\beta^2 e^{2\theta_0} - 2y\beta^2 e^{3\theta_0} - 2\beta\tau^3\omega^2 + 3\beta\tau^2\omega^2$$
$$- \beta\tau^2(\beta^2 e^{\theta_0} - \omega) - \beta\tau\omega^2 + \beta\tau(\beta^2 e^{\theta_0} - \omega)$$
$$+ 2\beta^2\tau^2\omega e^{\theta_0} - 2\beta^2\tau\omega e^{\theta_0} + \beta^2\tau e^{\theta_0} - 2\beta^3\tau e^{2\theta_0}$$

$$d^4l/d\gamma^2 d\theta_0 d\theta_1 = \frac{d^4l}{d\gamma^2 d\theta_0 d\theta_2} = 0$$

$$d^4l/d\gamma^2 d\theta_0^2 = -y\beta e^{\theta_0} + 7y\beta^2 e^{2\theta_0} - 12y\beta^3 e^{3\theta_0} + 6y\beta^4 e^{4\theta_0} - 2\beta\tau^3\omega^2$$
$$+ 3\beta\tau^2\omega^2 - \beta\tau^2(\beta^2 e^{\theta_0} - \omega) - \beta\tau\omega^2 + \beta\tau(\beta^2 e^{\theta_0} - \omega)$$

(35 terms)

$$+ 6\beta^2\tau^4\omega^2 + 2\beta^2\tau^3\omega^2 e^{\theta_0} - 12\beta^2\tau^3\omega^2 + 2\beta^2\tau^3(\beta^2 e^{\theta_0} - \omega)$$
$$- 3\beta^2\tau^2\omega^2 e^{\theta_0} + 7\beta^2\tau^2\omega^2 + 4\beta^2\tau^2\omega e^{\theta_0} + \beta^2\tau^2 e^{\theta_0}(\beta^2 e^{\theta_0} - \omega)$$

$$-3\beta^2\tau^2\left(\beta^2 e^{\theta_0}-\omega\right)+\beta^2\tau\omega^2 e^{\theta_0}-\beta^2\tau\omega^2-4\beta^2\tau\omega e^{\theta_0}$$
$$-\beta^2\tau e^{\theta_0}\left(\beta^2 e^{\theta_0}-\omega\right)+\beta^2\tau\left(\beta^2 e^{\theta_0}-\omega\right)+2\beta^2\tau e^{\theta_0}$$
$$-8\beta^3\tau^3\omega e^{\theta_0}-4\beta^3\tau^2\omega e^{2\theta_0}+12\beta^3\tau^2\omega e^{\theta_0}-2\beta^3\tau^2 e^{\theta_0}$$
$$+4\beta^3\tau\omega e^{2\theta_0}-4\beta^3\tau\omega e^{\theta_0}-8\beta^3\tau e^{2\theta_0}+2\beta^3\tau e^{\theta_0}+6\beta^4\tau^2 e^{2\theta_0}$$
$$+6\beta^4\tau e^{3\theta_0}-6\beta^4\tau e^{2\theta_0}$$

$$\frac{d^4\ell}{d\gamma^3 d\theta_0}=-y\beta e^{\theta_0}+7y\beta^2 e^{2\theta_0}-12y\beta^3 e^{3\theta_0}+6y\beta^4 e^{4\theta_0}-\beta\tau^2\omega+\beta\tau\omega$$
$$+6\beta^2\tau^3\omega+3\beta^2\tau^2\omega e^{\theta_0}-9\beta^2\tau^2\omega-3\beta^2\tau\omega e^{\theta_0}+3\beta^2\tau\omega$$
$$+4\beta^2\tau e^{\theta_0}-6\beta^3\tau^4\omega-6\beta^3\tau^3\omega e^{\theta_0}+12\beta^3\tau^3\omega-2\beta^3\tau^2\omega e^{2\theta_0}$$
$$+9\beta^3\tau^2\omega e^{\theta_0}-7\beta^3\tau^2\omega-9\beta^3\tau^2 e^{\theta_0}+2\beta^3\tau\omega e^{2\theta_0}-3\beta^3\tau\omega e^{\theta_0}$$
$$+\beta^3\tau\omega-10\beta^3\tau e^{2\theta_0}+9\beta^3\tau e^{\theta_0}+6\beta^4\tau^3 e^{\theta_0}+9\beta^4\tau^2 e^{2\theta_0}$$
$$-9\beta^4\tau^2 e^{\theta_0}+6\beta^4\tau e^{3\theta_0}-9\beta^4\tau e^{2\theta_0}+3\beta^4\tau e^{\theta_0}.$$

as $\gamma\to-\infty$ $\quad(\beta\to 0,\ \beta\tau\to 1,\ \tau\omega\to 0)$

$$d^2\ell/d\gamma d\theta_0 \to -0+0-\lim_{\gamma\to-\infty}\beta\tau^2\omega+\underbrace{\lim_{\gamma\to-\infty}\beta\tau\omega}_{=0}+0$$
$$=\lim_{\gamma\to-\infty}(\beta\tau)\lim_{\gamma\to-\infty}(\tau\omega)=1\cdot 0=0$$

$$d^3\ell/d\gamma^2 d\theta_0\to 0,\quad \frac{d^3\ell}{d\gamma\, d\theta_0^2}\to 0$$

$$d^4\ell/d\gamma^2 d\theta_0^2\to 0 \quad\text{and}\quad d^4\ell/d\gamma^3 d\theta_0\to 0$$

as $\gamma\to+\infty$ $\quad\left(\beta\to 1/\alpha=e^{-\theta_0},\ \tau\to 1,\ \tilde{x}:=\lim_{\gamma\to\infty}x\right)$

$$\frac{d^2\ell}{d\gamma d\theta_0}\to -y e^{-\theta_0}e^{\theta_0}+y e^{-2\theta_0}e^{2\theta_0}-\widetilde{\beta\omega}+\widetilde{\beta\omega}+e^{\theta_0}e^{-2\theta_0}$$
$$=-y+y-\widetilde{\beta\omega}+\widetilde{\beta\omega}+e^{-\theta_0}=e^{-\theta_0}=1/\alpha$$

$$d^3\ell/d\gamma^2 d\theta_0\to \cancel{-y}+\cancel{3y}-\cancel{2y}-\cancel{\widetilde{\beta\omega}}+\cancel{\widetilde{\beta\omega}}+\cancel{2\beta^2\omega}+\cancel{e^{\theta_0}\widetilde{\beta^2\omega}}$$
$$-\cancel{3\widetilde{\beta^2\omega}}-\cancel{e^{\theta_0}\widetilde{\beta^2\omega}}+\cancel{\widetilde{\beta^2\omega}}+2e^{\theta_0}\widetilde{\beta^2}-\cancel{2e^{\theta_0}\widetilde{\beta^3}}$$
$$-2e^{2\theta_0}\widetilde{\beta^3}+\cancel{2e^{\theta_0}\widetilde{\beta^3}}$$

$$=2e^{\theta_0}e^{-2\theta_0}-2e^{2\theta_0}e^{-3\theta_0}$$
$$=2e^{-\theta_0}-2e^{-\theta_0}=0$$

$$\frac{d^4 l}{d\gamma^2 d\theta_0^2} \to -\cancel{y} + \cancel{7y} - \cancel{12y} + \cancel{6y} - 2\cancel{\tilde\beta\omega^2} + 3\cancel{\tilde\beta\omega^2}$$

$$-\cancel{\tilde\beta^3 e^{\theta_0}} + \cancel{\tilde\beta\omega} - \cancel{\tilde\beta\omega^2} + \cancel{\tilde\beta^3 e^{\theta_0}} - \cancel{\tilde\beta\omega}$$

$$+ 6\cancel{\tilde\beta^2\omega^2} + 2\cancel{\tilde\beta^2\omega^2 e^{\theta_0}} - 12\cancel{\tilde\beta^2\omega^2} + 2\cancel{\tilde\beta^4 e^{\theta_0}}$$

$$- 2\cancel{\tilde\beta^2\omega} - 3\cancel{\tilde\beta^2\omega^2 e^{\theta_0}} + 7\cancel{\tilde\beta^2\omega^2} + 4\cancel{\tilde\beta^2\omega e^{\theta_0}}$$

$$+ \cancel{\tilde\beta^4 e^{2\theta_0}} - \cancel{\tilde\beta^2\omega e^{\theta_0}} - 3\cancel{\tilde\beta^4 e^{\theta_0}} + 3\cancel{\tilde\beta^2\omega} + \cancel{\tilde\beta^2\omega^2 e^{\theta_0}}$$

$$- \cancel{\tilde\beta^2\omega^2} - 4\cancel{\tilde\beta^2\omega e^{\theta_0}} - \cancel{\tilde\beta^4 e^{2\theta_0}} + \cancel{\tilde\beta^2\omega e^{\theta_0}} + \cancel{\tilde\beta^4 e^{\theta_0}} - \cancel{\tilde\beta^2\omega}$$

$$+ 2\tilde\beta^2 e^{\theta_0} - 8\cancel{\tilde\beta^3\omega e^{\theta_0}} - 4\cancel{\tilde\beta^3\omega e^{2\theta_0}} + 12\cancel{\tilde\beta^3\omega e^{\theta_0}} - 2\cancel{\tilde\beta^3 e^{\theta_0}}$$

$$+ 4\cancel{\tilde\beta^3\omega e^{2\theta_0}} - 4\cancel{\tilde\beta^3\omega e^{\theta_0}} - 8\tilde\beta^3 e^{2\theta_0} + 2\cancel{\tilde\beta^3 e^{\theta_0}} + 6\cancel{\tilde\beta^4 e^{2\theta_0}}$$

$$+ 6\tilde\beta^4 e^{3\theta_0} - 6\cancel{\tilde\beta^4 e^{2\theta_0}}$$

$$= 2\tilde\beta^2 e^{\theta_0} - 8\tilde\beta^3 e^{2\theta_0} + 6\tilde\beta^4 e^{3\theta_0}$$

$$= 2e^{-\theta_0} - 8e^{-\theta_0} + 6e^{-\theta_0} = 0$$

$$\frac{d^4 l}{d\gamma^3 d\theta_0} \to -\cancel{y} + \cancel{7y} - \cancel{12y} + \cancel{6y} - \cancel{\tilde\beta\omega} + \cancel{\tilde\beta\omega} + 6\cancel{\tilde\beta^2\omega} + 3\cancel{\tilde\beta^2\omega e^{\theta_0}}$$

$$- 9\cancel{\tilde\beta^3\omega} - 3\cancel{\tilde\beta^2\omega e^{\theta_0}} + 3\cancel{\tilde\beta^2\omega} + 4\tilde\beta^2 e^{\theta_0} - 6\cancel{\tilde\beta^2\omega} - 6\cancel{\tilde\beta^3\omega e^{\theta_0}}$$

$$+ 12\cancel{\tilde\beta^3\omega} - 2\cancel{\tilde\beta^3\omega e^{2\theta_0}} + 9\cancel{\tilde\beta^3\omega e^{\theta_0}} - 7\cancel{\tilde\beta^3\omega} - 9\cancel{\tilde\beta^3 e^{\theta_0}} + 2\cancel{\tilde\beta^3\omega e^{2\theta_0}}$$

$$- 3\cancel{\tilde\beta^3\omega e^{\theta_0}} + \cancel{\tilde\beta^3\omega} - 10\tilde\beta^3 e^{2\theta_0} + 9\cancel{\tilde\beta^3 e^{\theta_0}} + 6\cancel{\tilde\beta^4 e^{\theta_0}} + 9\cancel{\tilde\beta^4 e^{2\theta_0}}$$

$$- 9\cancel{\tilde\beta^4 e^{\theta_0}} + 6\tilde\beta^4 e^{3\theta_0} - 9\cancel{\tilde\beta^4 e^{2\theta_0}} + 3\cancel{\tilde\beta^4 e^{\theta_0}}$$

$$= 4\tilde\beta^2 e^{\theta_0} - 10\tilde\beta^3 e^{2\theta_0} + 6\tilde\beta^4 e^{3\theta_0}$$

$$= 4e^{-\theta_0} - 10e^{-\theta_0} + 6e^{-\theta_0} = 0$$

For use in 'predict' (se-fit)

Recall $\mathbb{E}[Y] = q\tau e^{\gamma} = q\dfrac{e^{\gamma}}{1-(1+\alpha e^{\gamma})^{-1/\alpha}}$; $Y \sim$ ZTNB.

Let the ZTNB mean be $\bar{\mu}$,

$$\bar{\mu} = \frac{e^{\gamma}}{1-(1+\alpha e^{\gamma})^{-1/\alpha}}, \quad d\bar{\mu}/d\gamma = e^{\gamma}\left[\frac{d\tau}{d\gamma}\right] + \tau e^{\gamma}$$

$$= e^{\gamma}(\beta\tau - \beta\tau^2) + \tau e^{\gamma}$$

$$= e^{\gamma}\beta\tau - e^{\gamma}\beta\tau^2 + \tau e^{\gamma}$$

as $\gamma \to -\infty$

$$e^{\gamma}\tau \to 1$$

$$e^{\gamma}\beta\tau \to \beta, \quad e^{\gamma}\beta\tau^2 \to e^{\gamma}\tau \text{ (since } \beta\tau \to 1)$$
$$\to 1 \quad (\text{as } \tau \to e^{-\gamma})$$

$$\Rightarrow \bar{\mu} \to \beta - 1 + 1 = \beta$$

as $\gamma \to \infty$, $\bar{\mu} = (e^{\gamma}\beta\tau - e^{\gamma}\beta\tau^2 + \tau e^{\gamma})$

$$\to e^{\gamma}/\alpha - e^{\gamma}/\alpha + e^{\gamma}$$

$$= e^{\gamma}$$

$q = 1 - e^{-e^{\eta}}$, $\quad dq/d\gamma = dq/d\eta \cdot d\eta/d\gamma$

$dq/d\eta = e^{-e^{\eta}} e^{\eta}$, $\quad \eta = \theta_1 + (b + e^{\theta_2})\gamma$

$$\Rightarrow d\eta/d\gamma = b + e^{\theta_2}$$

$\Rightarrow dq/d\gamma = e^{-e^{\eta}} e^{\eta}(b + e^{\theta_2})$, (we don't do asymptotics with η b/c it is endogenous i.e. it depends on γ already)

$$\Rightarrow \frac{d\mathbb{E}[Y]}{d\gamma} = dq/d\gamma \cdot \bar{\mu} + q \cdot \frac{d\bar{\mu}}{d\gamma}$$