

Negative Binomial PMF:

γ is counting y failures given r successes. Total # of trials

$$n = y+r$$

$$f(y; p, r) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

where p is the probability of success in each trial

$$\binom{y+r-1}{r-1} = \binom{y+r-2}{y}$$

check:

$$\begin{aligned} \binom{y+r-1}{r-1} &= \frac{(y+r-1)!}{(r-1)!((y+r-1)-r+1)!} = \frac{(y+r-1)!}{(r-2)!y!} \\ &= \frac{(y+r-1)!}{y!(r-1)!} = \frac{(y+r-2)!}{y!(y+r-2-y)!} \\ &= \binom{y+r-2}{y} \end{aligned}$$

□

$\Sigma (\tilde{U} - \bar{U})^2$

$\bar{U} = \frac{\sum U}{n}$
Dev. result

Therefore

$$f(y; p, r) = \binom{y+r-1}{y} p^r (1-p)^y$$

$$\text{Mean: } \frac{r(1-p)}{p} \approx \mu, \quad \text{Variance: } \frac{r(1-p)}{p^2} = \text{Var}(\gamma)$$

- s. II

$$\Rightarrow \mu p = r - rp$$

$$\Rightarrow p(r+\mu) = r$$

$$\Rightarrow p = r/\mu = \text{Var}(\gamma) = \mu/r = \mu/\mu(r+\mu) = \mu/\mu(r+\mu) = \mu + \mu^2/r$$

$$f(y; \mu, r) = \binom{y+r-1}{y} \left(\frac{r}{\mu+r}\right)^r \left(\frac{\mu}{\mu+r}\right)^y$$

$$\Gamma(n) = (n-1)!$$

if $n \in \mathbb{N}, n \geq 1$
if n is a real
the $\Gamma(\cdot)$

$$= \frac{\Gamma(y+r)}{y! \Gamma(r)} \left(\frac{r}{\mu+r}\right)^r \left(\frac{\mu}{\mu+r}\right)^y$$

is well defined

Define $r := \frac{1}{\alpha}$, $\Rightarrow p = \frac{1}{r} \left(\frac{1}{1+\alpha} + \mu \right) = \frac{1}{1+\alpha} + \mu$

$$f(y; \mu, \alpha) = \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha \mu}{1+\alpha} \right)^y$$

\Rightarrow The single data log-likelihood is

$$\begin{aligned} l_i^B &= y_i \log \left(\frac{\alpha \mu}{1+\alpha \mu} \right) - \frac{1}{\alpha} \log(1+\alpha \mu_i) + \log \Gamma(y_i + \frac{1}{\alpha}) \\ &\quad - \log \Gamma(y_i + 1) - \log \Gamma(\frac{1}{\alpha}) \end{aligned} \quad \dots \quad (1)$$

Zero-truncated NB.

The probability of a zero count is

$$f(0; \mu, \alpha) = \frac{\Gamma(\frac{1}{\alpha})}{\Gamma(1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}} = \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}}$$

The zero-truncated density is

$$\begin{aligned} f(y|y>0, \mu, \alpha) &= \frac{f(y; \mu, \alpha)}{1-f(0; \mu, \alpha)} \\ &= \frac{\Gamma(y + \frac{1}{\alpha})}{\Gamma(y+1) \Gamma(\frac{1}{\alpha})} \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{\alpha \mu}{1+\alpha} \right)^y \\ &\quad \frac{}{(1 - \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}})} \end{aligned}$$

This gives us the single data log-likelihood;

$$l_i^{ZT} = l_i^B - \log \left\{ 1 - \left(\frac{1}{1+\alpha} \right)^{\frac{1}{\alpha}} \right\} \quad \dots \quad (2)$$

Where (2) is the ZTB single data log-likelihood as given in (1)

Zero Inflated Negative Binomial (Hurdle model in Tutz (2012))

We use a similar parameterization as in Hlood et al. (2017) SA I. written by a

$$g(y; \mu, \alpha) = \begin{cases} 1-q & y=0 \\ q f(y|y>0, \mu, \alpha) & \text{otherwise} \end{cases}$$

where q is the probability of a positive counts

Similar to Hlood et al. (2017), we adopt the unconstrained parameterization

$$\sigma = \log \mu, \quad \eta = \log \hat{\pi} - \log(1-q)$$

This gives us the single-data log-likelihood (we drop the data index i)

$$l^{z^1} = \begin{cases} \log(1-q) & y=0 \\ \log q + \log f(y|y>0, \mu, \alpha) & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{HB} - \log \left\{ 1 - (1+e^{-e^\eta})^{-\frac{1}{\alpha}} \right\} & \text{otherwise} \end{cases}$$

$$= \begin{cases} -e^\eta & y=0 \\ \log(1-e^{-e^\eta}) + l^{z^1} & \text{otherwise} \end{cases}$$

where $L^{AB} = L^{AB} - \log \{1 - ((1 + \alpha e^y)^{-\frac{1}{\alpha}})\}$ is the Z^{AB} log-likelihood.
and $L^{AB} = y \log \left(\frac{\alpha e^y}{1 + \alpha e^y} \right) - \frac{1}{\alpha} \log (1 + \alpha e^y) + \log \Gamma(y + \frac{1}{\alpha})$
 $- \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

$$\Rightarrow L^{Z^A} = y \log \left(\frac{\alpha e^y}{1 + \alpha e^y} \right) + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$$
 $+ \log \left\{ \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}} \right\} + \log \left\{ \frac{1}{1 - \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}}} \right\}$

looking at the last two terms basically is

$$\log \left\{ \frac{1}{(1 + \alpha e^y)^{\frac{1}{\alpha}}} - 1 \right\} = - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$$
 $\Rightarrow L^{Z^A} = y \log \left\{ \frac{\alpha e^y}{1 + \alpha e^y} \right\} + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1)$
 $- \log \Gamma(\frac{1}{\alpha}) - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$
 $= y \log \alpha + y \gamma - y \log (1 + \alpha e^y) - \log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \}$
 $+ \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{1}{\alpha})$

Here as well, some care is required to evaluate L^{Z^A} without unnecessary overflow, since it is easy for $1 - e^{-\frac{1}{\alpha}}$ and $(1 + \alpha e^y)^{\frac{1}{\alpha}} - 1$ to evaluate to zero in finite precision arithmetic. Hence the limiting results $\log(1 - e^{-\frac{1}{\alpha}}) \rightarrow \log(e^n - e^{\frac{1}{\alpha}n} + \frac{e^{\frac{1}{\alpha}n}}{6}) \rightarrow y$ as $n \rightarrow \infty$.
Similar as $y \rightarrow -\infty$, $\log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \} \rightarrow \infty$ and as $\alpha \rightarrow 0$
 $\log \{ (1 + \alpha e^y)^{\frac{1}{\alpha}} - 1 \} \rightarrow \log(e^y - 1) \xrightarrow[\text{provided } y > -\infty]{} e^y$

Implemented
taking α as
fixed.

Proof:

$$\textcircled{1} \quad \log(1 - e^{-\frac{1}{\alpha}}) \rightarrow \log(e^n - e^{\frac{1}{\alpha}n} + \frac{e^{\frac{1}{\alpha}n}}{6}) \rightarrow y \quad n \rightarrow \infty$$

as $y \rightarrow -\infty$, $e^{-\frac{1}{\alpha}} \rightarrow 0$,
we know that $e^n \rightarrow 1 + n + \frac{n^2}{2} + \frac{n^3}{6}$ as $n \rightarrow \infty$

 $\Rightarrow e^{-\frac{1}{\alpha}} \rightarrow 1 - e^y + e^{\frac{1}{\alpha}y} - e^{\frac{1}{\alpha}y} / 6$

$$= 0 \text{ as } \eta \rightarrow -\infty, \log(1 - e^{-e^\eta})$$

$$\rightarrow \log\left(1 - 1 + e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6}\right)$$

$$\text{as } \eta \rightarrow -\infty, e^{2\eta} \text{ and } e^{3\eta} \rightarrow 0 \text{ faster than } e^\eta$$

$$= \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$= 0 \quad \log(e^\eta - \frac{e^{2\eta}}{2} + \frac{e^{3\eta}}{6})$$

$$\rightarrow \log e^\eta \rightarrow \eta$$

② $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow r \text{ as } \alpha \rightarrow -\infty$

as $r \rightarrow -\infty$, $e^r \rightarrow 0 \Rightarrow \alpha e^r \rightarrow 0$ (fixed α)
 We know that $(1+x)^{\frac{1}{x}} \rightarrow 1 + \frac{1}{x}x$ as $x \rightarrow 0$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow 1 + \frac{1}{\alpha}(\alpha e^r) = 1 + e^r$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} - \log\{(1 + e^r) - 1\}$$

$$= \log(e^r) = r$$

as $r \rightarrow -\infty$

③ $\log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow e^r \text{ as } \alpha \rightarrow 0$ for fixed r

remember that $(1 + \alpha e^r)^{\frac{1}{\alpha}} = \left(1 + \frac{e^r}{r}\right)^r$,
 so as $\alpha \rightarrow 0$, $r \rightarrow \infty$

$$\text{and } \lim_{r \rightarrow \infty} \left(1 + \frac{e^r}{r}\right)^r = e^{e^r} \text{ for } r \in \mathbb{R}$$

$$\Rightarrow (1 + \alpha e^r)^{\frac{1}{\alpha}} \rightarrow e^{e^r} \text{ as } \alpha \rightarrow 0$$

$$= \log\{(1 + \alpha e^r)^{\frac{1}{\alpha}} - 1\} \rightarrow \log\{e^{e^r} - 1\} \text{ as } \alpha \rightarrow 0$$

also, for very large r , $e^{e^r} - 1$ is dominated by e^{e^r}

$$\Rightarrow \log\{e^{e^r} - 1\} \rightarrow \log e^{e^r} = e^r \quad \square$$

On the expectation of y in the ZINB.

Recall the model's density

$$g(y; \mu, \kappa) = \begin{cases} 1 - q & y = 0 \\ q f(y > 0; \mu, \kappa) & y > 0 \end{cases}$$

$$\text{where } f(y | y > 0; \mu, \kappa) = \frac{f(y; \mu, \kappa)}{1 - f(0; \mu, \kappa)}$$

Therefore, the model adopts the form of hurdle models discussed in Tufy, section 7.8.

$$\text{Let } v := \frac{P(y > 0)}{1 - f(0; \mu, \kappa)} \stackrel{\text{ZINB}}{=} q^z, \text{ since in ZINB } P(y > 0) = q^z, \mu = e^x \text{ and } f(0; \mu, \kappa) = \frac{1}{(1 + e^x)^{\kappa}}$$

Therefore, the mean is given by

$$\begin{aligned} E[y] &= \sum_{s=1}^{\infty} s f(s; \mu, \kappa) v = \sum_{i=1}^{\infty} s f(s; \mu, \kappa) \cdot \frac{P(y > 0)}{1 - f(0; \mu, \kappa)} \\ &= P(y > 0) \sum_{s=1}^{\infty} s f(s; \mu, \kappa) \end{aligned}$$

$$= P(y > 0) E[y | y > 0, \mu, \kappa]$$

(y_2 the underlying
NB r.v.)

$$\text{or } = \frac{P(y > 0)}{1 - f(0; \mu, \kappa)} E[y_2] \leftarrow \text{doesn't depend on } s$$

For the ZINB model, which as $f(s; \mu, \kappa)$ is the PMF of a NB distribution with mean μ ;

$$= P E[y] = v \mu = q^z \mu = q^z e^x \quad (\mu = e^x).$$

$$\text{where } z = \frac{1}{1 - ((1 + e^x)^{-\kappa})} = \frac{y}{1 - ((1 + e^x)^{-\kappa})^{-1/\kappa}}$$

$$\text{also } \lim_{r \rightarrow \infty} e^{tr} = \lim_{r \rightarrow \infty} e^r / r$$

$$\begin{aligned} (\text{Using Hopital}) &= \lim_{r \rightarrow \infty} \frac{e^r}{\frac{1}{\alpha} \alpha e^r (1 + \alpha e^r)^{-1/\alpha - 1}} \quad (\text{for fixed } \alpha) \\ &= \lim_{r \rightarrow \infty} \frac{1}{(1 + \alpha e^r)^{-1/\alpha - 1}} = 1 \end{aligned}$$

$$\Rightarrow E[Y] \rightarrow q \text{ as } r \rightarrow \infty$$

as $r \rightarrow \infty$, $t \rightarrow 1$ for fixed α

$$\Rightarrow E[Y] \rightarrow q e^{\mu} = q \mu.$$

On the variance of y in the ZINB.

$$\begin{aligned}
 \text{Var}(y) &= \sum_{s=1}^{\infty} s^2 f(s; \mu, \kappa) v - \left(\sum_{r=1}^{\infty} r f(r; \mu, \kappa) v \right)^2 \\
 &= P(y>0) \sum_{s=1}^{\infty} \frac{s^2 f(s; \mu, \kappa)}{1-f(0; \mu, \kappa)} - \left(P(y>0) E[y|y>0, \mu, \kappa] \right)^2 \\
 &= P(y>0) \left[\text{var}(y|y>0, \mu, \kappa) + E[y|y>0, \mu, \kappa]^2 \right] \\
 &\quad - P(y>0)^2 E[y|y>0, \mu, \kappa]^2 \\
 &= P(y>0) \text{var}(y|y>0, \mu, \kappa) + P(y>0) (1 - P(y>0)) E[y|y>0, \mu, \kappa]^2
 \end{aligned}$$

which in the ZINB case gives, for $f(y; \mu, r)$ the underlying NB,
 y_r , with $\text{var}(y_r; \mu, r) = \mu + \frac{\mu^2}{r} = \mu + \frac{\mu^2}{\alpha}$, we have,
 $E[y_r^2; \mu, r] = \mu + \frac{\mu^2}{r} + \frac{r}{\alpha}$

$$\begin{aligned}
 \text{Since } r = \frac{1}{\kappa} \Rightarrow E[y_r^2; \mu, r] &= \mu + \frac{\mu^2}{\alpha} + \alpha \mu^2 = \mu + \frac{\mu^2}{\alpha} + \overbrace{E[y_r; \mu, r]}^{\text{and } \text{var}(y_r; \mu, r) = \mu + \frac{\mu^2}{\alpha}} \\
 \Rightarrow \text{Var}(y) &= \frac{P(y>0)}{1-f(0; \mu, \kappa)} \left\{ \overbrace{\text{var}(y_r; \mu, r)}^v + E[y_r; \mu, r] \right\} \\
 &\quad - \left(\frac{P(y>0)}{1-f(0; \mu, \kappa)} \right)^2 \mu^2 \\
 &= v(\mu + \mu^2 + \alpha \mu^2) - v^2 \mu^2 \\
 &= v \mu (1 + \mu + \alpha \mu - v \mu)
 \end{aligned}$$

$$\Rightarrow \frac{\text{Var}(y)}{E[y]} = \frac{1 + \mu + \alpha \mu - v \mu}{\mu} = 1 + \mu (1 + \alpha - v), \text{ assume non-trivial } \mu$$

Overdispersion if $1 + \alpha - v > 0 \Rightarrow -v > -1 - \alpha$

$\Rightarrow v < 1 + \alpha$; but we know
 that $v > 0$ from its definition (provided $P(y>0) \neq 0$)

\Rightarrow overdispersion if $0 < v < 1 + \alpha$

For underdispersion, we have that

$$1 + \mu(1 + \alpha - v) < 1 \quad \text{and} \quad \frac{v}{\mu} \operatorname{var}(y) > 0, \text{ if } \mu(1 + \alpha - v) > 0$$

$$\Rightarrow 1 + \alpha - v < 0 \quad \text{and} \quad 1 + \alpha - v > -\frac{1}{\mu}$$

$$\Rightarrow -v < -1 - \alpha \quad \text{and} \quad -v > -\frac{1}{\mu} - 1 - \alpha$$

$$\Rightarrow v > 1 + \alpha \quad \text{and} \quad v < \frac{1}{\mu} + 1 + \alpha$$

\Rightarrow underdispersion, provided,

$$1 + \alpha < v < \frac{1}{\mu} + 1 + \alpha$$

Thus the model is indeed flexible and allows for both under- and over-dispersion.

Finally if $q = \frac{v}{\mu} = \frac{(1 + \alpha \mu)^{\frac{1}{\alpha}} - 1}{(1 + \alpha \mu)^{\frac{1}{\alpha}}} - 1$ then we recover the NB model.

Proof:

$$g(y; \mu, \alpha) = \begin{cases} 1 - q & y = 0 \\ \frac{q}{1 - f(0; \mu, \alpha)} \cdot f(y; \mu, \alpha) & y > 0 \end{cases}$$

$$\text{but } 1 - f(0; \mu, \alpha) = \frac{(1 + \alpha \mu)^{\frac{1}{\alpha}} - 1}{(1 + \alpha \mu)^{\frac{1}{\alpha}}} = q \text{ and } 1 - q = \frac{1}{(1 + \alpha \mu)^{\frac{1}{\alpha}}} = f(0; \mu, \alpha)$$

$$\Rightarrow g(y; \mu, \alpha) = \begin{cases} f(0; \mu, \alpha) & y = 0 \\ f(y; \mu, \alpha) & y > 0 \end{cases} = f(y; \mu, \alpha) \quad \text{The NB density}$$

□.

Implementation (Following Hood et al. (2017))

According to the framework proposed in Hood et al. (2017), we need the deviance and its derivatives to implement an extend GAM model (see section 3.3.1 of that paper);

Differentiating w.r.t. y_i , $\frac{\partial D_i}{\partial y_i}$, $\frac{\partial^2 D_i}{\partial y_i^2}$, $\frac{\partial^3 D_i}{\partial y_i^3}$, $\frac{\partial^4 D_i}{\partial y_i^4}$ (For \hat{p})

instead of m_i , $\frac{\partial^4 h_i}{\partial y_i^4}$, $\frac{\partial^5 D_i}{\partial y_i \partial \theta_j}$, $\frac{\partial^5 D_i}{\partial y_i \partial \theta_j \partial \theta_k}$, $\frac{\partial^4 D_i}{\partial m_i^4}$, $\frac{\partial^4 D_i}{\partial y_i^3 \partial \theta_j}$, $\frac{\partial^4 D_i}{\partial y_i^2 \partial \theta_j \partial \theta_k}$
as this is done in SAI. (For \hat{p} via full Newton).

As for the zip model, we first define the deviance as -2l for model estimation.

w.r.t. y for $y > 0$

$$\textcircled{1} \quad \frac{dL}{dy} = \frac{y}{e^y - 1}, \quad \textcircled{2} \quad \frac{d^2 L}{dy^2} = (1-e^y) \frac{dy}{dy} - \left(\frac{dy}{dy} \right)^2$$

exactly as for zip.

$$\textcircled{3} \quad \frac{d^3 L}{dy^3} = -e^y \frac{dy}{dy} + (1-e^y)^2 \frac{dy}{dy} - 3(1-e^y) \left(\frac{dy}{dy} \right)^2 + 2 \left(\frac{dy}{dy} \right)^3 \quad \text{and}$$

$$\textcircled{4} \quad \frac{d^4 L}{dy^4} = (3e^y - 4)e^y \frac{dy}{dy} + 4e^y \left(\frac{dy}{dy} \right)^2 + (1-e^y)^3 \frac{dy}{dy} - 7(1-e^y) \left(\frac{dy}{dy} \right)^2 + 12(1-e^y) \left(\frac{dy}{dy} \right)^3 - 6 \left(\frac{dy}{dy} \right)^4$$

See SAI of Hood et al. (2016) for dealing with these derivatives as $y \rightarrow \pm\infty$

w.r.t. γ for $y > 0$

$$\textcircled{5} \quad \frac{dy}{d\gamma} = \frac{dL}{d\gamma} = \frac{d}{d\gamma} \left(y - y \log(1+\alpha e^\gamma) - \log \left\{ (1+\alpha e^\gamma)^{\frac{1}{\alpha}} - 1 \right\} \right)$$

$$= y - \frac{y}{1+\alpha e^\gamma} - \frac{1-\alpha e^\gamma}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}}$$

$$= y - y \frac{e^\gamma}{1+\alpha e^\gamma} - \frac{e^\gamma (1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}}{(1+\alpha e^\gamma)^{\frac{1}{\alpha}-1}}$$

$$= y - \frac{y \alpha e^x}{1 + \alpha e^x} - \left(\frac{e^x}{1 + \alpha e^x} \right) \left(\frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1} \right)$$

Define $\beta := \frac{e^x}{1 + \alpha e^x}$, $\tau := \frac{(1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1}$

$$\Rightarrow \frac{dy}{dx} = y - y \alpha \beta - \beta \tau$$

$$\textcircled{2} \quad \frac{d^2 y}{dx^2} = \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

Proof: $\frac{d}{dx} \{ y - y \alpha \beta - \beta \tau \} = -y \alpha \frac{d\beta}{dx} - \frac{d\beta}{dx} \tau - \beta \frac{d\tau}{dx}$

But $\frac{d\beta}{dx} = \frac{(1 + \alpha e^x) e^x - e^x \cdot \alpha e^x}{(1 + \alpha e^x)^2}$

$$= \frac{e^x}{1 + \alpha e^x} - \frac{\alpha e^{2x}}{(1 + \alpha e^x)^2} = \beta - \alpha \beta^2$$

and $\frac{d\tau}{dx} = \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} \right] \left[\alpha e^x \cdot \frac{1}{\alpha} \cdot (1 + \alpha e^x)^{\frac{1}{\alpha}-1} \right]}$

$$= \frac{\left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}{\left((1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right)^2}$$

$$= \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]} - \frac{e^x (1 + \alpha e^x)^{\frac{1}{\alpha}}}{(1 + \alpha e^x) \left[(1 + \alpha e^x)^{\frac{1}{\alpha}} - 1 \right]^2}$$

$$= \beta \tau - \beta \tau^2$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -y \alpha (\beta - \alpha \beta^2) - \beta \tau + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= y \alpha^2 \beta^2 - y \alpha \beta + \alpha \beta^2 \tau - \beta \tau + \beta^2 \tau^2 - \beta \tau^2$$

$$= \beta \{ y \alpha^2 \beta - y \alpha + \alpha \beta \tau - \tau + \beta \tau^2 - \beta \tau^2 \}$$

□

We also need

$$\mathbb{E}\left[\frac{d^2Y}{d\delta^2}\right] = \mathbb{E}_Y \mathbb{E}\left[\frac{d^2Y}{d\delta^2} | Y\right]$$

$$= \mathbb{E}_Y \left\{ \mathbb{E}\left[\underbrace{\frac{d^2Y}{d\delta^2}}_{=0 \text{ for } y=0} | Y=0\right] \mathbb{P}(Y=0) + \mathbb{E}\left[\frac{d^2Y}{d\delta^2} | Y>0\right] \mathbb{P}(Y>0) \right\}$$

$$= \mathbb{E}_Y \left\{ \mathbb{E}\left[\frac{d^2Y}{d\delta^2} | Y>0\right] \underbrace{\mathbb{P}(Y>0)}_q \right\}$$

$$= \mathbb{E}_Y \left\{ q \left(y\alpha^2\beta^2 - y\alpha\beta + \alpha\beta^2\tau - \tau\beta + \beta^2\tau^2 - \beta^2\tau \right) \right\}$$

$$= q (\mathbb{E}[y]\alpha^2\beta^2 - \mathbb{E}[y]\alpha\beta + \alpha\beta^2\tau - \tau\beta + \beta^2\tau^2 - \beta^2\tau)$$

where $\mathbb{E}[y] = q\tau e^\delta$, as derived on page 6 .

$$\textcircled{3} \quad \frac{d^3L}{dt^3} = p \left\{ -8yx^3\beta^2 + 3yx^2\beta - 2x^2\beta^2z - yx + 3\alpha\beta z - 3\alpha\beta^2z^2 + 3\alpha\beta^2t - t + 3\beta t^2 - 3\beta z - 2\beta^2z^3 + 3\beta^2t^2 - \beta^2z^2 \right\}$$

Proof. Using a numerical linear algebra package, set ∞ . \square

$$\textcircled{4} \quad \frac{d^4L}{dt^4} = p \left\{ 6yx^4\beta^3 - 12yx^3\beta^2 + 6x^3\beta^3z + 7yx^2\beta - 12x^2\beta^2z + 11x^2\beta^3z^2 - 11x^2\beta^3t - yx + 7\alpha\beta z - 18\alpha\beta^2z^2 + 18\alpha\beta^2t + 12\alpha\beta^3z^3 - 18\alpha\beta^3t^2 + 6\alpha\beta^3z^2 - t + 7\beta t^2 - 7\beta z - 12\beta^2z^3 + 18\beta^2t^2 - 6\beta^2z + 6\beta^3t^2 - 12\beta^3z^3 + 7\beta^3t^2 - \beta^3z^2 \right\}$$

Proof. ∞ \square

As with L^{22} , some care is required to ensure that the derivatives evaluate accurately without overflow even as with a range of r and κ as possible. As $r \rightarrow \infty$, $\beta = \frac{1}{r-\kappa} \rightarrow \frac{1}{\kappa}$ for fixed κ , while

$$t = \frac{1}{1 - (1+\kappa r)^{-1/\kappa}} \rightarrow 1, \quad \text{since } 1+r^{\kappa} \rightarrow \infty \Rightarrow (1+r^{\kappa})^{-1/\kappa} \rightarrow 0$$

whereas as $r \rightarrow -\infty$, $\beta \rightarrow \frac{1}{r-\kappa} \rightarrow 0$ for fixed κ , while $r^{\kappa} \rightarrow 0$
 $\Rightarrow (1+r^{\kappa})^{-1/\kappa} \rightarrow 1 + \frac{1}{\kappa} r^{\kappa} = 1+r^{\kappa} \rightarrow 0$, $t = \frac{(1+r^{\kappa})^{-1/\kappa}}{(1+r^{\kappa})^{1/\kappa}}$

first
order
Taylor
expansion
about $r=\kappa=0$

$$\rightarrow \frac{1+r^{\kappa}}{1+r^{\kappa}-1} = \frac{1+r^{\kappa}}{r^{\kappa}} \\ = e^{-\kappa} + 1 \rightarrow e^{-\kappa}$$

* How do we deal with products: $\beta z, \beta^2 z, \beta^3 z$ etc. what does it converge to?

Since as $r \rightarrow \infty$, $\beta \rightarrow r^{\kappa}$ and $z \rightarrow e^{-r}$
 $\Rightarrow \beta z \rightarrow 1 \Rightarrow (\beta z)^{\alpha} \rightarrow 1 + \alpha r^{\kappa}$
 (alternatively),

$$\lim_{r \rightarrow \infty} \beta z = \lim_{r \rightarrow \infty} \frac{\beta}{1 - (1+r^{\kappa})^{-1/\kappa}}$$

$$\begin{aligned}
 (1^{\text{st}} \text{ Hopital}) &= \lim_{r \rightarrow -\infty} \frac{\beta - \alpha \beta^2}{\gamma_k \alpha r^k (1 + \alpha r^\alpha)^{-\gamma_k - 1}} \\
 &= \lim_{r \rightarrow -\infty} \frac{\beta (1 - \alpha \beta)}{r^k (1 + \alpha r^\alpha)^{-\gamma_k - 1}} \\
 &\Rightarrow \lim_{r \rightarrow -\infty} \frac{e^r (1 - \alpha \beta)}{r^k (1 + \alpha r^\alpha)^{-\gamma_k - 1}} \\
 &= \lim_{r \rightarrow -\infty} \frac{(1 - \alpha \beta)}{(1 + \alpha r^\alpha)^{-\gamma_k - 1}} = 1
 \end{aligned}$$

also ; $\beta^a/b \rightarrow 0$ if $a > b$ as $r \rightarrow -\infty$
as $r \rightarrow \infty$, $\beta \rightarrow \alpha^{-1}$ and $\alpha \rightarrow 1 \Rightarrow \beta \rightarrow \alpha^{-1}$

with these we get

$$\begin{aligned}
 \frac{\partial L}{\partial r} &\rightarrow y - 1 \text{ as } r \rightarrow -\infty, \quad \frac{\partial L}{\partial r} \rightarrow y - y - \frac{1}{\alpha} = -\frac{1}{\alpha} \text{ as } r \rightarrow \infty \\
 \frac{\partial^2 L}{\partial r^2} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^2 L}{\partial r^2} \rightarrow y - y + \frac{3}{\alpha^2} - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^2} \\
 &= \frac{1}{\alpha} - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^2} = 0 \text{ as } r \rightarrow \infty \\
 \frac{\partial^3 L}{\partial r^3} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^3 L}{\partial r^3} \rightarrow -2y + 3y - 2\frac{1}{\alpha} - y + \frac{3}{\alpha} \\
 &\quad - \frac{3}{\alpha^2} + \frac{3}{\alpha^2} - \frac{1}{\alpha} + \frac{3}{\alpha^2} \\
 &\quad - \frac{3}{\alpha^2} - \frac{2}{\alpha^3} \\
 &\quad + \frac{3}{\alpha^3} - \frac{1}{\alpha^3} \\
 &= \frac{1}{\alpha} - \frac{1}{\alpha} + \frac{3}{\alpha^2} - \frac{3}{\alpha^2} \\
 &\quad + 0 = 0 \text{ as } r \rightarrow \infty \\
 \frac{\partial^4 L}{\partial r^4} &\rightarrow 0 \text{ as } r \rightarrow -\infty, \quad \frac{\partial^4 L}{\partial r^4} \rightarrow 0 \text{ as } r \rightarrow \infty
 \end{aligned}$$

For extended GAsns in which γ is a fn of σ and extra parameters
 e.g.,

$$\gamma = \theta_1 + (\nu + e^{\theta_2}) \sigma \quad (\text{as for the zip model in blood at al (2016)})$$

$$\frac{d\gamma}{d\theta_1} = 1, \quad \frac{d\gamma}{d\theta_2} = e^{\theta_2} \sigma, \quad \frac{d\gamma}{d\sigma} = \nu + e^{\theta_2}, \quad \frac{d^2\gamma}{d\sigma^2} = 0$$

$$\frac{d^2\gamma}{d\theta_1^2} = \frac{d^2\gamma}{d\theta_1 d\theta_2} = \frac{d^2\gamma}{d\theta_2 d\theta_1} = 0, \quad \frac{d^2\gamma}{d\theta_2^2} = e^{\theta_2} \sigma, \quad \frac{d^2\gamma}{d\sigma d\theta_2} = e^{\theta_2}$$

$$\frac{d^3\gamma}{d\sigma d\theta_1^2} = 0, \quad \frac{d^3\gamma}{d\sigma d\theta_2 d\theta_1} = 0, \quad \frac{d^3\gamma}{d\sigma d\theta_2^2} = \nu,$$

$$\frac{d^3\gamma}{d\sigma^2 d\theta_1} = \frac{d^3\gamma}{d\sigma^2 d\theta_2} = \frac{d^3\gamma}{d\sigma^3} = 0$$

$$\frac{d^4\gamma}{d\sigma^4} = \frac{d^4\gamma}{d\sigma^3 d\theta_1}, \quad \frac{d^4\gamma}{d\sigma^3 d\theta_2} = \frac{d^4\gamma}{d\sigma^2 d\theta_1^2} = \frac{d^4\gamma}{d\sigma^2 d\theta_2^2}$$

$$= \frac{d^4\gamma}{d\sigma^2 d\theta_1 d\theta_2} = 0$$

Total derivatives.

I We focus on θ_i , for $i \in \{1, 2\}$. Derivatives involving θ_0 are considered in section 2.

$$\begin{aligned} \frac{dy}{dr} &= \frac{\partial y}{\partial r} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{\partial r}, \quad \frac{d^2L}{dr^2} = \frac{\partial^2 L}{\partial r^2} + \frac{\partial^2 L}{\partial \eta \partial r} \cdot \frac{\partial \eta}{\partial r} \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial r} \overset{\text{cancel}}{=} 0 \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial r} \right)^2 \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} - \frac{\partial^2 \eta}{\partial r^2} \\ &= \frac{\partial^2 L}{\partial r^2} + \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \eta}{\partial r} \right)^2 \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r^2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\theta_i} &= \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{d\theta_i}, \quad \frac{d^2L}{dr d\theta_i} = \frac{\partial^2 L}{\partial r \partial \eta} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial r} \\ &\quad \underset{\approx 0}{\sim} \\ &= \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial r} \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 L}{\partial r^2 d\theta_i} &= \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r^2} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial^2 \eta}{\partial r^2} \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^3 \eta}{\partial r^2 \partial \theta_i} \\ &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial \eta}{\partial r} \cdot \frac{\partial \eta}{\partial \theta_i} + \frac{\partial \eta}{\partial \theta_i} \cdot \frac{\partial^2 \eta}{\partial r^2} \right) \\ &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^3 \eta}{\partial r^2 \partial \theta_i} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^3 L}{\partial r^3} &= \frac{\partial^3 L}{\partial \eta^3} + \frac{\partial^3 L}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial r} \cdot \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial r^2} \cdot \frac{\partial^2 \eta}{\partial \eta^2} \right] \\
 &\quad + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^3 \eta}{\partial r^3} \\
 &= \frac{\partial^3 L}{\partial \eta^3} + \frac{\partial^3 L}{\partial \eta^2} \left(\frac{\partial \eta}{\partial r} \right)^3 + 3 \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial r^2} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^3 \eta}{\partial r^3}
 \end{aligned}$$

$$\frac{\partial^2 L}{\partial \eta_i \partial \eta_j} = \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \eta_j} \frac{\partial \eta}{\partial \eta_i} + \frac{\partial^2 L}{\partial \eta^2} \cdot \frac{\partial^2 \eta}{\partial \eta_i \partial \eta_j}$$

$$\begin{aligned}
 \frac{\partial^3 L}{\partial r \partial \eta_i \partial \eta_j} &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \eta_j} \frac{\partial \eta}{\partial \eta_i} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \eta_i \partial \eta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \eta_i \partial r} \frac{\partial \eta}{\partial \eta_j} \right] \\
 &\quad + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \eta_j} \frac{\partial^2 \eta}{\partial \eta_i \partial r} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^3 \eta}{\partial r \partial \eta_i \partial \eta_j} \\
 &= \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \eta_i} \frac{\partial \eta}{\partial \eta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \eta_i \partial \eta_j} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \eta_i \partial r} \frac{\partial \eta}{\partial \eta_j} \right] \\
 &\quad + \frac{\partial \eta}{\partial \eta_j} \frac{\partial^2 \eta}{\partial \eta_i \partial r} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^3 \eta}{\partial r \partial \eta_i \partial \eta_j}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^4 L}{\partial r \partial \eta_i \partial \eta_j \partial \eta_k} &= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \eta_j} \frac{\partial \eta}{\partial \eta_i} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^3 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \eta_i \partial \eta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial^2 \eta}{\partial \eta_i \partial r} \frac{\partial \eta}{\partial \eta_j} \right] \\
 &\quad + \frac{\partial^3 L}{\partial \eta^3} \frac{\partial \eta}{\partial \eta_j} \left(2 \frac{\partial^2 \eta}{\partial \eta_i \partial \eta_k} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \eta_i \partial r} \cdot \frac{\partial \eta}{\partial \eta_k} \right) \\
 &\quad + \frac{\partial^2 L}{\partial \eta^2} \left(\frac{\partial^3 \eta}{\partial \eta_i \partial \eta_j \partial \eta_k} \frac{\partial \eta}{\partial r} + \frac{\partial^2 \eta}{\partial \eta_i \partial \eta_k} \frac{\partial^2 \eta}{\partial \eta_j \partial r} + \frac{\partial^3 \eta}{\partial \eta_i \partial \eta_j \partial r} \frac{\partial \eta}{\partial \eta_k} \right. \\
 &\quad \left. + \frac{\partial^2 \eta}{\partial \eta_i \partial r} \frac{\partial \eta}{\partial \eta_j} \frac{\partial^2 \eta}{\partial \eta_k \partial r} \right) \\
 &\quad + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial \eta_i} \frac{\partial^3 \eta}{\partial \eta^2 \partial \eta_j \partial \eta_k} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^4 \eta}{\partial r^2 \partial \eta_i \partial \eta_j \partial \eta_k}
 \end{aligned}$$

$$= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^2$$

$$+ \frac{\partial^4 L}{\partial \eta^3} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} \left(\frac{\partial \eta}{\partial r} \right)^2 + 2 \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \right]$$

$$+ \frac{1}{2} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j} \frac{\partial^2 \eta}{\partial r^2}$$

$$+ \frac{\partial^4 L}{\partial \eta^2} \left[\frac{2 \partial^3 \eta}{\partial \theta_i \partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r} + \frac{2 \partial^2 \eta}{\partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_i} + \frac{\partial^3 \eta}{\partial \theta_i \partial \theta_j} \frac{\partial \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_j} + \frac{\partial^3 \eta}{\partial r^2 \partial \theta_i} \frac{\partial^2 \eta}{\partial \theta_j}$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i \partial \theta_j}$$

$$\frac{\partial^4 L}{\partial \theta_i^3} = \frac{\partial^4 L}{\partial \theta_i^3 \partial \eta} - \frac{\partial^2 \eta}{\partial \theta_i} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^3 L}{\partial \eta^3} - 3 \left(\frac{\partial \eta}{\partial r} \right) \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i}$$

$\xrightarrow{=0}$

$$+ \frac{3 \partial^3 L}{\partial \eta^3} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} \right.$$

$$\left. + \frac{\partial^2 \eta}{\partial r} \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^2 \eta}{\partial \theta_i} \frac{\partial^3 \eta}{\partial r^3} + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i}$$

$$= \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial \theta_i} \left(\frac{\partial \eta}{\partial r} \right)^3 + 3 \frac{\partial^3 L}{\partial \eta^3} \left[\left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} + \frac{\partial \eta}{\partial \theta_i} \frac{\partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial r^2} \right]$$

$$+ \frac{\partial^2 L}{\partial \eta^2} \left[\frac{\partial^2 \eta}{\partial \theta_i \partial \theta_i} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial \theta_i \partial \theta_i} + \frac{\partial^3 \eta}{\partial r^3} \frac{\partial \eta}{\partial \theta_i} \right]$$

$$+ \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2 \partial \theta_i}$$

$$\frac{\partial^4 L}{\partial r^4} = \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial \eta^4} \frac{\partial \eta}{\partial r} \left(\frac{\partial \eta}{\partial r} \right)^3 + \frac{\partial^2 L}{\partial \eta^2} \cdot 3 \left(\frac{\partial \eta}{\partial r} \right)^2 \frac{\partial^2 \eta}{\partial r^2}$$

$$+ 3 \frac{\partial^3 L}{\partial \eta^3} \cdot \frac{\partial \eta}{\partial r} \frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} + \frac{3 \partial^2 L}{\partial \eta^2} \left[\frac{\partial \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^2} \frac{\partial^2 \eta}{\partial r^2} + \right.$$

$$\left. \frac{\partial^2 \eta}{\partial r} \frac{\partial^2 \eta}{\partial r^3} \right] + \frac{\partial^2 L}{\partial \eta^2} \frac{\partial \eta}{\partial r} \frac{\partial^3 \eta}{\partial r^2} + \frac{\partial L}{\partial \eta} \frac{\partial^4 \eta}{\partial r^2}$$

$$\begin{aligned}
&= \frac{\partial^4 L}{\partial r^4} + \frac{\partial^4 L}{\partial y^4} \left(\frac{\partial y}{\partial r} \right)^4 + 6 \frac{\partial^3 L}{\partial y^3} \left(\frac{\partial y}{\partial r} \right)^2 \frac{\partial^2 L}{\partial r^2} \\
&\quad + \frac{\partial^2 L}{\partial y^2} \left[3 \left(\frac{\partial^2 y}{\partial r^2} \right)^2 + 4 \frac{\partial y}{\partial r} \frac{\partial^3 y}{\partial r^3} \right] + \frac{\partial L}{\partial y} \frac{\partial^4 y}{\partial r^4}
\end{aligned}$$

II Derivatives involving θ_0

Since $x = \gamma_r$ and $r \in R_{>0} \Rightarrow x \in R_{>0}$. It is however easier to work with unrestricted parameter. Therefore, we define $\alpha := e^{\theta_0}$ with $\theta_0 \in \mathbb{R}$ unrestricted.

Recall that for $y > 0$,

$$\begin{aligned}
L^{2x} &= \log(1 - e^{-2x}) + y \log \alpha + yx - y \log(1 + \alpha e^x) - \log\{(1 + \alpha e^x)^k - 1\} \\
&\quad + \log \Gamma(y + \frac{1}{\alpha}) - \log \Gamma(y+1) - \log \Gamma(\frac{y}{\alpha})
\end{aligned}$$

$$\begin{aligned}
(\alpha = e^{\theta_0}) &= \log(1 - e^{-2x}) + y \log e^{\theta_0} + yx - y \log(1 + e^{\theta_0} e^x) - \log\{(1 + e^{\theta_0} e^x)^k - 1\} \\
&\quad + \log \Gamma(y + e^{\theta_0}) - \log \Gamma(y+1) - \log \Gamma(\frac{y}{e^{\theta_0}})
\end{aligned}$$

Note that if $x = (1 + e^{\theta_0} e^x)^{\frac{1}{e^{-\theta_0}}}$, then
 $\log x = e^{\theta_0} \log(1 + e^{\theta_0} e^x)$

$$\begin{aligned}
0 \quad \frac{dy}{d\theta_0} &= \frac{e^{-\theta_0}}{1 + e^{\theta_0} e^x} \cdot \frac{e^{\theta_0} e^x}{e^{-\theta_0}} + \log(1 + e^{\theta_0} e^x) \cdot e^{-\theta_0} \cdot -1
\end{aligned}$$

$$= \frac{e^x}{1 + e^{\theta_0} e^x} - e^{\theta_0} \log(1 + e^{\theta_0} e^x)$$

$$\Rightarrow \frac{dy}{d\theta_0} = \left(1 + e^{\theta_0} e^x \right)^{\frac{1}{e^{-\theta_0}}} \left\{ \beta - e^{\theta_0} \log(1 + e^{\theta_0} e^x) \right\}$$

also, $\frac{d}{dz} \log \Gamma(z) =: \psi(z)$, where ψ is the digamma fn.

Now dropping the $2x$ superscript, we have,

$$\frac{dy}{d\theta_0} = 0 + y/e^{\theta_0} - e^{\theta_0} + 0 - y \cdot \frac{e^{\theta_0} e^x}{1 + e^{\theta_0} e^x} - \frac{1}{(1 + e^{\theta_0} e^x)^{e^{-\theta_0}}} - 1$$

$$\left. \left(1 + e^{\theta_0} e^r \right)^{e^{-\theta_0}} \right\} \beta - e^{-\theta_0} \log \left(1 + e^{\theta_0} e^r \right) \Big]$$

$$- \varphi(y + e^{-\theta_0}) e^{-\theta_0} + \varphi(e^{-\theta_0}) e^{-\theta_0}$$

Define $\omega := e^{-\theta_0} \log(1 + e^{\theta_0} e^r) - \beta$

$$= y - y \beta e^{\theta_0} + \tau \omega - e^{-\theta_0} \varphi(y + e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})$$

To compute $d^2 l / d \theta_0^2$, we need $d \beta / d \theta_0$, and $d \gamma / d \theta_0$,

$$\frac{d \beta / d \theta_0}{d \theta_0} = \frac{d}{d \theta_0} \frac{\beta r}{1 + e^{\theta_0} e^r} = \frac{1 + e^{\theta_0} e^r \cdot 0 - e^r \cdot e^r e^{\theta_0}}{(1 + e^{\theta_0} e^r)^2}$$

$$= - \frac{e^{2r}}{(1 + e^{\theta_0} e^r)^2} e^{\theta_0} = - \beta^2 e^{\theta_0}$$

Let leave the rest to the computer; see ..;

For $\varphi^{(1)}$, see
Polygamma
Fn on Wikipedia.

$$\begin{aligned} \frac{d^2 l}{d \theta_0^2} &= -y \beta e^{\theta_0} + y \beta^2 e^{2\theta_0} + \tau^2 \omega^2 - \tau \omega^2 + \tau (\beta e^{\theta_0} - \omega) \\ &\quad + e^{-\theta_0} \varphi(y + e^{-\theta_0}) - e^{-\theta_0} \varphi(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) \\ &\quad - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0}) \end{aligned}$$

* How do we deal with products $\tau \omega \dots$

Since as $r \rightarrow -\infty$, $\tau \rightarrow +\infty$ and $\log(1 + e^{\theta_0} e^r) \rightarrow \log(1) = 0$ we end up with the product $+\infty \cdot 0$, which is indeterminate. But we have hope if $\tau \rightarrow +\infty$ slower than the terms of interact with $\rightarrow 0$, in which case the interaction $\rightarrow 0$ as $r \rightarrow -\infty$.

Taking θ_0 as fixed, we have

$$(i) \lim_{r \rightarrow -\infty} \tau \log(1 + e^{\theta_0} e^r) = \lim_{r \rightarrow -\infty} \frac{\log(1 + e^{\theta_0} e^r)}{1 - (1 + e^{\theta_0} e^r)^{-\tau - \theta_0}}$$

$$\begin{aligned} (\text{L'Hopital}) &= \lim_{r \rightarrow -\infty} \frac{e^{\theta_0} e^r}{(1 + e^{\theta_0} e^r)(0 + e^{-\theta_0} (1 + e^{\theta_0} e^r)^{-\tau - \theta_0} \cdot e^{\theta_0} e^r)} \\ &= \lim_{r \rightarrow -\infty} \frac{e^{\theta_0} e^r}{e^r (1 + e^{\theta_0} e^r)^{-\tau - \theta_0}} \end{aligned}$$

$$= \lim_{r \rightarrow -\infty} \frac{e^{\theta_0}}{(1+e^{\theta_0} e^r)^{-e^{\theta_0}}} = e^{\theta_0}$$

$$\Rightarrow (r \log(1+e^{\theta_0} e^r))^a \rightarrow e^{a\theta_0} \text{ if } a \in \mathbb{R}$$

also, $r^a \log(1+e^{\theta_0} e^r)^b \rightarrow 0$ if $b > a$ as $r \rightarrow -\infty$, $a, b \in \mathbb{R}$

$$\begin{aligned} \text{(ii)} \quad & \lim_{r \rightarrow -\infty} r \left(e^{-\theta_0} \log(1+e^{\theta_0} e^r) - p \right), \stackrel{(i)}{=} \frac{e^{-\theta_0} - e^{\theta_0} - 1}{r} \xrightarrow[r \rightarrow -\infty]{\beta \rightarrow 1, \text{ as } r \rightarrow -\infty} 0 \\ & \Rightarrow \lim_{r \rightarrow -\infty} r w = 0, \quad \Rightarrow \lim_{r \rightarrow -\infty} (rw)^a = 0 \text{ if } a \in \mathbb{R} \\ & \Rightarrow \lim_{r \rightarrow -\infty} r^a w^b = 0 \text{ if } a, b \in \mathbb{R} \text{ if } b > a \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & \lim_{r \rightarrow -\infty} r \left(p + \beta^2 e^{\theta_0} - e^{-\theta_0} \log(1+e^{\theta_0} e^r) \right) = \lim_{r \rightarrow -\infty} r \left(\beta^2 e^{\theta_0} - w \right) \\ & = \lim_{r \rightarrow -\infty} (\beta^2 r e^{\theta_0} - rw) = 0 \end{aligned}$$

for fixed θ_0
 For $r \rightarrow -\infty$, $r \rightarrow 1$, $\beta \rightarrow e^{-\theta_0}$, $\log(1+e^{\theta_0} e^r) \approx \log(e^{\theta_0} e^r)$
 since $e^{\theta_0} e^r$ will dominate 1. and $\log(e^{\theta_0} e^r) \rightarrow \theta_0 + r$.
 We'll use this even though we don't have indeterminacy since
 the fit implementation also uses it. See how $\log(e^r - 1)$ is approximated
 as e^r as $r \rightarrow \infty$ for the gip model likelihood here 😎.

With these, we get, as $r \rightarrow -\infty$

$$\frac{dy}{d\theta_0} \rightarrow y - e^{-\theta_0} \varphi(y + e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})$$

$$\begin{aligned} \frac{d^2y}{d\theta_0^2} \rightarrow & e^{-\theta_0} \varphi(y + e^{-\theta_0}) - e^{-\theta_0} \varphi(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) \\ & - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0}) \end{aligned}$$

when $\gamma \rightarrow -\infty$

$$\frac{dV}{d\theta_0} \rightarrow y - y e^{-\theta_0} e^{\theta_0} - (-e^{-\theta_0} (\theta_0 + r) + e^{-\theta_0}) \\ - e^{-\theta_0} \varphi(y + e^{-\theta_0}) + e^{-\theta_0} \varphi(e^{-\theta_0})$$

$$= e^{-\theta_0} \{ \theta_0 + r - 1 - \varphi(y + e^{-\theta_0}) + \varphi(e^{-\theta_0}) \}$$

$$\frac{d^2 V}{d\theta_0^2} \rightarrow e^{-\theta_0} + e^{-2\theta_0} e^{\theta_0} - e^{-\theta_0} (\theta_0 + r) + e^{-\theta_0} \varphi(y + e^{-\theta_0}) \\ - e^{-\theta_0} \varphi(e^{-\theta_0}) + e^{-2\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) - e^{-2\theta_0} \varphi^{(1)}(e^{-\theta_0})$$

$$= e^{-\theta_0} \{ 2 - \theta_0 - r + \varphi(y + e^{-\theta_0}) - \varphi(e^{-\theta_0}) \\ + e^{-\theta_0} \varphi^{(1)}(y + e^{-\theta_0}) - e^{-\theta_0} \varphi^{(1)}(e^{-\theta_0}) \}$$

$$\frac{d^2 V}{d\theta_0 d\theta_1} = \frac{d^2 V}{d\theta_0 d\theta_2} = 0$$

$$\frac{d^3 V}{d\tau d\theta_0} = -y \beta e^{\theta_0} + y \beta^2 e^{2\theta_0} - \beta \tau^2 \omega + \beta \tau \omega + e^{\theta_0} \beta^2 \tau$$

$$\frac{d^3 V}{d\tau^2 d\theta_0} = -y \beta e^{\theta_0} + 3y \beta^2 e^{2\theta_0} - 2y \beta^3 e^{3\theta_0} - \beta \tau^2 \omega + \beta \tau \omega + 2\beta^2 \tau^3 \omega + e^{\theta_0} \beta^2 \tau^2 \omega \\ - 3\beta^2 \tau^2 \omega - e^{\theta_0} \beta^2 \tau \omega + \beta^2 \tau \omega + 2e^{\theta_0} \beta^2 \tau - 2e^{\theta_0} \beta^3 \tau^2 \\ - 2\tau^2 \beta^3 \omega + 2\tau^2 \beta^3 \omega$$

$$\frac{d^3 V}{d\tau d\theta_0 d\theta_1} = \frac{d^3 V}{d\tau d\theta_0 d\theta_2} = 0$$

$$\frac{d^3 V}{d\theta d\theta_0^2} = -y \beta e^{\theta_0} + 3y \beta^2 e^{2\theta_0} - 2y \beta^2 e^{3\theta_0} - 2\beta \tau^3 \omega^2 + 3\beta^2 \tau^2 \omega^2 \\ - \beta \tau^2 (\beta^2 e^{\theta_0} - \omega) - \beta \tau \omega^2 + \beta \tau (\beta^2 e^{\theta_0} - \omega) \\ + 2\beta^2 \tau^2 \omega e^{\theta_0} - 2\beta^2 \tau \omega e^{\theta_0} + \beta^2 \tau e^{\theta_0} - 2\beta^3 \tau e^{2\theta_0}$$

$$\frac{d^4 V}{d\tau^2 d\theta_0 d\theta_1} = \frac{d^4 V}{d\tau^2 d\theta_0 d\theta_2} = 0$$

$$\frac{d^4 V}{d\theta^2 d\theta_0^2} = -y \beta e^{\theta_0} + 7y \beta^2 e^{2\theta_0} - 12y \beta^3 e^{3\theta_0} + 6y \beta^4 e^{4\theta_0} - 2\beta \tau^5 \omega^2 \\ + 3\beta^2 \tau^2 \omega^2 - \beta \tau^4 (\beta^2 e^{\theta_0} - \omega) - \beta \tau \omega^2 + \beta \tau (\beta^2 e^{\theta_0} - \omega) \\ + 6\beta^2 \tau^4 \omega^2 + 2\beta^2 \tau^3 \omega^2 e^{\theta_0} - 12\beta^2 \tau^5 \omega^2 + 2\beta^2 \tau^3 (\beta^2 e^{\theta_0} - \omega) \\ - 3\beta^2 \tau^2 \omega^2 e^{\theta_0} + 7\beta^2 \tau^2 \omega^2 + 4\beta^2 \tau^2 \omega e^{\theta_0} + \beta^2 \tau^2 \tau^2 (\beta^2 e^{\theta_0} - \omega)$$

(15 terms)

$$\begin{aligned}
 & -3\beta^2\tau^2 (\beta^2 e^{0^\circ} - \omega) + \beta^2 \tau w^2 e^{0^\circ} - \beta^2 \tau \omega^2 - 4\beta^2 \tau w e^{0^\circ} \\
 & - \beta^2 \tau e^{0^\circ} (\beta^2 e^{0^\circ} - \omega) + \beta^2 \tau (\beta^2 e^{0^\circ} - \omega) + 2\beta^2 \tau e^{0^\circ} \\
 & - 8\beta^3 \tau^3 w \tau^{0^\circ} - 4\beta^3 \tau^2 \omega \tau^{0^\circ} + 12\beta^3 \tau^2 w e^{0^\circ} - 2\beta^3 \tau^2 e^{0^\circ} \\
 & + 4\beta^3 \tau w \tau^{0^\circ} - 4\beta^3 \tau w e^{0^\circ} - 8\beta^3 \tau \tau^{0^\circ} + 2\beta^3 \tau e^{0^\circ} + 6\beta^3 \tau^2 e^{0^\circ} \\
 & + 6\beta^4 \tau \tau^{0^\circ} - 6\beta^4 \tau e^{0^\circ}
 \end{aligned}$$

$$\begin{aligned}
 \frac{d^4 C}{dt^4 \theta_0} = & -y\beta^2 e^{80^\circ} + 7y\beta^2 e^{10^\circ} - 12y\beta^3 e^{30^\circ} + 6y\beta^4 e^{40^\circ} - \beta^2 \omega f_{\beta^2 \omega} \\
 & + 6\beta^2 \tau^3 \omega + 8\beta^2 \tau^2 \omega e^{80^\circ} - g\beta^2 \tau^2 \omega - 3\beta^2 \tau \omega e^{80^\circ} + 3\beta^2 \tau \omega \\
 & + 4\beta^2 \tau \omega e^{80^\circ} - 6\beta^3 \tau^9 \omega - 6\beta^3 \tau^3 \omega e^{80^\circ} + 12\beta^3 \tau^3 \omega - 12\beta^3 \tau^2 \omega e^{20^\circ} \\
 & + 9\beta^3 \tau^2 \omega e^{80^\circ} - 7\beta^3 \tau^2 \omega - 9\beta^3 \tau^2 \omega e^{80^\circ} + 2\beta^3 \tau \omega e^{20^\circ} - 3\beta^3 \tau \omega e^{80^\circ} \\
 & + \beta^3 \tau \omega - 10\beta^3 \tau \omega e^{10^\circ} + 9\beta^3 \tau \omega e^{80^\circ} + 6\beta^4 \tau^3 \omega e^{80^\circ} + 9\beta^4 \tau^2 \omega e^{20^\circ} \\
 & - 9\beta^4 \tau^2 \omega e^{80^\circ} + 6\beta^4 \tau \omega e^{30^\circ} - 9\beta^4 \tau \omega e^{20^\circ} + 3\beta^4 \tau \omega e^{80^\circ}.
 \end{aligned}$$

$$\text{as } \tau \rightarrow -\infty \quad (\beta \rightarrow 0, \quad \beta \tau \rightarrow 1, \quad \tau \omega \rightarrow 0)$$

$$\frac{d^2C}{dr d\theta_0} \rightarrow -0 \leftarrow -\lim_{r \rightarrow -\infty} \beta v^r w + \underbrace{\lim_{r \rightarrow -\infty} \beta v^r w}_{=0} + 0 \\ = \lim_{x \rightarrow -\infty} (\beta x) \lim_{x \rightarrow -\infty} (vw) = 1 \cdot 0 = 0$$

$$\frac{d^3V}{d\sigma^2 d\theta_0} \rightarrow 0 \quad , \quad \frac{d^3L}{d\tau d\theta_0^2} \rightarrow 0$$

$$\frac{d^4 C}{d\gamma^2 d\theta_0^2} \rightarrow 0 \quad \text{and} \quad \frac{d^4 C}{d\gamma^3 d\theta_0} \rightarrow 0$$

$$a) \quad \tau \rightarrow +\infty \quad (\beta \rightarrow \frac{1}{\tau} \alpha = e^{-\tau \beta}, \quad \tau \rightarrow 1, \quad \tilde{x} := \lim_{\tau \rightarrow +\infty} x)$$

$$\frac{d^2L}{dt^2} \rightarrow -y e^{-\theta_0 t} + y \tilde{e}^{-2\theta_0 t} - \tilde{\beta} \omega + \tilde{\beta} \omega + e^{\theta_0 t} e^{-2\theta_0 t}$$

$$= -y f_y - \tilde{\beta} u f_{\beta u} + e^{-\theta_0} = e^{-\theta_0} = 1/\alpha$$

$$\frac{d^3L}{dr^2 d\theta_0} \rightarrow -y' + \cancel{\beta y} - \cancel{\beta y} - \cancel{\beta^2 \omega} + \cancel{\beta \omega} + \cancel{\omega^2 \beta^2 \omega} \\ - 3 \cancel{\beta^2 \omega} - e^{\theta_0 \beta^2} \cancel{\omega^2 \beta^2 \omega} + \cancel{\beta^2 \omega} + e^{2r \theta_0} \cancel{\beta^2} - 2e^{\theta_0 \beta^2} \\ - 2e^{\theta_0 \beta^2} + 2e^{\theta_0 \beta^2}$$

$$= d \ell e^{\theta_0 - 2\theta_0} - d \ell e^{2\theta_0 - 3\theta_0}$$

$$= \alpha e^{-\theta_0} - \alpha e^{-\theta_0} = 0$$

$$\begin{aligned}
 \frac{d^4y}{dr^2 d\theta_0} &\rightarrow -y + 7y - 12y + 6y - 2\beta^2 w^2 + 3\beta^2 e^{2\theta_0} \\
 &\quad - \beta^2 e^{2\theta_0} + \beta^2 w - \beta^2 e^{2\theta_0} + \beta^2 e^{2\theta_0} - \beta^2 w \\
 &\quad + 6\beta^2 w^2 + 2\beta^2 w^2 e^{2\theta_0} - 12\beta^2 w^2 + 2\beta^2 e^{4\theta_0} \\
 &\quad - 2\beta^2 w - 3\beta^2 w^2 e^{2\theta_0} + 7\beta^2 e^{2\theta_0} + 4\beta^2 w^2 e^{2\theta_0} \\
 &\quad + \beta^2 e^{2\theta_0} - \beta^2 w^2 e^{2\theta_0} - 3\beta^2 e^{2\theta_0} + 3\beta^2 w + \beta^2 w^2 e^{2\theta_0} \\
 &\quad - \beta^2 w^2 - 4\beta^2 w^2 e^{2\theta_0} - \beta^4 e^{2\theta_0} + \beta^2 w e^{2\theta_0} + \beta^2 e^{2\theta_0} - \beta^2 w \\
 &\quad + 2\beta^2 e^{2\theta_0} - 8\beta^2 w e^{2\theta_0} - 4\beta^2 w^2 e^{2\theta_0} + 12\beta^2 w^2 e^{2\theta_0} - 2\beta^2 e^{2\theta_0} \\
 &\quad + 4\beta^3 w e^{2\theta_0} - 4\beta^3 w^2 e^{2\theta_0} - 8\beta^3 w^2 e^{2\theta_0} + 2\beta^3 w^2 e^{2\theta_0} + 6\beta^3 e^{2\theta_0} \\
 &\quad + 6\beta^4 e^{2\theta_0} - 6\beta^4 e^{2\theta_0} \\
 &= 2\beta^2 e^{2\theta_0} - 8\beta^3 e^{2\theta_0} + 6\beta^4 e^{2\theta_0} \\
 &= 2e^{-\theta_0} - 8e^{-2\theta_0} + 6e^{-3\theta_0} = 0
 \end{aligned}$$

$$\begin{aligned}
 \frac{d^4L}{d\theta^3 d\theta_0} &\rightarrow -y + 7y - 12y + 6y - \beta^2 w + 6\beta^2 w^2 + 3\beta^2 e^{2\theta_0} \\
 &\quad - 9\beta^2 w - 3\beta^2 w^2 e^{2\theta_0} + 3\beta^2 w + 4\beta^2 w^2 e^{2\theta_0} - 6\beta^2 w - 6\beta^3 w^2 e^{2\theta_0} \\
 &\quad + 12\beta^2 w - 2\beta^2 w^2 e^{2\theta_0} + 9\beta^2 w e^{2\theta_0} - 7\beta^2 w - 9\beta^2 w^2 e^{2\theta_0} + 2\beta^3 w^2 e^{2\theta_0} \\
 &\quad - 3\beta^3 w e^{2\theta_0} - \beta^3 w - 10\beta^3 w^2 e^{2\theta_0} + 9\beta^3 w^2 e^{2\theta_0} + 6\beta^3 w^2 e^{2\theta_0} + 9\beta^3 w^2 e^{2\theta_0} \\
 &\quad - 9\beta^4 w^2 e^{2\theta_0} + 6\beta^4 w^2 e^{2\theta_0} - 9\beta^4 w^2 e^{2\theta_0} + 3\beta^4 w^2 e^{2\theta_0} \\
 &= 4\beta^2 e^{2\theta_0} - 10\beta^3 e^{2\theta_0} + 6\beta^4 e^{2\theta_0} \\
 &= 4e^{-\theta_0} - 10e^{-2\theta_0} + 6e^{-3\theta_0} = 0
 \end{aligned}$$

On sampling.

For a zero-truncated density, f^* ;

$$f^*(x) = \begin{cases} \frac{f(x)}{1 - F(0)} & \text{for } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

where $F(0) = P(X \leq 0)$
in our case (discrete pmf)

the truncated CDF is

$$\begin{aligned} F(x) &= P(X = 0) \\ &\quad + P(X < 0) \\ &= f(0) + 0 = f(0) \end{aligned}$$

$$F^*(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \int_0^x \frac{f(u)}{1 - F(0)} du & \text{for } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{F(x) - F(0)}{1 - F(0)} & \text{for } x > 0 \end{cases}$$

To sample from this distribution, we proceed as follows;

- (p0) 1. Generate $U \sim U(0,1)$
2. Let $V = F(0) + (1 - F(0))U$
3. Return $X = F^{-1}(V)$

Proof. We show that $X \sim F^*$

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(V) \leq x) = P(Y \leq F(x)) \\ &= P(F(0) + (1 - F(0))U \leq F(x)) \\ &= P\left(U \leq \frac{F(x) - F(0)}{1 - F(0)}\right) \\ &= \begin{cases} 0 & \text{if } x < 0 \quad (\text{since } F(x) = 0) \\ \frac{F(x) - F(0)}{1 - F(0)} & \text{for } x > 0 \quad \text{and } P(U \leq -x) = 0 \end{cases} \\ &= f^*(x) \end{aligned}$$

□

$$\left(\begin{array}{l} \text{Since } U \sim U(0,1) \\ \Rightarrow P(U \leq u) = u \end{array} \right)$$

$$V \sim U(F(0), 1)$$

Proof.

$$P(V \leq v) = P(F(0) + (1-F(0))U \leq v)$$

$$= P\left(U \leq \frac{v - F(0)}{1 - F(0)}\right)$$

$$= \begin{cases} 0 & \text{if } v < F(0) \\ \frac{v - F(0)}{1 - F(0)} & \text{if } F(0) \leq v \leq 1 \\ 1 & \text{if } v > 1 \end{cases}$$

$$= U(F(0), 1) \quad \square$$

So, the procedure above can be reduced to
(P1)

1 Generate $V \sim U(F(0), 1)$

2 Return $X = F^{-1}(V)$

$$\text{again } P(X \leq x) = P(F^{-1}(V) \leq x) = P(V \leq F(x))$$

$$= \begin{cases} 0 & \text{if } F(x) \leq F(0) \Leftrightarrow x \leq 0 \\ \frac{F(x) - F(0)}{1 - F(0)} & \text{if } F(x) > F(0) \Rightarrow x > 0 \end{cases}$$

$$= F^*(x)$$

For the ZT negative binomial.

Recall the ZT NB density

$$f(y | y \geq 0, r, p) = \frac{\Gamma(y+r)}{\Gamma(y+1)\Gamma(r)} p^r (1-p)^y \quad \frac{1}{1-p^r}$$

Some care is need for sampling -

1. as $r \rightarrow 0$, $f(y|y \geq 0, r, p)$ is not defined b/c $1-p^r \rightarrow 0$

As $\Gamma(r)$, $\frac{\Gamma(y+r)}{\Gamma(y+1) \Gamma(r)} = \frac{\Gamma(y+r)}{y! \Gamma(r)}$

and using Euler's reflects formula $\bullet\bullet$

$$\Gamma(r) \Gamma(1-r) = \pi / \sin \pi r, \text{ as } r \rightarrow 0 \quad \Gamma(1-r) \rightarrow \Gamma(1) = 1! = 1$$

and $\sin \pi r \rightarrow \pi r$

$$\Rightarrow \Gamma(r) \rightarrow y_r$$

$$\Rightarrow \frac{\Gamma(y+r)}{\Gamma(y+r) \Gamma(r)} \rightarrow \frac{\Gamma(y)}{y! y_r} = \frac{r(y-1)!}{y!} = \frac{r}{y}.$$

as the first order MacLaurin expansion of p^r around $r=0$, &

$$p^r \approx p^0 + r \frac{dp^r}{dr} \Big|_{r=0} = 1 + r(p^r \ln p \Big|_{r=0}) \\ = 1 + r \ln p$$

$$\text{let } y = p^r$$

$$\Rightarrow \ln y = r \ln p \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow 1 - p^r \xrightarrow{r \rightarrow 0} -r \ln p$$

$$\Rightarrow \frac{dy}{dr} = \ln p$$

$$= p^r \ln p$$

$$\Rightarrow f(y|y \geq 0, r, p) \xrightarrow{r \rightarrow 0} \frac{y}{y!} \frac{(1-p)^y}{-r \ln p} = - \frac{(1-p)^y}{y \ln p}$$

This is exactly the logarithmic distribution with parameter $(1-p)$

so for small p , i.e., as $p \rightarrow 0$, sample from $\text{Log}(1-p)$

2. $p \rightarrow 1$,
again $f(y|y>0, r, p)$ is not defined;

Note that as $p \rightarrow 1$

$$P(y=1 | y>0, r, p) = f(1 | y>0, r, p)$$

$$= \frac{\Gamma(1+r)}{\Gamma(1+r) \Gamma(r)} \frac{p^r (1-p)^r}{1-p^r}$$

$$= \frac{r!}{(r-1)!} \frac{p^r (1-p)^r}{1-p^r} = \frac{rp^r (1-p)^r}{1-p^r} \quad 1-p \approx r(1-p)$$

Note that $1-p^r$ ^{first order} Taylor expanded about $p=1$ is

$$1-p^r \approx (1-r) + \frac{d(1-p^r)}{dp} \Big|_{p=1} (p-1) = 0 + -r p^{r-1} \Big|_{p=1} (p-1) \\ = -r(p-1) = r(1-p)$$

$$\Rightarrow \frac{r p^r (1-p)^r}{1-p^r} \approx \frac{r p^r (1-p)^r}{r(1-p)} = p^r \rightarrow 1 \text{ as } p \rightarrow 1$$

$$\Rightarrow P(y=1 | y>0, r, p) \rightarrow 1 \text{ as } p \rightarrow 1.$$

* Why do we take care of these cases?

One may wonder, since the truncated distribution F^* isn't used directly in the sampling procedure. So why did we derive these limits ($r \rightarrow 0$, $p \rightarrow 1$). If it is ∞ as $r \rightarrow 0$ or $p \rightarrow 1$, $F(0) (=f(0; r, p))$,

in our case) goes to 1 (p^0 or r^0). Thus the support of V would be very small $[[F(0), 1]]$. Thus the samples from $U(F(0), 1)$ would be fairly indistinguishable (and close to 1) due to machine precision and the quantile $F^{-1}(V)$ may generate values that are not 1 due to machine errors, but we would want 1s in these cases. Secondly, by resolving these limits we can directly set the value of our samples (e.g. for $p \rightarrow 1$) or sample directly from a known distribution (for $r \rightarrow 0$), which is more efficient than the procedure (lower runtime).

Sampling from hurdle model.

The hurdle is of the form

$$g(y) = \begin{cases} 1-p & \text{for } y=0 \\ \frac{P f(y)}{1-F(0)} & \text{for } y>0 \end{cases} \quad \xrightarrow{\substack{\text{CDF} \\ \rightarrow D}} G(y) = \begin{cases} 1-p & , y=0 \\ \frac{P F(T) - F(0)}{1-F(0)}, y>0 \end{cases}$$

To sample from this, we follow the following procedure.
 (P_2) , see Grimmett ch. 4.11 Example 7.

1. Generate $u \sim U(0, 1)$ and compute p .
2. If $u \leq 1-p$ set $y \leftarrow 0$
3. Otherwise generate $x \sim G(\cdot, u) = F^*$, by setting $x = F^{-1}(v)$, v being as in (P_1) and set $y \leftarrow x$

Prof. like prove that this procedure generates samples from the hurdle model.

$$P(Y \leq 0) = P(Y=0) = P(U \leq 1-p) = 1-p \quad \text{--- (1)}$$

and for $y \geq 0$

$$P(Y \leq y) = P(U > 1-p) \cdot P(X \leq y; U > 1-p)$$

$$= 1 - P(U \leq 1-p) F^*(y) \quad \xleftarrow{\text{2nd B.}} \quad \text{since } y \geq 0 \text{ (see def of } F^*)$$

$$= p F^*(y) = p \left(\frac{F(y) - F(0)}{1 - F(0)} \right) \quad \text{--- (2)} \quad \text{--- (1) and (2) gives (1)} \quad \square$$

(u here is different from u used in (P_1))

For us in 'predict' (se-fit)

$$\text{Recall } E[\tau] = q \tau e^r = q \frac{e^r}{1 - (1 + \kappa e^r)^{-\alpha}}, \quad r \sim ZTMB.$$

Let the ZTMB mean be $\bar{\mu}$,

$$\begin{aligned} \bar{\mu} &= \frac{e^r}{1 - (1 + \kappa e^r)^{-\alpha}}, \quad d\bar{\mu}/dr = e^r \left[\frac{1 + \kappa}{\alpha} \right] + \kappa e^r \\ &= e^r (\beta z - \beta z^2) + \kappa e^r \\ &= e^r \beta z - e^r \beta z^2 + \kappa e^r \end{aligned}$$

as $r \rightarrow -\infty$

$$e^r \beta z \rightarrow \beta, \quad e^r \beta z^2 \rightarrow e^r z \quad (\text{since } \beta z \rightarrow 1)$$

$$\rightarrow 1 \quad (\text{as } z \rightarrow e^{-r})$$

$$\Rightarrow \bar{\mu} \rightarrow \beta - 1 + 1 = \beta$$

$$\text{as } r \rightarrow \infty, \quad \bar{\mu} = (e^r \beta z - e^r \beta z^2 + \kappa e^r)$$

$$\rightarrow e^r/\alpha - e^r/\alpha + e^r$$

$$= e^r$$

$$q = 1 - e^{-e^r}, \quad \frac{dq}{dr} = \frac{dq}{dy} \cdot \frac{dy}{dr}$$

$$\frac{dq}{dy} = e^{-e^r} e^r, \quad y = \theta_1 + (y + e^{\theta_2}) \alpha \Rightarrow \frac{dy}{dr} = b + e^{\theta_2}$$

$$\Rightarrow \frac{dq}{dr} = e^{-e^r} e^r (b + e^{\theta_2}), \quad (\text{we don't do asymptotics with } y \text{ bc it is endogenous i.e. it depends on } r \text{ already})$$

$$\Rightarrow \frac{dE[\tau]}{dr} = \frac{dq}{dr} \cdot \bar{\mu} + q \cdot \frac{d\bar{\mu}}{dr}$$