Chapter 3

Matrix Algebra

MATRICES are 2-D arrays of numbers or functions that obey the laws that define matrix algebra. The subject is important for physics because it facilitates the description of linear transformations such as changes of coordinate systems, provides a useful formulation of quantum mechanics, and facilitates a variety of analysis in classical and relativistic mechanics, particle theory, and other areas. Note also that the development of a mathematics of two-dimensionally ordered arrays is a natural and logical extension of concepts involving ordered pairs of numbers (complex numbers) or ordinary vectors (one-dimensional arrays). The most distinctive feature of matrix algebra is the rule for the multiplication of matrices.

3.1 Matrix

$2-D$ array of numbers or functions that obey the laws which governs the matrix algebra. It helps the transformations of system of coordinate. Provides useful formulation of Quantum Mechanics and facilitates a variety of analysis in classical and relativistic mechanics.

Order $(m \times n)$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$a_{ij} \simeq i^{th}$ row, j^{th} column.

Equal Matrix

Matrix $A =$ Matric B if and only if $a_{ij} = b_{ij}$ for all values of i and j.

$$a_{ij} = b_{ij}$$

Addition & Subtraction

$A \pm B = C$ if and only if $a_{ij} \pm b_{ij} = c_{ij}$ $A \pm B = [a_{ij} + b_{ij}]$ for all values of i and j, the elements combining according to the laws of ordinary algebra. This means that $A + B = B + A$, commutation. Also, an associative law is satisfied $(A+B)+C = A+(B+C)$. If all elements are zero, the matrix called null matrix, is denoted by 0. For all A, $A + 0 = 0 + A = A$.

$$0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Multiplication

The multiplication of matrix A by the scalar quantity α is defined as

$$\alpha A = (\alpha A)$$

in which the element of αA are αa_{ij}; that is, each element of matrix A is multiplied by the scalar factor. This is in striking contrast to the behavior of determinants in which the factor α multiplies only one column or one row and not every element of entire determinant.

$$\alpha A = A\alpha, \qquad \text{commutation}$$

If A is a square matrix, then

$$\det(\alpha A) = \alpha^n \det(A)$$

Moreover, inner Product $\qquad [A, B] = AB - BA.$

Unit Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal Matrix

A square matrix all of whose elements are zero except those in the main diagonal, is called diagonal matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{11} & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Here; we simply note a significance property of diagonal matrix: multiplication of diagonal matrices is commutative. If A and B are each diagonal.

$$AB = BA$$

Inverse Matrix

$$(a^{-1})_{ij} = \frac{(-1)^{i+j} M_{ji}}{\det(A)}$$

Rank, Transpose, Adjoint, and Trace

$$\det(AB) = (\det(A))(\det(B))$$

$$[\det(A)^{-1}] = \frac{1}{\det(A)}$$

$\widetilde{A}$ is transpose of A i.e. $a_{ij} = a_{ji}$

1. Complex conjugate A^* formed by taking complex conjugate $i \to -i$ of each element.
2. Adjoint, $A^\dagger$ formed by transposing A^*

$$A^\dagger = \bar{A}^* = \tilde{A}^*$$

3. Hermitian matrix (self adjoint)

   If $A^\dagger = A$

   If A is real then $A^\dagger = \overline{A}$
4. Unitary matrix U If $U^\dagger = U^{-1}$

   If U is real then $U^\dagger = \bar{U}$. Real unitary matrix are orthogonal.
5. $(AB)^* = A^* B^*$ and $(AB)^t = B^t A^t$

Unitary Similarity Transformation

A is matrix of complex elements. As and equation $A' = UAU^\dagger$ transformation matrix is unitary.Unitary matrix are important in Quantum Mechanics because the leave length of similarity transformation.

$$a'_{ij} = \sum_{kl} b_{ik}\, a_{kl}\, B^{-1}_{i,j}$$

$$B^{-1}_{ij} = \overline{B_{ij}} = b_{ji}$$

$$a'_{ij} = \sum_{kl} b_{ik}\, b_{ji}\, a_{kl}$$

A is an operator; rotating coordinate axes relating angular momentum of angular velocity, crystal axes.

$$a'_{ij} = \sum_{kl} b_{ik}\, b_{ji}\, a_{kl}$$

is also definition of tensor of second Rank as tensor is,

$$A'_{ij} = \sum_{pq} a_{ip}\, a_{jp}\, A_{pq}$$

Hence the matrix which transform by rotation of system of a tensor of rank 2.

Orthogonal Matrices

Under the rotation of coordinate system from (x, y, z) to $(x', y' z')$ having sam origin.

$$x_i' = \sum_j a_{ij}\, x_j$$

Length is scaler quantity and is invarient under rotation of coordinate system.
For convarince squaring,

$$\begin{aligned}
\sum x_i^2 &= \sum (x_i)'^2 \\
&= \sum_j a_{ij}\, x_j \sum_k a_{ik}\, x_k \\
&= \sum_j \sum_k a_{ij}\, a_{ik}\, x_j\, x_k \\
&= \sum_k \delta_{jk}\, x_j\, x_k \qquad \text{(Orthogonality)}
\end{aligned}$$

a_{ij} can be written in the form of orthogonal matrix. In matrix form,

$$\{x'\} = A\,\{x\}$$

A describe the rot of coordinate system.

$$\text{In 2-D} \qquad A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\text{In 3-D} \qquad A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{11} & a_{11} \\ a_{11} & a_{11} & a_{11} \\ a_{11} & a_{11} & a_{11} \end{pmatrix}$$

Inverse Matrix (A^{-1})

$$\{x\} = A^{-1}\,\{x'\}$$

A^{-1} describe the reverse of rotation of coordinate given by A and return the coordinate system.

$$\{x\} = A^{-1}A\{x\}$$
$$A^{-1}A = I$$
$$\{x\} = \{x\}$$

$$A = \{a_{ij}\} \qquad A^t = \{a_{ji}\}$$

Interacting row and column under orthogonality,

$$A^tA = I \quad \text{but} \quad A^{-1}A = I$$

$$\Rightarrow \quad A^t = A^{-1}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$A^t = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$AA^t = \begin{pmatrix} \cos\phi + \sin^2\phi & -\cos\phi\sin\phi + \sin\phi\cos\phi \\ -\cos\phi\sin\phi + \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Unitary Matrix

A square matrix is said to be unitary if $(\overline{A})^t = A^{-1}$. Now multiplying by A

$$A(\overline{A})^t = AA^{-1}$$
$$A(\overline{A})^t = I$$

For real matrix,

$$\overline{A} = A \quad \because \quad AA^t = I$$

Again multiplying A^{-1}

$$A^{-1}A^t = A^{-1}$$

$$A^tA = AA^{-1} = I \quad \Rightarrow \quad A^t = A^{-1}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Is unitary matrix.

Example 3.1.1 *A real unitary matrix is orthogonal. Determinant of unitary matrix is either* $+1$ *or* -1. *Inverse of product equal to product of in reverse order,*

$$(ST)^{-1} = T^{-1}S^{-1}$$

Solution

(ST) is matrix.

$$(ST)(ST)^{-1} = I$$

Multiplying by S^{-1}

$$S^{-1}(ST)(ST)^{-1} = S^{-1}I$$

$$S^{-1}ST(ST)^{-1} = S^{-1}I$$

$$IT(ST)^{-1} = S^{-1}$$

Multiplying by T^{-1}

$$T^{-1}T(ST)^{-1} = T^{-1}S^{-1}$$

$$I(ST)^{-1} = T^{-1}S^{-1}$$

$$(ST)^{-1} = T^{-1}S^{-1}$$

$$(ST)^t = T^tS^t$$

Product of two orthogonal matrices is orthogonal, i.e. $(ST)^t = (ST) - 1$. For orthogonal matrix $T^{-1} = T^t$, $\quad S^{-1} = S^t$. So, $T^{-1}S^{-1} = T^tS^t$. We know that $(T^{-1}S^{-1}) = (ST)^{-1}$

$$(ST)^{-1} = T^t S^t$$

$$(ST)^{-1} = (TS)^t$$

$$T^t S^t = (ST)^t$$

Product of two unitary matrices ϕ is unitary. Inverse unitary matrix is unitary.

3.2 Jaccobi Identity

$$[A,[B,C]] = [B,[A,C]] - [C,[A,B]]$$

Jaccobian for $[A,B] = AB - BA$

$$\begin{aligned} L.H.S = [A,[B,C]] &= [A, BC - CB] \\ &= A(BC - CB) - (BC - CB)A \\ &= ABC - ACB - BCA + CBA \end{aligned} \tag{3.1}$$

$$\begin{aligned} R.H.S &= [B,[A,C]] - [C,[A,B]] \\ &= [B,(AC - CA)] - [C,(AB - BA)] \\ &= [B,(AC - CA) - (AC - CA)B] - [C(AB - BA) - (AB - BA)C] \\ &= BAC - BCA - ACB + CAB - CAB + CBA + ABC - BAC \\ &= ABC - ACB - BCA + CBA \end{aligned} \tag{3.2}$$

Comparing Eq.(3.1) and Eq.(3.2), Hence proved.

3.3 Successive Rotation

$$\{x'\} = a\{x\} \tag{3.3}$$

A describes the Rotation. A is also called transformation matrix and it is orthogonal. Now we assume that it is followed by second rotation give by matrix B. i.e.

$$\{x''\} = B\{x'\} \tag{3.4}$$

In component form

$$x_j' = \sum_j a_{jk}\, x_k$$

$$x_i'' = \sum_j b_{ij}\, x_j'$$

$$x_j' = \sum_k a_{jk}\, x_k$$

$$x_i'' = \sum_j b_{ij} \sum_k a_{jk}\, x_k$$

$$= \sum_j \sum_k b_{ij}\, a_{jk}\, x_k$$

$$= \sum_k \left(\sum_j b_{ij}\, a_{jk}\right) x_k$$

$\sum_j b_{ij}\, a_{jk}$ is matrix multiplication defining a matrix $C = BA$ such that

$$x_i'' = \sum_k c_{ik}\, x_k \tag{3.5}$$

Physical interpretation of Eq.(3.5) is that the matrix product of two matrices BA is rotation that carries the unprime system directly into double prime coordinate system.

3.4 Symmetry Properties

A is a matrix rotating vector r into a new position r_1

$$r_1 = Ar$$

Now applying B rotate r_1 into r_1' i.e.

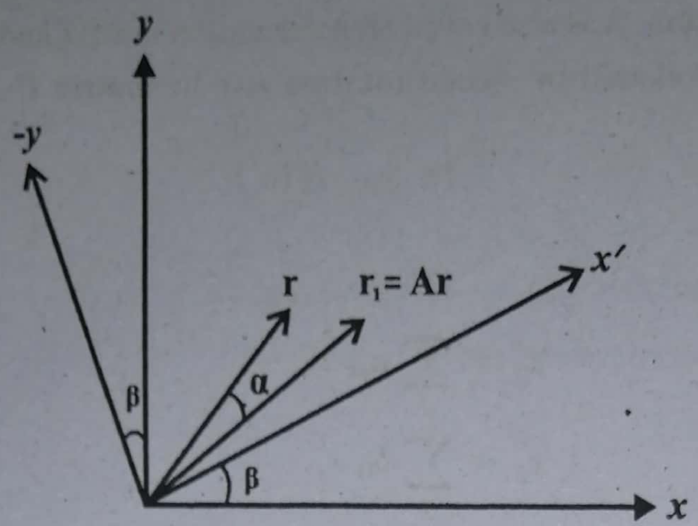

Fig. 3.1. The schametic diagram of Stern and Gerlach experiments.

$$\begin{aligned} r_1' &= Br_1 \\ &= B(Ar) \\ &= BAB^{-1}Br \\ &= (BAB^{-1})Br \end{aligned}$$

$$r_1' = A'r' \qquad \because \quad BAB^{-1} = A' \;;\; Br = r'$$

A' operating in space x', y', z' as A operating in space x, y, z. Transformation carried out by $A' = BAB^{-1}$ with B any matrix not necessarily be orthogonal matrix is known as **Similarity Transformation**.

3.5 Properties of Matrix

1. A+B=B+A — Commutative law.
2. A+(B+C)=(A+B)+C — Associative law.
3. (cd)A=c(dA)=(cdA) — Associative with respect to set.
4. A+O=O+A=A — Additive identity (O is null).
5. IA=AI=A — I is unit identity.
6. C(A+B)=CA+CB — Distributive law.
7. A(BC)=(AB)C — Associative law.
8. Left distribution, A(B+C)=AB+AC

9. Right distribution, C(AB)=(CA)B
10. AB $\neq$ BA

$$L = IW$$

$$[L_1, L_2, L_3] = \begin{pmatrix} I_{11} & I_{12} & \cdots \\ I_{21} & I_{22} & \cdots \\ \cdots & \cdots & I_{nm} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11}w_1 + I_{13}w_2 + I_{13}w_3 + \cdots + I_{14}w_4 \\ \vdots + \vdots + \vdots + \cdots + \vdots \\ I_{n1}w_1 + \cdots \cdots + i_{nn}w_n \end{pmatrix}$$

$$\begin{aligned}
I_{xx} &= m_i((r_i)^2(x_i)^2) \\
I_{11} &= m_1((r_1)^2 - x_1) + m_2((r_2)^2(x_2)^2) + m_3((r_3)^2(x_3)^2) \\
I_{22} &= m_1((r_1)^2 - y_1) + m_2((r_2)^2(y_2)^2) + m_3((r_3)^2(y_3)^2) \\
I_{33} &= m_1((r_1)^2 - z_1) + m_2((r_2)^2(z_2)^2) + m_3((r_3)^2(z_3)^2) \\
&= 1(6+2)+??? \\
I_{xy} &= m_1((r_1)^2 - (x_1)^2) = m_i x_i y_i \\
&= m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3
\end{aligned}$$

3.6 Matrix Matrices

$$x_i' = \sum_j a_{ij}\, x_j$$

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

3.7 Orthogonal Transpose Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Matrix of d' cosine $a_{i,j}$ i.e. cosine of angle between x_i' and x_j'. So, a_{11}, a_{12}, a_{13} are d' cosine of x_1' relative to x_1, x_2, x_3. i.e.

$$\widehat{x_1'} = a_{11}\widehat{x_1'} + a_{12}\widehat{x_2'} + a_{13}\widehat{x_3'}$$

$\widehat{x_1'}, \widehat{x_2'}, \widehat{x_3'}$ are orthogonal. Orthogonal transformation matrix A transform one orthogonal coordination system into second orthogonal coordinate system by rotation unit vector in spherical polar coordinate.

$$\begin{pmatrix} \widehat{r} \\ \widehat{\theta} \\ \widehat{\phi} \end{pmatrix} = C \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix}$$

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix} = C^{-1} \begin{pmatrix} \widehat{r} \\ \widehat{\theta} \\ \widehat{\phi} \end{pmatrix} = \overline{C} \begin{pmatrix} \widehat{r} \\ \widehat{\theta} \\ \widehat{\phi} \end{pmatrix}$$

For orthogonal matrix $\overline{C} = C^{-1}$

3.8 Pauli and Dirac Matrix

Set of three 2×2 pauli matrix are,

$$\delta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \delta_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \delta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

having property

$$\delta_i\delta_j + \delta_j\delta_i = 2\delta_i\delta_j I_2$$
$$\delta_i\delta_j = i\delta_k \qquad i, j, k \text{ cyclic permutation} 1, 2, 3$$
$$\delta_i^2 = I_2$$

Dirac Matrices

$$\gamma^o = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & \delta_1 \\ -\delta_1 & 0 \end{pmatrix} \quad ; \quad \gamma^3 = \begin{pmatrix} 0 & \delta_3 \\ -\delta_3 & 0 \end{pmatrix}$$

Properties of γ^0 and γ^1

1. From the properties of the α and β matrices immediately obtain:

$$(\gamma^0)^2 = \beta^2 = 1$$

and

$$
\begin{aligned}
(\gamma^1)^2 =& \beta\alpha_x\beta\alpha_x \\
=& \alpha_x\beta\beta\alpha_x \\
=& -\alpha_x^2 = -1
\end{aligned}
$$

2. The full set of relation is

$$
\begin{aligned}
(\gamma^0)^2 =& 1 \\
(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 =& -1 \\
\gamma^0\gamma^j + \gamma^j\gamma^0 =& 0 \\
\gamma^j\gamma^k + \gamma^k\gamma^j =& 0 \quad (j \neq k)
\end{aligned}
$$

Which can be expressed as

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

3. Are the gamma matrices Hermitian ?
   - β is Hermitian so γ^0 is Hermitian.
   - The α matrices also Hermitian, giving

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta a_x = -\gamma^1$$

item Hence $\gamma^1, \gamma^2, \gamma^3$ are anti-Hermitian

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3$$

Properties of Pauli Matrix

$$\delta x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \delta y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \delta z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. $\delta^2 x = \delta^y = \delta^2 z = 1$

$$\delta^2 x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\partial^2 x = I_2 = 1$$

$$\delta^2 x = \delta^y = \delta^2 z = I$$

- The Pauli Matrices $(\delta x, \delta y, \delta z)$ square to identity.
- Eigen values of $(\delta x, \delta y, \delta z) = +1$ and -1

2. Pauli Matrix Anti-Commute

(i)

$$\{\delta x, \delta y\} = 0$$

$$\delta x \delta y + \delta y \delta x = 0$$

$$L.H.S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0 = R.H.S$$

So,

$$\{\delta x, \delta y\} = \{\delta y, \delta z\} = \{\delta z, \delta x\} = 0$$

(ii)

$$\{\delta i, \delta j\} = 2\delta_{ij} I_2$$

Where $i, j = x, y$

$$\delta_{ij} = \begin{cases} 0 \ ; & i \neq j \\ 1 \ ; & i = j \end{cases}$$

These form of relation is also known as Clifford Algebra.

3. Pauli Matrix Are Troceless

$$T_r(\delta i) = 0 \qquad where\ i = x, y, z$$

3.9 Eigen Vectors and Eigen Values

A be $n \times n$ matrix X be column matrix AX will be column matrix AX only depends upon X

$$AX \propto X$$

$$AX = \lambda X$$

X is eigen vector of A belonginig to eigen values λ. If $\overline{r}$ be vector then

$$A\overline{r} = \lambda \overline{r}$$

$\overline{r}$ Eigen vector of A and λ Eigen vector of A.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} x_1\lambda \\ x_2\lambda \\ x_3\lambda \\ \vdots \\ x_n\lambda \end{pmatrix}$$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & \lambda x_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & \lambda x_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & \lambda x_3 \\ \vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & \lambda x_n \end{array}$$

$$\begin{array}{ccccccccc} (a_{11}-\lambda)x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & (a_{22}-\lambda)x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & = & 0 \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & (a_{nn}-\lambda)x_n & = & 0 \end{array}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$$(A - \lambda I)X = 0$$

This is called eigen value equation. The solution of these linear homogeneous equation will be non-trivial only when the determinant of coefficient is zero. i.e.

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad |A - \lambda I| = 0$$

It's degree is n so it has n roots. So $\lambda = \lambda_1, \lambda_2, \ldots.\lambda_n$ are roots of eigen values.

Example 3.9.1 *Find eigen value of* $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|A-\lambda I| = 0$$

$$\left|\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} -\lambda & i \\ i & -\lambda \end{pmatrix}\right| = 0$$

$$\lambda^2 + i^2 = 0 \quad \Rrightarrow \quad \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\lambda_1 = 1, \lambda_2 = -1$$

$$\begin{bmatrix} A - \lambda I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$-x \qquad -iy = 0$$

$$ix \qquad -y = 0$$

$$y = ix \quad \Rrightarrow \quad x = -iy$$

$$y = 1 \quad \Rrightarrow \quad x = -i$$

$$r_1 = (-i - 1)$$

For $\lambda = -1$

$$\begin{vmatrix} 1 & -i \\ i & 1 \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$x - iy = 0 \quad \Rrightarrow \quad ix + y = 0$$

For $y = 1$ $x = i$, $r_2(i, 1)$. r_1, r_2 are eigen vectors λ_1, λ_2 are eigen values,

$$R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$|R| = -i - i = -2i$$

$$R^{-1} = \frac{1}{|R| \operatorname{adj} R}$$

$$R^{-1} = \frac{1}{-2i} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{-2i} & \frac{1}{2i} \\ \frac{i}{2i} & \frac{-i}{-2i} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} i & -1 \\ 1 & 1 \end{pmatrix}$$

$$R^{-1}AR = \frac{1}{2} \begin{pmatrix} i & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} i & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0-i & 0-i \\ -1+0 & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -i^2+1 & -i^2-i \\ 1-1 & 1-i \end{pmatrix}$$

A is Hermitian matrix. It's given eigen values are real and eigen vectors are orthogonal.

Proof

Show that eigen value of Hermitian matrix are real and eigen vectors are orthogonal.

solution: λ_i is an eigen value of r_i is corresponding eigen vectors of matrix A.

$$A\,r_i = \lambda_i\,r_i \tag{3.6}$$

$$A\,r_j = \lambda_j\,r_j \tag{3.7}$$

Multiplying Eq.(3.6) by $r_j^\dagger$

$$r_j^\dagger\,A\,r_i = r_j^t\,\lambda_i\,r_i \tag{3.8}$$

Multiplying Eq.(3.7) by $r_i^\dagger$

$$r_i^\dagger\,A\,r_j = r_i^\dagger\,\lambda_j\,r_j$$

$$r_i^\dagger A r_j = \lambda_j r_i^\dagger r_j$$

Taking it's adjoint

$$(r_i^\dagger A r_j)^\dagger = (\lambda_j r_i^\dagger r_j)^\dagger$$

$$r_j^\dagger A^\dagger r_i = \lambda_j^* r_j^\dagger r_i \tag{3.9}$$

Compairing Eq.(3.8) and Eq.(3.9)

$$(\lambda_i - \lambda_j^*) r_j^\dagger r_i = 0$$

If $i = j$

$$(\lambda_i - \lambda_i^*)|r_i^2| = 0$$

$$|r_i^2| \neq= 0$$

so $\lambda_i = \lambda_i^*$, so λ_i are real $\forall i$. For $i \neq j$

$$(\lambda_i - \lambda_j^*) r_j^\dagger r_i = 0$$

$$\text{as } \lambda_i^* = \lambda_i \qquad \lambda_j^* = \lambda_j$$

$$(\lambda_i - \lambda_j) r_j^t r_i = 0$$

Since $\lambda_i \neq \lambda_j$

$$\Rightarrow \quad r_j^\dagger) r_i = 0$$

Which shows that eigen vectors of distinct eigen values are orthogonal.

3.10 Diagonalization of Matrix

A matrix is said to be diagonalize if it is similar to diagonal matrix. A is diagonal if there exist an invertile matrix B such that $B^{-1}AB$ is diagonalizable matrix. If B is orthogonal i.e $B^{-1} = B^{\dagger}$

$$B^{-1}AB = B^{\dagger}AB$$

is also diagonal matrix then A is called orthogonally diagonalizable and B is said to be orthogonally diagonalizable which can be done by orthogonal similarity transformation which reduce the matrix to diagonal form nondiagonal element all are zero.

Let r_1, r_2, r_3 eigen vectors of matrix.

$$R = (r_1, r_2, r_3) = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

For n eigen vectors $r_1, r_2, r_3.....r_n$. So $R = (r_1, r_2, r_3.....r_n)$. A is matrix of eigen values λ_i

$$A\,r_i = \lambda_i r_i$$

$$\begin{aligned} A\,R &= (Ar_1, Ar_2..., Ar_n) \\ &= (\lambda_1 r_1, \lambda_2 r_2, ...\lambda_n r_n) \\ &= (r_1, r_2, \cdots, r_n) \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \\ &= R \text{ diag } (\lambda_1, \lambda_2, \cdots \lambda_n) \end{aligned}$$

Multiplying b by side R^1

$$R^{-1}AR = R^{-1}R\,\text{diag}\,(\lambda_1, \cdots \lambda_n)$$
$$= \text{diag}\,(\lambda_1, \lambda_2, \cdots \lambda_n)$$

$R^{-1}AR$ is diagonal matrix formed by eigen value of matrix A. Multiplying both sides by R^{-1}

$$R^{-1}AR = R^{-1}R\,\text{diag}\,(\lambda_1, \lambda_2, ..., \lambda_n)$$
$$R^{-1}AR = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Hence $R^{-1}AR$ is a diagonal matrix formed by the eigen values of matrix A.

Example 3.10.1 *Diagonalize* $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ *we have to find* $R^{-1}AR$.

Solution

Step1: We Find R

R is matrix formed by eigen vectors of given matrix A. First we find eigen values and then eigen vectors. Secular equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)^2 - 0] - 0 - 1(2-\lambda) = 0$$
$$(2-\lambda)[4 + \lambda^2 - 4\lambda - 1] = 0$$
$$(2-\lambda)[\lambda^2 - 4\lambda + 3] = 0$$
$$(2-\lambda)(\lambda - 1)(\lambda - 3) = 0$$

For $\lambda = 1, 2, 3$

Case-I

$$\lambda_1 = \lambda, \qquad r_1 = ?$$

$$(A - \lambda I)r = 0$$

$$\begin{pmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x - z = 0 \quad \text{and} \quad y = 0$$

$$-x + z = 0 \quad \text{and} \quad x = z, y = 0$$

we choose $x = 1, z = 1, y = 0$ $\quad r_1 = (1, 1, 0)$
We get,

$$r_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$$

Case-II

$$\lambda_2 = 2, r_2 = ?$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-z = 0 \quad \Rightarrow \quad x = 0$$

$$-x = 0 \quad \Rightarrow \quad z = 0$$

we chose $y = 1$

$$r_2 = (0, 1, 0)$$

Case-III

$$\lambda_3 = 3, r_3 = ?$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x - z = 0 \Rightarrow x = -z$$

$$-y = 0 \Rightarrow y = 0$$

Suitable choice is $(1, 0, -1)$

$$\vec{r_3} = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$$

Matrix formed by Eigen vectors is

$$R = (r_1, r_2, r_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Step2: We Find R^{-1}.

We know that $R^{-1} = \frac{1}{|R|}\text{adj } R$
Adj R=transpose of matrix of co-factors of R.

$$\text{Adj}R = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & +\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|R| = -1$$

$$R^{-1} = \frac{AdjR}{|R|} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Step3

$$R^{-1}AR = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$R^{-1}AR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Which is required diagonal matrix.

Example 3.10.2 *Diagonalize* $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Solution

For $\lambda = -1, \;\; x - iy = 0, \;\; y = 1$

$\lambda = \pm 1, \;\; r_1 = (i, 1), \;\; ix + y = 0, \;\; x = i$

$$\lambda = \pm 1$$

$$x = -iy \qquad x = -i$$

$$y = ix, \quad y = 1$$

$$r_2 = (-i, 1)$$

$$R = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$R^{-1}\Lambda R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \text{diag}\,\Lambda$$

Example 3.10.3 *A certain rigid body may be represented by three point masses.*
$m_1 = 1$ *at* $(1, 1, -2)$ $m_2 = 2$ *at* $(-1, -1, 0)$
$m_3 = 3$ *at* $(1, 1, 2)$
Find inertia matrix?

Solution

$L = I\omega$ $L =$ angular momentum $\omega =$ angular velocity
Inertia matrix I is found to have diagonal component.

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2)$$

i refers to the mass m_i. For nondiagonal components.

$$I_{xy} = -\sum_i m_i(x_i y_i) = I_{yx}$$

Step-1

Diagonal components

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2)$$

$$I_{11} = m_1(r_1^2 - x_1^2) + m_2(r_2^2 - x_2^2) + m_3(r_3^2 - x_3^2) \tag{3.10}$$

$$r_1 = \sqrt{1+1+4} = \sqrt{6} \Rightarrow r_1^2 = 6 \ ; \ m_1 = 1$$

$$r_2 = \sqrt{1+1+0} = \sqrt{2} \Rightarrow r_2^2 = 2 \ ; \ m_2 = 2$$

$$r_3 = \sqrt{1+1+4} = \sqrt{6} \Rightarrow r_3^2 = 6 \ ; \ m_3 = 1$$

$$I_{11} = 1(6-1) + 2(2-1) + 1(6-1) = 12$$

$$I_{22} = m_1(r_1^2 - y_1^2) + m_2(r_2^2 - y_2^2) + m_3(r_3^2 - y_3^2) = 12$$

$$I_{33} = m_1(r_1^2 - z_1^2) + m_2(r_2^2 - z_2^2) + m_3(r_3^2 - z_3^2) = 8$$

Step-2

Nondiagonal components

$$I_{xy} = -\sum_i m_i x_i y_i = I_{yx}$$

$$\begin{aligned}
I_{12} &= -m_1 x_1 y_1 - m_2 x_2 y_2 - m_3 x_3 y_3 \\
&= -(1)(1)(1) - 2(-1)(-1) - (1)(1)(1) \\
&= -1 - 2 - 1 = -4 = I_{21} \\
I_{13} &= -m_1 x_1 z_1 - m_2 x_2 z_2 - m_3 x_3 z_3 \\
&= 2 + 0 - 2 = 0 = I_{31} \\
I_{23} &= -m_1 y_1 z_1 - m_2 y_2 z_2 - m_3 y_3 z_3 \\
&= -(1)(1)(-2) - (2)(-1)(0) - (1)(1)(2) \\
&= 2 - 0 - 2 = 0 = I_{32}
\end{aligned}$$

Now required inertia matrix becomes [1]

$$I = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

We want to orient the coordinate axes in space so that I_{xy} and other nondiagonal elements will vanish. A consequence of this orientation is that if angular velocity is along one such realigned axis the angular velocity and angular momentum will be parallel.

If R be unitary then we form transformation of matrix which will convert our hermitian matrix A into diagonal form as $R^t AR$. Here

$$R = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = (r_1, r_2, r_3)$$

[1] Diagonalize the inertia matrix see previous part and do your self

$$R^t A R = \begin{pmatrix} r_1^* \\ r_2^* \\ r_3^* \end{pmatrix} (A)((r_1)(r_2)(r_3)) = \begin{pmatrix} r_1^* \\ r_2^* \\ r_3^* \end{pmatrix} ((\lambda_1 r_1)(\lambda_2 r_2)(\lambda_3 r_3)) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Hence $R^t A R$ is a diagonal matrix with eigen values λ_i

3.11 Inertia Matrix

$$L = I\omega$$

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2) \quad \text{Diagonal element}$$

$$I_{xy} = -\sum_i m_i x_i y_i = I_{yx} \quad \text{Nondiagonal element}$$

Rigid body of mass $m_1 = 1$ at $(1, 1, -2)$

$$m_2 = 2 \text{ at } (-1, -1, 0) \quad ; \quad m_3 = 1 \text{ at } (1, 1, 2)$$

$$\text{As} \quad I_{xx} = \sum_i m_i(r_i^2 - x_i^2)$$

$$= m_1(r_1^2 - x_1^2) + m_2(r_2^2 - x_2^2) + m_3(r_3^2 - x_3^2)$$

$$r_1 = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$r_1^2 = 6; m_1 = 1$$

$$r_2^2 = 6; m_2 = 2$$

$$r_3^2 = 6; m_3 = 1$$

$$I_{11} = 1(6-1) + 2(2-1) + 1(6-1) = 12 \Rrightarrow 5 + 2 + 5 = 12$$

$$I_{22} = 1(6-1) + 2(2+1) + 1(6-1) = 17$$

$$I_{33} = 1(6+1) + 2(2+1) + 1(6-1) = 18$$

$$I_{xy} = -\sum_i m_i x_i y_i$$

$$\begin{aligned} I_{12} &= -m_1 x_1 y_1 - m_2 x_2 y_2 - m_3 x_3 y_3 \\ &= -(1)(1) - 2(-1)(-1) - (1)(1)(1) \end{aligned}$$

$$I_{21} = -1 - 2 - 1 = -4$$

$$\begin{aligned} I_{13} &= -m_1 x_1 z_1 - m_2 x_2 z_2 - m_3 x_3 z_3 \\ &= 0 = I_{13} \end{aligned}$$

$$\begin{aligned} I_{23} &= -m_1 y_1 z_1 - m_2 y_2 z_2 - m_3 y_3 z_3 \\ &= 0 = I_{32} \end{aligned}$$

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 17 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$