Chapter 2

Tensor

Tensor is an entity comprising many components which obey a definite law of transformation between different coordinate system. A point in space of n dimensions can be specified by a set of n coordinates. This set can be written in the form of $x_1, x_2, x_3, ...x_n$. If a transformation of coordinate is made, the x_i becomes x_i' in the new coordinate system. Similarly, a set of n components of any vector A i.e. $A_1, A_2, ...A_n$ denoted as A_i, when transformation occurs these change to A_i', such a set is a **Tensor**.
If certain transformation laws hold. Tensors are very important in physics e.g. general theory of relativity and electrodynamics. Scalars and vectors are special cases of tensor.

2.1 Scalar, Vector and Dyadic

Scalar

Scalar is a tensor of rank zero. It is single real number or component. In three dimensional (3-D) space the number of components of a scalar is $3^{\circ} = 1$. Scalar is a quantity which do not change under rotation of coordinates i.e. it is invariant and transformation law for scalar is $A_i' = A$. Example of tensor of rank zero (i.e. scalar) are mass, charge,speed etc.

Vector

A tensor of rank one. In three dimensional space the number of components of a vector is $3'$(i.e.3). Components transform under rotation like those of the distance of a point from a chosen origin. The transformation law for vector's component is

$$A_i' = \sum_{p=1}^{3} a_{ip} A_p = \sum \frac{\partial x_p}{\partial x_i'} A_p \tag{2.1}$$

Examples: momentum, electric field, velocity.

Dyadic

Tensor of rank two. In three dimensional space the number of components of a 2^{nd} rank tensor is given by 3^2 (i.e. 9)

$$A_{ij}' = \sum_p \Sigma_q a_{ip} a_{jq} A_{pq}$$

Examples: electric field, magnetic field and thermal conductivity of anisotropic media. Tensors of rank higher than two have no name. A tensor of rank 2 has 9 components in 3 dimensional which can be expressed as

$$A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Covariant Tensor of rank 2 (A_{pq})

$$A_{ij}' = \sum_{pq} \frac{\partial x_p}{\partial x_i'} \frac{\partial x_q}{\partial x_j'} A_{pq} \tag{2.2}$$

Contravariant Tensor of rank 2 (A^{pq})

$$A'^{ij} = \sum_{pq} \frac{\partial x_i'}{\partial x_p} \frac{\partial x_j'}{\partial x_q} A^{pq} \tag{2.3}$$

Mixed Tensor of rank 2 (A_q^p)

$$A_j'^i = \sum_{pq} \frac{\partial x_i'}{\partial x_p} \frac{\partial x_q}{\partial x_j'} A_q^p \tag{2.}$$

A_q^p transformation contravariantly w.r.t. index p but covariantly w.r.t. q.

Example 2.1.1 *Show that velocity of a fluid at any point is a contravariant vector of rank one.*

Solution

Let $x_\alpha(t)$ is coordinates of a morning particle with the time t, we have

$$V^\alpha = \frac{dx_\alpha}{dt} = \text{velocity of particle}$$

In transformed coordinates, the components of velocity are

$$V'^\alpha = \frac{dx'_\alpha}{dt} = \frac{\partial x'_\alpha}{\partial x_B}\frac{\partial x_B}{\partial t}$$

$$V'^\alpha = \frac{\partial x'_\alpha}{\partial x_B} V^B$$

Hence Proved

$$dx^k = ?$$

$$dx'^j = \frac{\partial x'^j}{\partial x^k} dx^k$$

$\Rightarrow dx^k$ is a contravariant tensor of rank 1 or a contravariant vector.

2.2 Kronecker Delta

$$\delta_{pq} = \begin{cases} 1 & \text{when } p = q \\ 0 & \text{when } p \neq q \end{cases}$$

$\delta_{11} = \delta_{22} = \delta_{33} = 1$. If $p, q = 1; 2, 3$ then $\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$. If $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are unit vectors perpendicular to each other, then $\hat{e}_p.\hat{e}_q = \delta_{pq}$. Kronecker delta δ_{kl} is really a mixed tensor of second rank δ_l^k. Our criterion is

$$B_j'^i = \sum_{kl} \frac{\partial x'_i}{\partial x_k}\frac{\partial x_l}{\partial x'_j} B_l^k \tag{2.5}$$

By using the summation convention, we can write

$$\delta_l^k \frac{\partial x_i'}{\partial x_k}\frac{\partial x_l}{\partial x_j'} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_k}{\partial x_j'}$$

As $\delta_l^k = 1$ when $l = k$

$$\frac{\partial x_i'}{\partial x_k}\frac{\partial x_k}{\partial x_j'} = \frac{\partial x_i'}{\partial x_j'}$$

But x_i' and x_j' are independent coordinates, therefore the variation of one w.r.t. other must be zero if they are different and unit if they coincide i.e. $\frac{\partial x_i'}{\partial_j'} = \delta_j'^i$. Hence,

$$\delta_j'^i = \frac{\partial x_i'}{\partial_k}\frac{\partial x_l}{\partial_j'}\delta_l^k \tag{2.6}$$

It shows that δ_l^k are indeed the components of a mixed second-rank tensor. When

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 = 1 + 1 + 1 = 3$$

Similarly, for $i = j = n$, thus $\delta_i^i = n$.

Example 2.2.1 *Prove that* $\delta_j^i\, A^{jk} = A^{ik}$

Solution

Consider L.H.S, If $i = 1$,

$$\begin{aligned}
\delta_j^1\, A^{jk} &= \delta_1^1\, A^{1k} + \delta_2^1\, A^{2k} + \cdots \\
\delta_j^1\, A^{jk} &= A^{1k} + 0 + \cdots
\end{aligned}$$

$$\begin{aligned}
\text{If } i = 2 \quad \delta_j^2\, A^{jk} &= A^{2k} \\
\delta_j^3\, A^{jk} &= A^{3k} \\
\vdots &= \vdots \\
\delta_j^n\, A^{jk} &= A^{nk}
\end{aligned}$$

$$\begin{aligned}
\delta_j^i\, A^{jk} &= A^{1k} + A^{2k} + A^{3k} + \cdots + A^{nk} \\
\delta_j^i\, A^{jk} &= A^{ik}
\end{aligned}$$

Example 2.2.2 *Prove that* $\delta^i_j\,\delta^j_k = \delta^i_k$

Solution

Consider L.H.S. For $i = 1$

$$\delta^1_j\,\delta^j_k = \delta^1_1\,\delta^1_k + \delta^1_2\,\delta^{2+\cdots+\delta^1_n\,\delta^n_k}_k$$
$$= \delta^1_k + 0 + 0\cdots$$
$$i = 2\ ,\ \delta^2_j\delta^j_k = \delta^2_1\delta^1_k + \delta^2_2\delta^2_k + ...\delta^2_n\delta^n_k$$
$$= \delta^2_k + 0 + 0$$
$$i = 3\ ,\ \delta^3_j\delta^j_k = \delta^3_k$$
$$\delta^n_j\delta^j_k = \delta^n_k$$

Generalizing

$$\delta^i_j\delta^j_k = \delta^1_k + \delta^2_k + \delta^3_k + ...\delta^n_k = \delta^i_k$$

2.3 Symmetric and Anti-symmetric Tensor

Summation Convention

When a subscript (letter, not number) appears twice on one side of an equation, summation with respect to that subscript is implied.

Symmetric Tensor

A tensor called symmetric w.r.t. two contravariant or covariant indices if its components remains unchanged upon the interchange of indices. If $A^{mn} = A^{nm}$ then tensor is symmetric in m and n. If A^{mn} and A^{nm} becomes A'^{mn} and A'^{nm} in other system then symmetry will be maintained in this system if $A'^{mn} = A'^{nm}$. As

$$A'^{mn} = \frac{\partial x'_m}{\partial x_i}\frac{\partial x'_n}{\partial x_j}A^{ij}$$

$$= \frac{\partial x'_m}{\partial x_i}\frac{\partial x'_n}{\partial x_j}A^{ji} \qquad [A^{ij} = A^{ji}]$$

$$= \frac{\partial x'_n}{\partial x_j}\frac{\partial x'_m}{\partial x_i}A^{ji}$$

$$A'^{mn} = A'^{nm}$$

It shows that contravariant tensor is symmetric. For covariant tensor

$$A'_{mn} = \frac{\partial x_i}{\partial x'_m}\frac{\partial x_j}{\partial x'_n}A_{ij}$$

$$= \frac{\partial x_i}{\partial x'_m}\frac{\partial x_j}{\partial x'_n}A_{ji} \qquad [A_{ij} = A_{ji}]$$

$$= \frac{\partial x_j}{\partial x'_n}\frac{\partial x_i}{\partial x'_m}A_{ji}$$

$$A'_{mn} = A'_{nm}$$

It shows that covariant tensor is symmetric.

Anti-Symmetric or Skew-Symmetric Tensor

A tensor called skew-symmetric w.r.t. two contravariant or covariant indices if its components change sign upon the interchange of indices.

$$A^{mn} = -A^{nm}$$

$$A'^{mn} = \frac{\partial x'_m}{\partial x_i}\frac{\partial x'_n}{\partial x_j}A^{ij}$$

$$= -\frac{\partial x'_m}{\partial x_i}\frac{\partial x'_n}{\partial x_j}A^{ji} \qquad [A^{ij} = -A^{ji}]$$

$$= -\frac{\partial x'_n}{\partial x_j}\frac{\partial x'_m}{\partial x_i}A^{ji}$$

$$A^{mn} = -A'^{nm}$$

Similarly, for covariant tensor

$$A'_{mn} = \frac{\partial x_i}{\partial x'_m}\frac{\partial x_j}{\partial x'_n}A_{ij}$$

$$= -\frac{\partial x_j}{\partial x'_n}\frac{\partial x_i}{\partial x'_m}A^{ji} \qquad [A_{ij} = -A_{ji}]$$

$$A'_{mn} = -A'_{nm}$$

2.4 Levi-civita symbol/Permutation symbol/Eplison Tensor

$$\epsilon_{pqr} = \left.\begin{matrix} +1 \\ -1 \\ 0 \end{matrix}\right\} \Rightarrow \begin{matrix} \text{according to} \\ \text{whether} \\ p, q, r \end{matrix} \left\{\begin{matrix} \text{form an even} \\ \text{form an odd} \\ \text{do not form} \end{matrix}\right\} \textit{ or } \left\{\begin{matrix} \text{are in cyclic order} \\ \text{are in noncyclic order} \\ \text{any two or more are repeated} \end{matrix}\right\}$$

A permutation is called even (or odd) if an even (or odd) number of transpositions (i.e. interchange of two indices) produces the permutation 1,2,3.

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$$

$$\epsilon_{111} = \epsilon_{112} = \epsilon_{122} = \cdots = 0$$

Example 2.4.1 *Show that* $\delta_{ij}\epsilon_{ijk} = 0$. *For* $i = j, \delta_{ij} = 1$ *and* $\epsilon_{ijk} = 0$.

Solution

Hence

$$\delta_{ij}\epsilon_{ijk} = 0$$

For $i \neq j, \delta_{ij} = 0$ and $\epsilon_{ijk} = +1$ or -1. Hence

$$\delta_{ij}\epsilon_{ijk} = 0$$

Example 2.4.2 *Prove that* $A_{pq} = \begin{bmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{bmatrix}$ *are components of a* 2^{nd} *rank tensor in two dimensions.*

Solution

A 2^{nd} rank tensor in two dimensions have 4 components,

$$p = 1, 2\ ,\ q = 1, 2$$

A_{pq} represents 4 components namely

$$\begin{aligned} A_{11} &= x_2^2 & A_{12} &= -x_1x_2 \\ A_{21} &= -x_1x_2 & A_{22} &= x_1^2 \\ \text{So,}\quad A'_{11} &= x_2'^2 & A'_{12} &= -x'_1x'_2 \\ A'_{21} &= -x'_1x'_2 & A'_{22} &= x_1'^2 \end{aligned}$$

Now we evaluate components of a tensor in primed system. Transformation equation fo a second rank tensor is

$$A'_{ij} = \sum_{p=1}^{2}\sum_{q=1}^{2} \frac{\partial x_p}{\partial x'_i}\frac{\partial x_q}{\partial x'_j} A_{pq} = \sum_{p=1}^{2}\sum_{q=1}^{2} a_{ip}a_{jq}A_{pq}$$

$$\begin{aligned} A'_{11} &= \sum_{p=1}^{2}\sum_{q=1}^{2} a_{1p}a_{1q}A_{pq} = \sum_{p=1}^{2} [a_{1p}a_{11}A_{p1} + a_{1p}a_{12}A_{p2}] \\ &= a_{11}a_{11}A_{11} + a_{11}a_{12}A_{12} + a_{12}a_{11}A_{21} + a_{12}a_{12}A_{22} \\ A'_{12} &= \sum_{p=1}^{2}\sum_{q=1}^{2} a_{1p}a_{2q}A_{pq} \\ &= a_{11}a_{21}A_{11} + a_{11}a_{22}A_{12} + a_{12}a_{21}A_{21} + a_{12}a_{22}A_{22} \end{aligned}$$

$$A'_{21} = \sum_{p=1}^{2}\sum_{q=1}^{2} a_{2p}a_{1q}A_{pq}$$
$$= a_{21}a_{11}A_{11} + a_{21}a_{12}A_{12} + a_{22}a_{11}A_{21} + a_{22}a_{12}A_{22}$$
$$A'_{22} = a_{21}a_{21}A_{11} + a_{21}a_{22}A_{12} + a_{22}a_{21}A_{21} + a_{22}a_{22}A_{22}$$

x_1, x_2 are related with x'_1, x'_2 as

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \underbrace{\cos\theta}_{a_{11}} & \underbrace{\sin\theta}_{a_{12}} \\ \underbrace{-\sin\theta}_{a_{21}} & \underbrace{\cos\theta}_{a_{22}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} A'_{11} &= x'^2_2 = x_2^2\cos^2\theta - 2x_1x_2\sin\theta\cos\theta + x_1^2\sin^2\theta \\ A'_{12} &= -x'_1x'_2 = -x_2^2\sin\theta\cos\theta - x_1x_2(\cos^2\theta - \sin^2\theta) + x_1^2\sin\theta\cos\theta \\ A'_{21} &= -x'_1x'_1 = -x_2^2\sin\theta\cos\theta - x_1x_2(\cos^2\theta - \sin^2\theta) + x_1^2\sin\theta\cos\theta \\ A'_{22} &= x'^2_1 = x_1^2\cos^2\theta + 2x_1x_2\sin\theta\cos\theta + x_2^2\sin^2\theta \end{aligned} \tag{2.7}$$

$$x'_1 = x_1\cos\theta + x_2\sin\theta$$
$$x'_2 = -x_1\sin\theta + x_2\cos\theta$$
$$(x'_2)^2 = (-x_1\sin\theta + x_2\cos\theta)^2$$

$$\begin{aligned} A'_{11} &= x'^2_2 = (-x_1\sin\theta + x_2\cos\theta)^2 \\ &= x_1^2\sin^2\theta - 2x_1x_2\sin\theta\cos\theta + x_2^2\cos^2\theta \\ A'_{12} &= -x'_1x'_2 = -(x_1\cos\theta + x_2\sin\theta)(-x_1\sin\theta + x_2\cos\theta) \\ &= x_1^2\sin\theta\cos\theta - x_1x_2(\cos^2\theta - \sin^2\theta) - x_2^2\sin\theta\cos\theta \\ A'_{21} &= -x'_1x'_2 = -(x_1\cos\theta + x_2\sin\theta)(-x_1\sin\theta + x_2\cos\theta) \\ &= x_1^2\sin\theta\cos\theta - x_1x_2(\sin^2\theta - \cos^2\theta) - x_2^2\sin\theta\cos\theta \\ A'_{22} &= x'^2_1 = x_1^2\cos^2\theta + 2x_1x_2\sin\theta\cos\theta + x_2^2\sin^2\theta \end{aligned} \tag{2.8}$$

The corresponding values of the components of tensor given by Eq.(2.7) and Eq.(2.8) are same. So A_{pq} follows the transformation equation. Hence it is a tensor.

Example 2.4.3 *Show that every tensor can be expressed as sum of two tensors one of which is symmetric and other is skew symmetric in a pair of covariant and contravariant indices.*

Solution

For covariant indices, let

$$A_{mn} = \frac{1}{2}\Big[A_{mn} + A_{nm}\Big] + \frac{1}{2}\Big[A_{mn} - A_{nm}\Big]$$
$$= B_{mn} + C_{mn}$$

We show that B_{mn} is symmetric and C_{mn} is skew symmetric. Now,

$$B_{mn} = \frac{1}{2}\Big[A_{mn} + A_{nm}\Big]$$
$$= \frac{1}{2}\Big[A_{nm} + A_{mn}\Big]$$
$$B_{nm} = \frac{1}{2}\Big[A_{mn} + A_{nm}\Big] = B_{mn}$$

$\Rightarrow B_{nm} = B_{mn}$ $\quad \therefore B_{mn}$ is symmetric.

$$C_{mn} = \frac{1}{2}\Big[A_{mn} - A_{nm}\Big]$$
$$= \frac{1}{2}\Big[A_{nm} - A_{mn}\Big]$$
$$C_{nm} = \frac{1}{2}\Big[A_{mn} - A_{nm}\Big] = -C_{mn}$$

$\Rightarrow C_{mn}$ is skew symmetric. Similarly, we can prove for contravariant indices.

2.5 Fundamental Operation with Tensors

There are some operation with tensors;

Addition

Two or more tensors can be added provided that they are of the same rank and type same number of contravariant indices and same number of convariant indices. The s of two or more tensors is a tensor of the same rank and type. If A^{mp}_q and B^{mp}_q are tensors then $C^{mp}_q = A^{mp}_q + B^{mp}_q$ is also tensor.

Subtraction

Two tensors of the same rank and type can be subtracted and difference of two tensors is also a tensors of same rank and type.
If A^{mp}_q and B^{mp}_q are tensors then

$$D^{mp}_q = A^{mp}_q - B^{mp}_q$$

Example 2.5.1 *If A^{pq}_r and B^{pq}_r are tensors then their sum and differences are also tensors.*

Solution

$$A'^{ij}_k = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_q}\frac{\partial x_r}{\partial x'_k}A^{pq}_r \tag{2.9}$$

$$B'^{ij}_k = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_q}\frac{\partial x_r}{\partial x'_k}B^{pq}_r \tag{2.10}$$

By adding Eq.(2.9) and Eq.(2.10)

$$A'^{ij}_k + B'^{ij}_k = C'^{ij}_k = \left[\frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_q}\frac{\partial x_r}{\partial x'_k}\right]\left[A^{pq}_r + B^{pq}_r\right]$$

$$C'^{ij}_k = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_q}\frac{\partial x_r}{\partial x'_k}C^{pq}_r$$

It implies that C^{pq}_r is also a tensor. Similarly by subtracting Eq.(2.9) and Eq.(2.10)

$$D'^{ij}_k = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_q}\frac{\partial x_r}{\partial x'_k}D^{pq}_r$$

$\Rightarrow D^{pq}_r$ is also a tensor as it satisfies transformation laws.

Contraction

If two indices, one covariant and the other contravariant are set equal to each other and we sum over this repeated index, then the resulting sum is a tensor of rank 2 less than the original tensor. It is called contraction.

Let A^{mpr}_{qs} be tensor of rank 5. We set $r = q$ then $A^{mpq}_{qs} = B^{mp}_{s}$ (A tensor of rank 3). Again let $m = s$ in B^{mp}_{s}, then we get $B^{sp}_{s} = C^{p}$ (A tensor of rank 1). Now contract the second rank mixed tensor B'^{i}_{j}.

$$B'^{i}_{j} \rightarrow B'^{i}_{i} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_l}{\partial x'_i} B^{k}_{l}$$
$$= \frac{\partial x_l}{\partial x_k} B^{k}_{l}$$

Since x_l and x_k are independent coordinates so variation of one with respect to other is zero. We can write

$$\frac{\partial x_l}{\partial x_k} = \delta^{l}_{k}$$

$$B'^{i}_{i} = \delta^{l}_{k} B^{k}_{l}$$
$$= B^{k}_{k}$$

$\Rightarrow$ Contracted second rank mixed tensor is invariant and therefore a scalar.

Outer Multiplication

The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product involves ordinary multiplication of the components of the tensor.

$$A^{pr}_{q} B^{m}_{s} = C^{prm}_{qs}$$

Example 2.5.2 *Prove that* $A^{pr}_{q} B^{m}_{s} = C^{prm}_{qs}$ *is a tensor.*

Solution

$$A'^{ij}_{k} = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_j}{\partial x_r}\frac{\partial x_q}{\partial x'_k} A^{pr}_{q} \quad (2$$

$$B_t^{'l} = \frac{\partial x_l'}{\partial x_m}\frac{\partial x_s}{\partial x_t'}B_s^m \tag{2.12}$$

Multiplying Eq.(2.11) and Eq.(2.12)

$$A_k^{'ij}B_t^{'l} = \frac{\partial x_i'}{\partial x_p}\frac{\partial x_j'}{\partial x_r}\frac{\partial x_q}{\partial x_k'}A_q^{pr}\frac{\partial x_l'}{\partial x_m}\frac{\partial x_s}{\partial x_t'}B_s^m$$

$$C_{kt}^{'ijl} = A_q^{pr}B_s^m\frac{\partial x_i'}{\partial x_p}\frac{\partial x_j'}{\partial x_r}\frac{\partial x_q}{\partial x_k'}\frac{\partial x_l'}{\partial x_m}\frac{\partial x_s}{\partial x_t'}$$

$$C_{kt}^{'ijl} = C_{qs}^{prm}\frac{\partial x_i'}{\partial x_p}\frac{\partial x_j'}{\partial x_r}\frac{\partial x_q}{\partial x_k'}\frac{\partial x_l'}{\partial x_m}\frac{\partial x_s}{\partial x_t'}$$

Example 2.5.3 *Prove that outer product of two vectors is a tensor of rank 2.*

Solution

Let C_j^i be outer product of two vectors A^i and B_j

$$C_j^i = A^iB_j$$

$$A'^i = \frac{\partial x_i'}{\partial x_\alpha}A^\alpha\ ,\ B_j' = \frac{\partial x_B}{\partial x_j'}B_\beta$$

$$A'^iB_j' = \frac{\partial x_i'}{\partial x_\alpha}A^\alpha\frac{\partial x_\beta}{\partial x_j'}B_\beta$$

$$= \frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_\beta}{\partial x_j'}A^\alpha B_\beta$$

$$C_j'^i = \frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_\beta}{\partial x_j'}C_\beta^\alpha$$

Direct Product

It is another name of outer multiplication. i.e. direct product of two tensors is a tensor of rank equal to the sum of two initial ranks.

$$A_j^iB^{kl} = C_j^{ikl}$$

Inner Product

If the outer multiplication of two tensors is followed by contraction, the process is called inner multiplication and resulting tensor is called inner product of two tensors.

Example 2.5.4 *Let a_i and b^j be the components of a covariant and contravariant tensor respectively. Then their outer product may be written as $a_i b^j$.*

Solution

This is actually second rank tensor

$$\begin{aligned}
a'_i b'^j &= \frac{\partial x_k}{\partial x'_i} a_k \frac{\partial x'_j}{\partial x_l} b^l \\
&= \frac{\partial x_k}{\partial x'_i} \frac{\partial x'_j}{\partial x_l} a_k b^l \\
a'_i b'^i &= \frac{\partial x_k}{\partial x_l} a_k b^l \qquad \text{Contracting (set i=j)} \\
&= \delta^k_l a_k b^l \\
&= a_k b^k \quad \text{(A scalar product)}
\end{aligned}$$

Quotient Rule

If inner product of a quantity with an arbitrary tensor is itself a tensor, then this quantity is also a tensor.

Example 2.5.5 *Let A_{ij} is a quantity such that $A_{ij} B^i = C_j$. Where B^i is an arbitrary contravariant vector and C_j is a covariant vector. We have to prove that A_{ij} is a covariant tensor of rank 2.*

Solution

$$A_{ij} B^i = C_j$$

Let A'_{pq}, B'^p, C'_q be their corresponding components then

$$A'_{pq}B'^p = C'_q \tag{2.13}$$

$$B'^p = \frac{\partial x'^p}{\partial x^i}B^i \quad ; \quad C'_q = \frac{\partial x^j}{\partial x'^q}C_j$$

Eq.(2.13) becomes

$$A'_{pq}\frac{\partial x'^p}{\partial x^i}B^i = \frac{\partial x^j}{\partial x'^q}C_j$$

$$A'_{pq}\frac{\partial x'^p}{\partial x^i}B^i = \frac{\partial x^j}{\partial x'^q}A_{ij}B^i$$

$$\left[A'_{pq}\frac{\partial x'^p}{\partial x^i} - \frac{\partial x^j}{\partial x'^q}A_{ij}\right]B^i = 0$$

Since B^i is an arbitrary contravariant vector, so

$$A'_{pq}\frac{\partial x'^p}{\partial x^i} - \frac{\partial x^j}{\partial x'^q}A_{ij} = 0$$

$$A'_{pq}\frac{\partial x'^p}{\partial x^i} = \frac{\partial x^j}{\partial x'^q}A_{ij}$$

Multiplying both side by $\frac{\partial x'}{\partial x'^m}$

$$A'_{pq}\frac{\partial x'^p}{\partial x^i}\frac{\partial x'}{\partial x'^m} = \frac{\partial x'}{\partial x'^m}\frac{\partial x^j}{\partial x'^q}A_{ij}$$

$$A'_{pq}\delta^p_m = \frac{\partial x'}{\partial x'^m}\frac{\partial x^j}{\partial x'^q}A_{ij}$$

$$A'_{qm} = \frac{\partial x'}{\partial x'^m}\frac{\partial x^j}{\partial x'^q}A_{ij}$$

Replacing index m by p

$$A'_{pq} = \frac{\partial x'}{\partial x'^p}\frac{\partial x^j}{\partial x'^q}A_{ij}$$

Hence proved.

Example 2.5.6 *Show that matrix* $T = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$ *is a contravariant 2nd rank tensor in 2D.*

Solution

We will use x' and y'. So their values are $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$

1.

$$\begin{aligned}
T'^{11} &= -x'y' \\
&= -(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta) \\
&= -\Big[(y^2 - x^2)\cos\theta\sin\theta + xy(\cos^2\theta - \sin^2\theta)\Big] \qquad (2.14)\\
T'^{11} &= \sum_{kl} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_1'}{\partial x_l}T^{kl} \\
&= \sum_{kl}^{2} a_{1k}a_{1l}T^{kl} \\
&= a_{11}a_{11}T^{11} + a_{12}a_{11}T^{21} + a_{11}a_{12}T^{12} + a_{12}a_{12}T^{22} \\
&= \cos^2\theta T^{11} + \sin\theta\cos\theta T^{21} + \sin\theta\cos\theta T^{12} + \sin^2\theta T^{22} \\
&= \cos^2\theta(-xy) + \sin\theta\cos\theta(x^2) + \sin\theta\cos\theta(-y^2) + \sin^2\theta(xy) \\
&= -[(y^2 - x^2)\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)] \qquad (2.15)
\end{aligned}$$

As Eq.(2.14) and Eq.(2.15) are equal so T'^{11} satisfied transformation law.

2.

$$\begin{aligned}
T'^{12} &= -y'^2 = -(-x\sin\theta + y\cos\theta)^2 \\
&= -[x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta] \qquad (2.16)\\
T'^{12} &= \sum_{kl} \frac{\partial x_1'}{\partial x_k}\frac{\partial x_2'}{\partial x_l}T^{kl} \qquad i = 1, j = 2 \\
&= \sum_{kl}^{2} a_{1k}a_{2l}T^{kl} \\
&= a_{11}a_{21}T^{11} + a_{11}a_{22}T^{12} + a_{12}a_{21}T^{21} + a_{12}a_{22}T^{22} \\
&= \sin\theta\cos\theta(xy) + \cos^2\theta(-y^2) + \sin^2\theta(-x^2) + \sin\theta\cos\theta(xy) \\
&= -[x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta] \qquad (2.17)
\end{aligned}$$

Eq.(2.16) and Eq.(2.17) are same.

3.

$$T'^{21} = x'^2 = (x\cos\theta + y\sin\theta)^2$$
$$= x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta \tag{2.18}$$

$$T'^{21} = \sum_{kl} \frac{\partial x_2'}{\partial x_k}\frac{\partial x_1'}{}\partial x_l T^{kl} \quad i = 2, j = 1$$
$$= \sum_{kl}^{2} a_{2k}a_{1l}T^{kl}$$
$$= a_{21}a_{11}T^{11} + a_{21}a_{12}T^{12} + a_{22}a_{11}T^{21} + a_{22}a_{12}T^{22}$$
$$= \cos\theta\sin\theta(xy) + \cos^2\theta(x^2) + \sin^2\theta(y^2) + \sin\theta\cos\theta(xy)$$
$$= x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta \tag{2.19}$$

Eq.(2.18) and Eq.(2.19) are same.

4.

$$T'^{22} = x'y'$$
$$= (y^2 - x^2)\cos\theta\sin\theta + xy(\cos^2\theta - \sin^2\theta) \tag{2.20}$$

$$T'^{22} = \sum_{kl} \frac{\partial x_2'}{\partial x_k}\frac{\partial x_2'}{\partial x_l} T^{kl}$$
$$= \sum_{kl} a_{2k}a_{2l}T^{kl}$$
$$= a_{21}a_{21}T^{11} + a_{21}a_{22}T^{12} + a_{22}a_{21}T^{21} + a_{22}a_{22}T^{22}$$
$$= \cos^2\theta(xy) + \sin\theta\cos\theta(-x^2) + \sin\theta\cos\theta(y^2) - \sin^2\theta(xy)$$
$$= (y^2 - x^2)\cos\theta\sin\theta + xy(\cos^2\theta - \sin^2\theta) \tag{2.21}$$

Eq.(2.20) and Eq.(2.21) are same.

Hence, we see that $T = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$ is a tensor of rank = 2.