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Journal:	*European Journal of Physics*
Manuscript ID:	EJP-100085
Manuscript Type:	Paper
Date Submitted by the Author:	18-Dec-2013

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Parametric Instability in the Watt Governor with Periodic Loading

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Classification

PACS : 45.30.+s 89.20.Kk

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Abstract

In this work we examine a potential source of instability in the Watt governor, which can occur when the governor is subjected to periodic variation of the load torque. Floquet analysis is used to obtain the condition for existence of the unstable solutions, and results are obtained which supplement the usual Maxwell-Vyshnegradskii conditions for governor stability. Pedagogically, this paper is relevant for senior undergraduate students. It presents a detailed yet simple exposition of classical Floquet theory, a subject often ignored in undergraduate education even though the related concept of Bloch's theorem is popular in quantum mechanics courses. At the same time, it touches on certain practical engineering applications of physical concepts, with its discussion on mechanical governors.

Introduction

A standard class of problems in classical mechanics features systems where one or more of the parameters are time-dependent. An example is a spring whose force constant changes with time (say it is being heated) or a rocket whose mass changes with time as the fuel is thrown out the nozzle. An important sub-class of these systems is one where the time dependence is periodic, i.e. the parameter(s) has (have) the same value at time t and time $t+T$ for some fixed T. One example of this is a toy consisting of a bob attached to the end of a string which is wound around a spool. The system is like an ordinary pendulum, except that its length can be changed by winding or unwinding more of the string. It is found that when the string is wound and unwound in a periodic manner with a certain frequency then the bob tends to swing violently. For handling systems of this type, Floquet analysis provides a systematic approach which can be used to obtain results of arbitrary precision.

A topic apparently unrelated to Floquet analysis is control theory. The aim of this theory is to regulate the behaviour of dynamic systems by curbing any tendency of the system to depart from the set point. For instance in an oven which we have set for 200^{o}C, the power to the coils is cut off if the temperature increases beyond say 205^{o}C and is switched on again if the temperature falls beyond 195^{o}C. This is one of the simplest control strategies called hysteresis, bang-bang or two limit control (despite its simplicity, it can be amazingly powerful – the strategy called Direct Torque Control of induction motor is based on this). Another strategy is called proportional control in which the controller output is proportional to the error signal. This is generally more accurate than hysteresis control and can be used for precise regulation of the process variable.

While feedback and control are often associated with circuits and electronics, an association which derives from the constant application of these concepts in electrical engineering, they are also applicable to mechanical, chemical and biological systems. In mechanical engineering, the classic example is that of the centrifugal governor [1] which is used to control the speed of an engine. Originally invented by unknown person/s in the context of water pumps, the governor was significantly improved and applied in steam engines by Sir James Prescott Watt. It consists of a pair of metal balls mounted on arms attached to a rotating shaft which is connected to the engine. The rotation of the shaft causes the balls to swing outward, and a sleeve mechanism is used to link the position of the balls to the amount of fuel flowing into the engine.

The condition for stability of the governor had first been obtained by Maxwell [2] and later improved by Vyshnegradskii [3-4]. Hence it is called MV stability condition. An extension of the stability condition has been carried out by Denny [5] who has included nonlinear terms in the analysis. An augmentation may be found in Sotomayor et. al. [6] who have discovered a Hopf bifurcation in the system. These works however are confined to the case where the engine is tractable i.e. its torque output is smooth and stable. In reality, engines often have imperfections which give rise to fluctuations in the output. Here we consider the stability of the governor in response to such fluctuations. In particular we treat the case where the fluctuation is periodic in time, so that Floquet analysis becomes applicable. We find that there exists an upper bound on the amplitude of the fluctuation, exceeding which the operation of the governor becomes unstable even when the MV conditions are not violated.

II. The Watt Governor

The schematic diagram of a Watt governor is shown in Fig. 1 below, adapted from Ref. [1].

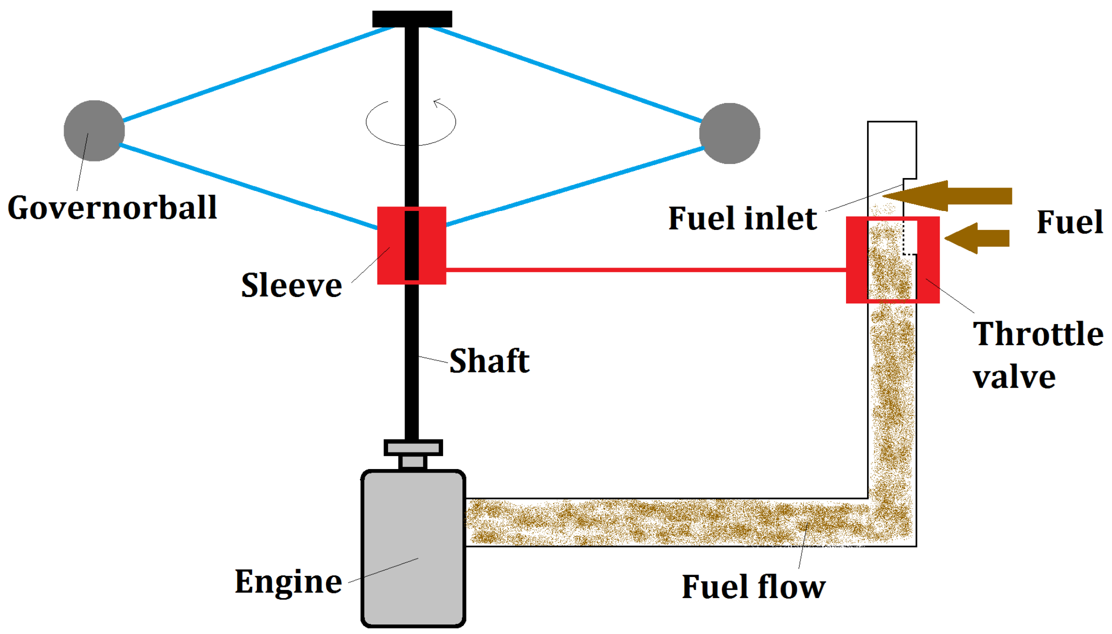


Figure 1 : *The Watt governor.*

Due to symmetry of the system, the equation of motion of the system can be obtained by considering only one governorball. Since in a Watt governor, this is assumed infinitely heavier than the sleeve, its dynamics will in fact be that of a simple conical pendulum. This can be written as

$$l\mathrm{p}^2\theta + k\mathrm{p}\theta + \sin\theta\left(g - \omega^2 l\cos\theta\right) = 0 \quad , \tag{1}$$

where *l* is the length of the arms leading to the balls, θ is the angle made by the arms with the vertical, *g* the acceleration due to gravity, ω the rate of rotation of the shaft about its axis, *k* the damping constant and p denotes differentiation with respect to time (Haveside notation). There are of course two possible equilibrium postions, θ=0 and

$\theta = \arccos\left(g / \omega^2 l\right)$. Of these, the second position is stable whenever it is defined (i.e. $\omega^2 l > g$) and the first is stable otherwise. Since in a governor we want the height of the sleeve to depend on the speed of rotation, we must operate it above the critical speed determined above.

The next step is to linearize (1) about the operating equilibrium, which we label by the angle θ^*. Writing $\Delta\theta = \theta - \theta^*$ we have the following linear equation for $\Delta\theta$

$$\mathrm{p}^2\Delta\theta + k\mathrm{p}\Delta\theta + (\omega^2 \sin^2\theta^*)\Delta\theta = 0 \quad . \tag{2}$$

From this point, the MV analysis proceeds by constructing a model for the torque of the engine as a function of the governor position. The usual expression chosen to represent the fuel flow regulation is a proportionality relation

$$\Gamma = -\beta\Delta\theta \quad , \tag{3}$$

where Γ is the torque of the engine and β a positive constant. In terms of the moment of inertia I_e of the engine and the load, the above can be written as

$$I_e \mathrm{p}\omega + \beta\Delta\theta = 0 \quad . \tag{4}$$

In this work however we will not concern ourselves with the dynamics of the fuel flow. Rather we will treat the case where the system is at steady state i.e. the engine is operating at the rated speed but due to fluctuations, it also has a sinusoidal component superimposed on this speed. The speed of the engine can then be written as

$$\omega = \omega^* + \Delta\omega\cos\alpha t \quad , \tag{5}$$

in which α denotes the frequency of the ripples in the speed. Substituting (5) into (2) and retaining terms upto first order in $\Delta\omega/\omega$ we have the equation

$$\mathrm{p}^2\Delta\theta + k\mathrm{p}\Delta\theta + \omega^{*2}\sin^2\theta^*\left(1 + \frac{\Delta\omega}{\omega^*}\cos\alpha t\right)\Delta\theta = 0 \quad , \tag{6}$$

which is called Mathieu equation. In the next section of this paper we shall show how to solve Mathieu equation by Floquet analysis.

II. The Mathieu Equation

The treatment here follows that of Kumar [7], which, along with Refs. [8-10] may be consulted as an additional reference. The standard form of the Mathieu equation is as follows for a function $x(t)$:

$$\mathrm{p}^2 x + 2\lambda \mathrm{p} x + \omega_0^2 \left(1 + a\cos\omega t\right) x = 0 \quad , \tag{7}$$

where ω_0 denotes the natural frequency of the system and ω is the (variable) frequency of the external excitation. Now Floquet's theorem states that the solution $x(t)$ can be written in the following form:

$$x(t) = \exp(st)\exp(j\gamma\omega t)\sum_{-\infty\ldots-3,-2,-1}^{0,1,2\ldots\infty} X_n \exp(jn\omega t) \tag{8}$$

where γ is some real number (as yet undetermined), s is a real number which may be positive or negative and j is the imaginary unit. We now obtain a restriction on the value of γ. by imposing the condition that $x(t)$ be real. We can show that only for the two values γ=0 and 1/2 it is possible to arrange the series in such a way that the imaginary components of the terms cancel pairwise. The case where γ=0 is called the harmonic case as the solution function has the same periodicity as the excitation. The case of γ=1/2 is called the subharmonic case for then the solution is periodic with half the frequency, or double the period of the excitation.

Substituting (8) into (7) we can attempt a solution by equating the coefficient of the term $\exp(s+j(k+\gamma)\omega)t$ to zero for each k. However, by writing the $\cos\omega t$ in (7) as the sum of two complex exponentials we can see that the coefficient of the term $\exp\left(s+j(k+\alpha)\omega\right)t$ will feature not only X_k but also $X_{k\pm1}$. Hence, we will not get a set of isolated equations for each X_k; rather we will find a recursion relation connecting X_k, X_{k+1} and X_{k-1}. From this point on, the derivation of the recursion is straightforward and we directly write the relation as

$$a\left(X_{k+1} + X_{k-1}\right) = -\frac{2}{\omega_0^2}\left[\left(s + j(k+\gamma)\omega\right)^2 + 2\lambda\left(s + j(k+\gamma)\omega\right) + \omega_0^2\right] X_k \quad . \tag{9}$$

Denoting the coefficient of X_k in the right hand side (RHS) of the above by A_k, we can express the same as a matrix equation

$$\begin{bmatrix} \dots & & & & & & \\ & A_{-2} & & & & & \\ & & A_{-1} & & & & \\ & & & A_0 & & & \\ & & & & A_1 & & \\ & & & & & A_2 & \\ & & & & & & \dots \end{bmatrix} \begin{bmatrix} \dots \\ X_{-2} \\ X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ \dots \end{bmatrix} = a \begin{bmatrix} \dots & \dots & & & & & \\ \dots & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & \dots \\ & & & & & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots \\ X_{-2} \\ X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ \dots \end{bmatrix}, \qquad (10)$$

which is seen to correspond exactly to (9). The amplitude *a* is kept separate from the coefficients A_k for a reason which we will see a little later. The empty spaces in the above matrices are of course to be filled by ciphers. Basically, the matrix on the left hand side (LHS) is diagonal with the nonzero elements being the coefficients A_k and the matrix on the right has zero along the diagonal and unity along the immediate super- and sub-diagonals. In theory the matrices are infinite, but in practice one can truncate them at any point one desires. The greater the number of terms retained, the higher will be the accuracy of the calculation. Now the A_k's will follow the same conjugation rule as the X_k's and these rules imply that while truncating the matrices for the harmonic case, an equal number of terms on both sides of zero must be retained. On the other hand, while truncating a subharmonic matrix, the number of terms to the left of zero (i.e. A_k for k less than zero) must be one more than the number of terms on the right of zero. This in turn means that the harmonic matrix has an odd number of rows and columns while the subharmonic matrix has an even number of them.

Now (10) can be written as

$$\mathbf{AX} = a\mathbf{BX} \qquad (11)$$

where **A** and **B** denote the matrices on the LHS and RHS of (10) and **X** denotes the vector of coefficients. Let us now decide on what exactly we want to determine from the analysis. As of yet there are two variable quantities s and ω implicit in each of the coefficients A_k and there is a third variable, the amplitude a, appearing explicitly on the RHS of (11). It is reasonable to assume that if any two of these three quantities are fixed, then the third will be determined uniquely. Now the value of s is crucial in determining the nature of the solution : s<0 implies a damped solution while s>0 implies an exponentially growing solution. The boundary between the two regimes is the s=0 case which describes perfectly periodic solutions. When s is non-zero, its specific value is not really of too much interest; *any* negative value denotes a damped solution which is generally desirable or at least harmless, while *any* positive value denotes a growing solution, which is always undesirable, however slow the growth rate. Hence it makes the maximum sense to determine the s=0 curve in the a—ω plane. This curve will be the separatrix between the bounded and unbounded solutions and will yield, for each value of *ω*, the corresponding amplitude necessary to generate perfectly periodic motion. Any smaller amplitude will produce a damped solution while any larger amplitude will give rise to the unwanted growing solution.

In accordance with the above logic we set *s*=0 and then fix a value of *ω* to obtain the corresponding amplitude *a*. Now (11) can be written as

$$\mathbf{A}^{-1}\mathbf{B}\mathbf{X} = \frac{1}{a}\mathbf{X} \quad , \tag{12}$$

which implies that 1/*a* is an eigenvalue of the matrix $\mathbf{A}^{-1}\mathbf{B}$. Now when the matrix dimensions are large, there will be a large number of eigenvalues. So which one(s) is(are) relevant ? Firstly, complex eigenvalues are junk as the oscillation amplitude cannot be complex, hence these must be rejected at once. Negative eigenvalues are also meaningless as they imply negative amplitudes which, though not physically absurd, are entirely equivalent to phase-shifted oscillations of positive amplitude. Among the positive amplitudes we will generally be interested in the *smallest* amplitude which can produce perfectly periodic solutions (and exceeding which can take us into the danger area) hence the *largest eigenvalue* will the one of greatest practical utility. By now it is clear why the amplitude *a* was given 'special treatment' in (10) and (11). Since accurate analyses require large dimensions of matrix, and determining their eigenvalues is impossible by hand and trivial by computer, we will compute the eigenvalues numerically. In addition to the largest eigenvalue we will also compute the second largest eigenvalue and plot that too. The result of the computation is shown in Fig. 1, where the system has very little damping (λ=0.01). A set of tongue-like structures appear – alternate ones correspond to harmonic and subharmonic solutions. The minimum amplitude for exciting harmonic oscillations occurs (not surprisingly) at the natural frequency of the pendulum and that for exciting subharmonic oscillations occurs (again no surprise here) at double the natural frequency.

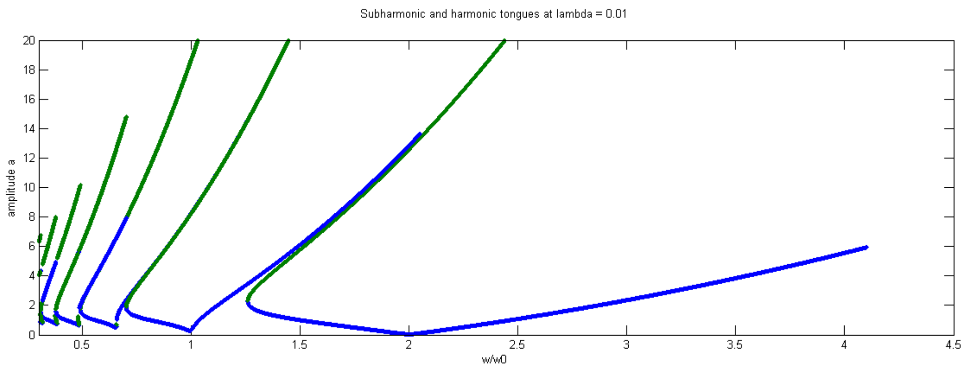


Figure 2 : *Floquet tongues for the lightly damped pendulum, λ=0.01. The* x*-axis shows the excitation frequency* ω *normalized to the natural frequency* ω_0 *while the* y *axis shows the amplitude* a*. Two eigenvalues (largest and second-largest) have been plotted per frequency to elucidate the nature of the tongues.*

Further computations are carried out by varying λ. As expected, the amplitude required to excite perfectly periodic motion becomes higher as λ increases; also, the positions of the minima get shifted from their undamped positions, similar to what happens in an ordinary resonance in a damped driven pendulum. We present the plot of λ=1 below, Fig. 3.

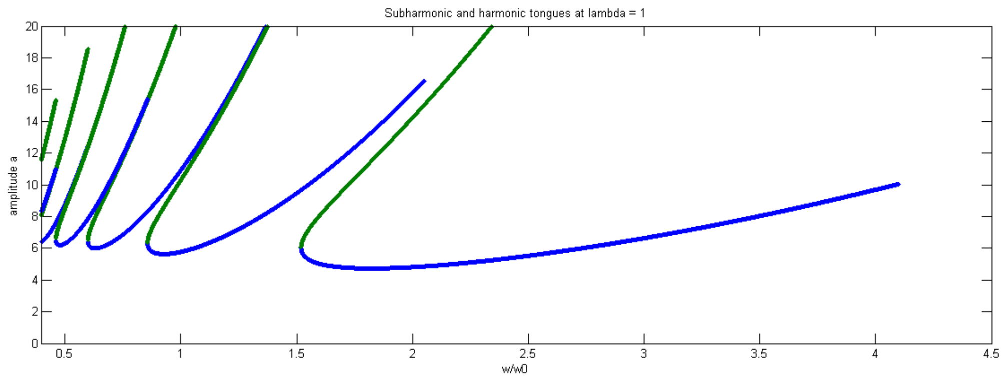
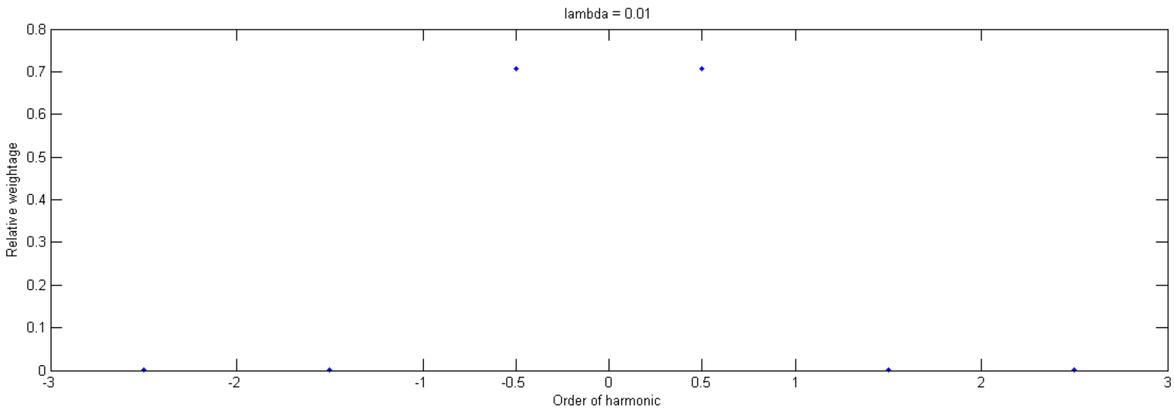


Figure 3 : *Same as Fig. 2 except for the change in λ.*

These calculations indicate that the subharmonic mode is easier to excite than the harmonic mode. Hence while designing the system, it should be kept in mind that the amplitude of the ripple be less than the critical value required to excite the subharmonic oscillations. In Fig. 4 we show the harmonic content of the oscillations at the critical point. We see that very low damping causes virtually pure fundamental oscillations, while higher values of damping give rise to higher harmonic content.



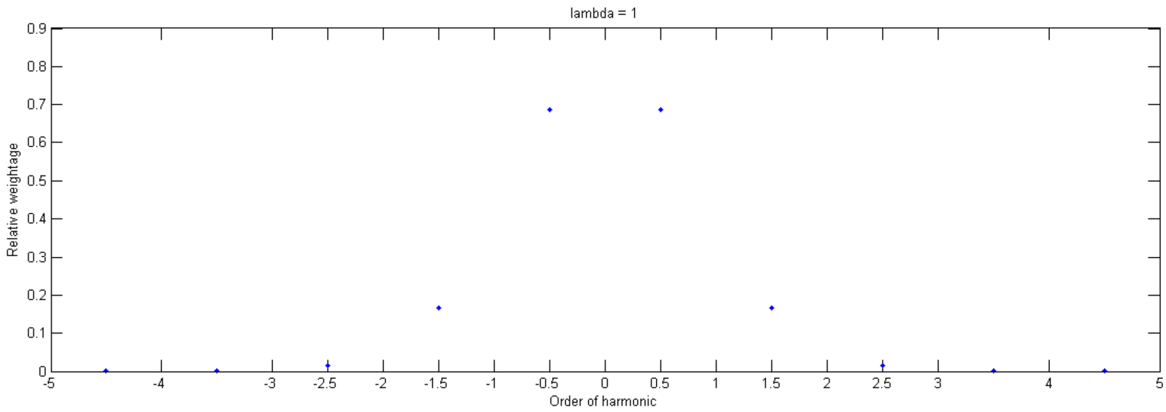


Figure 4 : *Upper panel shows oscillations for the lightly damped case (λ=0.01) while the lower panel shows a stronger damped case (λ=1). The fundamental harmonics in the subharmonic mode are of course* exp(±jωt/2) *; these are characterized by the x-labels ±1/2. The higher harmonics are labelled as 3/2, 5/2 etc on x axis. The y axis shows the amplitude (in arbitrary units) of the various harmonic components when the excitation is applied on the separatrix between bounded and growing motions.*

It should of course be remembered that the condition for the Floquet instability depends on the amplitude and frequency of the external excitation, unlike the MV conditions which are functions of the system internal parameters. Hence, even in regions where the MV conditions are satisfied, the Floquet resonance can still take place. Our work thus extends the MV conditions to the case of periodic loading.

Conclusions

In this work we have used basic Floquet theory to examine the stability of the Watt governor to a sinusoidal variation in the engine speed. The greatest chance of instability will occur if the excitation frequency in the engine be double of the conical pendulum's natural frequency. The easiest way to kill off the Floquet resonance will be to make the damping of the pendulum arbitrarily high. Since however an overdamped pendulum is in general sluggish, critical damping is the ideal scenario to go for and it must be ensured that the engine normally does not show such high amplitude ripples. For safety reasons, a ripple detector should be installed which will shut the system off in the event of unacceptable ripple. One possible implementation of this is by introducing a fuel cut-off feature in the throttle valve such that the fuel flow is completely stopped if the pendulum angle increases beyond a certain limiting value.

Acknowledgement

This paper is the result of the project done by the first author under the aegis of Kishore Vaigyanik Protsahan Yojana (KVPY) Government of India in the summer of 2013. The second author acted as advisor to the first author.

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