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CSC 6013
WEEK 5 WORKSHEET

BACK SUBSTITUTION

1) $T(n) = 2T(n-1)+1, \; T(0)=1$

$T(1) = 2T(0)+1 = 2(1)+1 = 3$

$T(2) = 2T(1)+1 = 2(3)+1 = 7$

$T(3) = 2T(2)+1 = 2(7)+1 = 15$

$T(4) = 2T(3)+1 = 2(15)+1 = 31$

$T(n) = \boxed{2^{(n+1)}} - 1$

$T(1) = 2^{(1+1)} - 1 = 3$

BASE CASE

$T(0) = 2^{(0+1)} - 1 = 1$

DOMINANT TERM

$T(n)$ is $O(2^{(n+1)})$

PATTERN

$T(0)=1$

$T(1) = 2T(0)+1) = 3$

$T(2) = 2T(1)+1 = 2(2T(0)+1)+1 = 2^{2T(0)} + 2^1 + 1 = 2^2+2+1 = 7$

$T(3) = 2T(2)+1 = 2(2^{2T(0)} + 2^1 + 1) + 1 = 2^{3T(0)} + 2^2 + 2^1 + 1 = 2^3 + 2^2 + 2 + 1 = 31$

$\boxed{\text{TIGHT UPPER BOUND} = T(n) \text{ is } O(2^{(n+1)})}$

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2) $T(n) = T(n-2)+n^2, \; T(0)=1$

$T(2) = T(0) + 2^2 = 1+4 = 5$

$T(4) = T(2) + 4^2 = 5+16 = 21$

$T(6) = T(4) + 6^2 = 21+36 = 57$

$T(2k) = 1 + 2^2 + 4^2 + 6^2 \ldots + (2k-2)^2$

$T(2k) = 1 + (2k-2(k-1))^2 + (2k-2(k-2))^2 + \ldots (2k-2)^2 + (2k)^2$

$T(2k) \leq 1 + (2k)^2$

$\boxed{T(n) \leq 1 + \boxed{n^2} = O(n^2)}$

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3) $T(n) = T(n-1) + \frac{1}{n}, \; T(1)=1$

$T(2) = T(2-1) + \frac{1}{2} = T(1) + \frac{1}{2} = 1.5$

$T(n) = T(n-1) + \frac{1}{n}$

$T(n) = (T(n-2) + \frac{1}{n-1}) + \frac{1}{n}$

$T(n) = (T(n-3) + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$

$H(n) = 1 + 1/2 + 1/3 + \ldots 1/n$

$H(n) \approx \ln(n) + \gamma$

$T(n) = T(1) + H(n-1) \approx 1 + \ln(n-1) + \gamma$

$T(n) \approx \ln(n) + C$ (c is a constant, like γ)

$\boxed{T(n) \text{ is } O(\ln(n))}$

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MASTER METHOD

$T(n) = aT(n/b) + f(n)$

if $a < b^d$ then $T(n) = O(n^d)$
if $a = b^d$ then $T(n) = O(n^d \cdot \lg(n))$
if $a > b$ then $T(n) = O(n^{\log_b a})$

4) $T(n) = 2T(\frac{n}{4}) + 1$, $T(0) = 1$

$a = 2$, $b = 4$, $f(n) = 1 = n^0$ or $d = 0$

$n^{\log b(a)} = n^{\log 4(2)} \approx n^{0.5}$

$\boxed{T(n) = 2T(n/4) + 1 \text{ is } T(n) = \Theta(n^{0.5})}$

5) $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$, $T(0) = 1$

$T(n) = 2T(n/4) + \sqrt{n}$

$a = 2$, $b = 4$, $f(n) = \sqrt{n}$

$f(n) = n^{0.5}$

$n^{\log b(a)} = n^{\log 4(2)} \approx n^{0.5}$

$T(n) = \Theta(f(n)) = \Theta(\sqrt{n})$

$\boxed{T(n) = 2T(n/4) + \sqrt{n} \text{ is } T(n) = \Theta(\sqrt{n})}$

6) $T(n) = 2T(\frac{n}{4}) + n^2$, $T(0) = 1$

$a = 2$, $b = 4$, $f(n) = n^2$

$f(n) = n^2 = n^{\log 4(2)} = d$

$n^{\log 4(2)} \approx n^{0.5}$

$T(n) = \Theta(f(n)) = \Theta(n^2)$

7) $T(n) = 10T(\frac{n}{3}) + n^2$, $T(0) = 1$

$a = 10$, $b = 3$, $f(n) = n^2$

$f(n) = n^2 = n^{\log 3(10)}$

$n^{\log 3(10)} \approx n^{2.19}$

$T(n) = \Theta(n^{\log b(a)} \cdot \log n = \Theta(n^{2.19} \cdot \log(n)$

$\boxed{T(n) = 10T(n/3) + n^2 \text{ is } T(n) = \Theta(n^{2.19} \cdot \log n)}$

$\boxed{T(n) = \Theta(n^{2.19} \log(n))}$

8) $T(n) = 2T(\frac{2n}{3} + 1$, $T(0) = 1$

$T(n) = 2T(\frac{n}{3/2}) + 1$, $T(0) = 1$

$a = 2$, $b = 3/2$, $f(n) = n^0 = 1$

$\log b(a) = \log_{(\frac{3}{2})} 2 \approx 1.71$

$n^{\log b(a)} = n^{1.71}$

$\boxed{T(n) = \Theta(n^{\log b(a)}) = \Theta(n^{1.71})}$