Problem Set 5 - Group Actions, Dihedral Group, Odd/Even Permutations, Homomorphisms

Problem 1 Let $a = (1\ 2\ 5\ 7)(2\ 4\ 6)$. Let $b = (5\ 3\ 1\ 2)$. Let $c = (5\ 4\ 1\ 2\ 3)$.

a) Find the order of ab

b) Find the order of $(ac)^{-1}b(ac)$

c) Find the order of $(a^{-1}ba)^2$ and $(a^{-1}ba)^4$

d) Find the order of $(a^{-1}ca)^3$ and $(a^{-1}ca)^5$

e) For each of a, b, c, ab, ac determine whether the element is odd or even.

f) Find an element d such that $d^{-1}bd = (1\ 2\ 3\ 4)$.

Problem 2 Recall that D_4 denotes the symmetry group of the square. Let $R_0, R_{90}, R_{180}, R_{270}$ denote rotations counterclockwise. Let F_a, F_b, F_c, F_d denote reflections across the lines $y = 0, y = x, x = 0, y = -x$.

a) Compute $F_a R_{90} F_b$, compute $F_c F_a$, and compute $F_a F_c$.

b) What is the order of each of the elements in D_4?

c) What is the order of $F_a R_{270} F_b F_c$?

d) D_4 acts on the set of vertices of the square. Find the elements in S_4 corresponding to each of the elements in D_4 through this action.

e) Illustrate the orbit-stabilizer theorem by considering the number of elements in D_4 that fix a vertex and the number of elements in the orbit of the vertex.

Problem 3 a) Let G be the group of rigid symmetries, in $\mathbb{R}^3$, of the cube. Compute the number of elements in G.

b) By considering the action of G on the faces of the cube, find generators for the permutation representation of G in S_6.

c) By considering the action of G on the vertices of the cube, find generators for the permutation representation of G in S_8.

d) By considering the action of G on the edges of the cube, find generators for the permutation representation of G in S_{12}.

e) Repeat part a) to compute the order of the symmetry group for each of the other platonic solids.

Problem 4: Consider the group S_3. There is an action of S_3 on the set of elements of S_3 by $g.x = gx$. This leads to a homomorphism of S_3 into S_6.

a) Find the elements in the image of this homomorphism.

b) How many elements are in the kernel of this homomorphism?

c) How many elements are in the image of this homomorphism?

d) Write down generators for the kernel of the homomorphism.

e) Write down generators for the image of the homomorphism.

Problem 5: Consider the group S_4. There is an action of S_4 on the set of 2-cycles in S_4 by $g.x = g^{-1}xg$. This leads to a homomorphism of S_4 into a symmetric group.

a) Which symmetric group?

b) How many elements are in the kernel of this homomorphism?

c) How many elements are in the image of this homomorphism?

d) Write down generators for the kernel of the homomorphism.

e) Write down generators for the image of the homomorphism.