M366 Some Problems on Rings and Fields

Problem 1. *Let* $A = \begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$ *considered as an element of the ring* $M_2(Z_{11})$.

a) Determine $A + A$ *and* $A * A$.

b) Determine the additive and multiplicative inverse of A.

Problem 2. *Let* J *be the ideal* $< x^2 + 3x + 1 >$. *Let* $R = \mathbb{Z}_5[x]/J$.

a) Let $A = (x + 1) + J$. *The element* A^2 *can be written in the form* $(ax + b) + J$. *Find* a *and* b.

b) What is the inverse of $(x + 1) + J$*?*

c) How many distinct elements are in R*?*

Problem 3. *Let* $\mathbb{Z}_{12}$ *be the ring of integers modulo* 12.

a) What are the units in $\mathbb{Z}_{12}$*?*

b) Do the units in $\mathbb{Z}_{12}$ *form an ideal? Why or why not?*

c) Consider the ideal $J =< 3 >$ *in* $\mathbb{Z}_{12}$. *Is the quotient ring* $\mathbb{Z}_{12}/J$ *a field? Why or why not?*

Problem 4. *Give an example of each of the following:*

a) A non-abelian group.

b) An infinite cyclic group.

c) A group with a proper normal subgroup.

d) A group which has no proper normal subgroups.

e) A non-commutative rng (like a ring but no multiplicative identity).

f) A field with an infinite number of elements.

g) A commutative ring, not an integral domain, infinitely many elements.

h) A commutative ring, not an integral domain, finitely many elements.

i) An integral domain which is not a field.

j) A non-commutative ring with a finite number of elements.

Problem 5. *Let J be the ideal $< x^2+2x+4 >$. Consider the map $\phi : \mathbb{Z}[x] \to \mathbb{Z}[x]/J$ given by $\phi(a) = a + J$ for each $a \in \mathbb{Z}[x]$.*

a) Show that ϕ is a homomorphism.

b) What is the kernel of ϕ?

Problem 6. *Let $A = \mathbb{Z}_{15} \oplus \mathbb{Z}$. Let $B = \mathbb{Z} \oplus \mathbb{Q}$.*

a) Find a prime ideal in A which is not a maximal ideal.

b) Find a maximal ideal in A.

c) What are the units in A?

d) What are the zero-divisors in A?

e) Is B an integral domain?

f) Find a prime ideal in B which is not a maximal ideal.

g) Find a maximal ideal in B.

Problem 7. *Let I be the ideal $< x^2 + 2x + 4 >$. Let $R = \mathbb{Z}_5[x]/I$. Let $f = (3x + 1) + I$. Write $f \cdot f$ in the form $(ax + b) + I$.*

Problem 8. *a) Is $< 6 >$ a maximal ideal in $\mathbb{Z}$? Why or why not?*

b) Is $< 6 >$ a prime ideal in $\mathbb{Z}$? Why or why not?

c) Is the ring $\mathbb{Z}/ < 7 >$ an integral domain? Why or why not?

d) Is the ring $\mathbb{Z}/ < 7 >$ a field? Why or why not?

e) Find a zero-divisor in $\mathbb{Z}_5[i]$.

f) What are the zero-divisors and units of $\mathbb{Z} \oplus \mathbb{R}$?

g) How many solutions are there to $x^2 + 5x = 0$ in $\mathbb{Z}_{12}$?

h) What is the characteristic of the ring $\mathbb{Z}_6 \oplus \mathbb{Z}_{10}$?