Effects of increasing crowds on the Millenium Bridge

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Abstract

Londoners experienced a frightening wobbling on the Millenium Bridge the day it opened to pedestrian traffic in the year 2000. The large lateral oscillations were caused by a resonance between the bridge's vibrations and the stepping frequencies of walkers, and they had nothing to do with the design of the bridge. Although several models have been proposed to describe this phenomenon, they focus mainly on the wobbling, leaving the synchronization of pedestrians aside. In [7] the well-known Kuramoto model used to study synchronization of biological oscillators is adapted to that particular problem, so that synchrony and resonance are no longer separable. In this document we attempt to reproduce the results of that study, including also a brief and clear review of the most relevant methods.

I Introduction

On the opening day of the Millennium Bridge there were observed side to side vibrations on the bridge that were not noticeable with a few walkers but, as the number of walkers increased, the vibrations amplified to such an extent that the bridge had to be closed and repaired afterwards. This work will not focus on the details of the innovative design of the bridge as it is done in many other works like [2] since today it is widely accepted that the bridge oscillations were simply a matter of bridge-walkers synchronization and resonance. More specifically, we will model the walkers as forcing agents whose frequencies will be synchronized between them and with the bridge's, as reported in [7]. Initially what happens is that a small oscillation of the bridge induces some walkers to synchronize their steps with the bridge's oscillations, hence exerting an increasing external force to the bridge. However, these oscillations are not observed for a small number of walkers but there is a critical value from which the oscillations amplitude rapidly increases until it reaches a constant value corresponding to the steady-state. This critical value will be estimated both numerically and theoretically in this work.

These vibrations amplification can indeed be modeled the same way as the synchronization of biological oscillators such as neurons or pacemakers cells in the heart. Furthermore, the Millennium Bridge oscillations serves as an example of emergence of collective behaviour in systems made of many coupled components and one of the most basic models for such phenomena is the Kuramoto model, whose equations are intrinsically related to the ones used in this work. However, a realistic model of the Millennium Bridge requires more advanced mathematical analysis than Kuramoto's formulation since we need to address the mechanism by which walkers synchronize with the bridge. Nevertheless, this is not explained in the main paper whose results we aim to reproduce, [7], but a more elaborate analysis can be found in [1].

Throughout this work we will numerically solve the second order differential equation modelling the bridge as a weakly damped and forced harmonic oscillator and the N differential equations modelling each walker's stepping frequency and coupling with the bridge (N is the number of walkers). These equations will be solved with different increasing values of N in order to simulate a realistic case in which more and more people walk on the bridge. A graphical representation of the oscillations amplitude will reveal the critical value N_c from which the oscillations quickly grow, whereas a graphical representation of the order parameter in terms of N will quantify the synchronization and the collective movement of the walkers.

II Methods

As already said, the bridge is modeled as a weakly damped and forced oscillator (equation (1)), being $X(t)$ the displacement of the lateral vibrations from the resting position, M the bridge's mass, B the damping and K the stiffness, which accounts for the extent to which an oscillator resists deformation when submitted to an external force. The force imparted by each pedestrian is of the form $G\sin(\theta_i)$, being θ_i a phase restricted in the range $[0, 2\pi)$ for each walking cycle.

Furthermore, equation (2) models the effects of the bridge's oscillation on each pedestrian's steps. We consider that each walker has its own walking frequency Ω_i, with a given probability distribution $P(\Omega)$. Since we are interested in the resonance phenomenon, we assume $P(\Omega)$ to be a Gaussian distribution with mean value Ω_0 and a standard deviation to be chosen (the reason for this particular choice will become clear in Subsection II.I). In addition, C is another constant to be estimated from real data that measures the effect of the bridge's vibrations amplitude $A(t)$ on each pedestrians steps. Ψ is defined in such a way that $X = A\sin(\Psi)$ and $dX/dt = A\Omega_0\cos(\Psi)$, where Ω_0 is the natural frequency of the bridge, $\Omega_0 = \sqrt{K/M}$. Finally, α is simply a phase parameter to be set at the beginning,

$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N}\sin\theta_i \tag{1}$$

$$\frac{d\theta_i}{dt} = \Omega_i + CA\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \ldots, N \tag{2}$$

Before integrating numerically equations (1) and (2) it is convenient to express (1) in such a way that the coupling with (1) is more evident. For that let's expand the sinusoidal term of (2) using the widely used rule for $\sin(A + B)$:

$$\frac{d\theta_i}{dt} = \Omega_i + CX\cos(\alpha - \theta_i) + \frac{C}{\Omega_0}\frac{dX}{dt}\sin(\alpha - \theta_i), \qquad i = 1, \ldots, N, \tag{3}$$

where we have used $X = A\sin(\Psi)$ and $dX/dt = A\Omega_0\cos(\Psi)$. The amplitude $A(t)$ can be deduced from the sum of the square of these two last expressions:

$$A(t) = \sqrt{X^2 + \left(\frac{1}{\Omega_0}\frac{dX}{dt}\right)^2}. \tag{4}$$

Finally, using (3) and (1) we can properly write a system of coupled oscillators and in order to deal with first order differential equations we can introduce a new variable $Y = \frac{dX}{dt}$. The final equations are thus:

$$\begin{gathered}\frac{dX}{dt} = Y \\ M\frac{dY}{dt} + BY + KX = G\sum_{i=1}^{N}\sin(\theta_i) \\ \frac{d\theta_i}{dt} = \Omega_i + CX\cos(\alpha - \theta_i) + \frac{C}{\Omega_0}\frac{dX}{dt}\sin(\alpha - \theta_i), \qquad i = 1, \ldots, N.\end{gathered} \tag{5}$$

Since our aim is to reproduce the results in [7] as closely as possible, we will use the same parameters as in the paper, namely: $M = 1.13 \times 10^5$ kg, $B = 1.10 \times 10^4$ kg/s, $K = 4.73 \times 10^6$

kg/s^2, $G = 30$ kgm/s^2, $C = 16$ m^{-1}s^{-1}, $\alpha = \pi/2$. Pedestrians' frequencies are distributed according to a Gaussian with mean $\Omega_0 = 6.47$ rad/s and standard deviation $\sigma = 0.63$ rad/s. The $N + 2$ system of equations will be solved for increasing values of walkers: we will begin with $N = 50$ and at each step it will be increased $\Delta N = 10$. The initial values of θ_i are simply chosen randomly in the interval $[0, 2\pi)$. For the first non-zero value of N ($N = 50$) we need as well initial conditions for the X and Y and we simply considered $X_0 = 0,\ Y_0 = 0$ (there are no vibrations initially). For the remaining values of N the initial values of X and Y are the last values of the previous N in order to simulate a continuous and real situation in which the number of pedestrians gradually increases (we also do that for the phases of pedestrians already on the bridge).

Finally, the equations were solved numerically using the package deSolve, [5], from the software R, [3]. More precisely, the function `lsoda` was used, which chooses among different integration methods (mainly, 4th order Runge-Kutta methods) depending on whether the system is stiff or not. A Python code was also used to check if both programs provide the same results, but we decided to stick with the R codes.

II.I Theoretical critical crowd size

In this subsection we briefly explain how to obtain the theoretical crowd size, namely,

$$N_c = \frac{4\zeta}{\pi}\left(\frac{K}{GCP(\Omega_0)}\right). \tag{6}$$

In the previous equation ζ can be computed as $\zeta = B/\sqrt{4MK}$, and $P(\Omega)$ is the probability density function of the stepping frequency of the walkers, as already explained.

To do that, we follow the supplementary information of [7]. However, we try to make a much stronger connection with the Kuramoto model for the synchronization of biological oscillators exposed in [6], which saves us lots of calculations.

The first step consists on introducing a different time scale, using the bridge's resonant frequency $\Omega_0 = \sqrt{K/M}$. Thus, defining $\tau = \Omega_0 t$ and using equations (1) and (2) we are left with

$$\frac{d^2X}{d\tau^2} + 2\zeta\frac{dX}{d\tau} + X = \frac{NG}{K}\langle\sin\theta_i\rangle,$$
$$\frac{d\theta_i}{d\tau} = \frac{\Omega_i}{\Omega_0} + \frac{C}{\Omega_0}A\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \ldots, N.$$

This shows that two different spatial scales also coexist in the model, $L_1 = NG/K$ (spatial scale associated to bridge's oscillations) and $L_2 = \Omega_0/C$ (spatial scale associated to the walkers' stepping frequency). We shall assume that $L_1 \ll L_2$, so that $\varepsilon = \sqrt{L_1/L_2}$ can be thought of as a small parameter.[1] Defining the spatial scale of the problem as the geometric mean, $L = \sqrt{L_1 L_2}$, the previous equations are now

$$\frac{d^2x}{d\tau^2} + 2\zeta\frac{dx}{d\tau} + x = \varepsilon\langle\sin\theta_i\rangle,$$
$$\frac{d\theta_i}{d\tau} = \frac{\Omega_i}{\Omega_0} + \varepsilon a\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \ldots, N,$$

where $x = X/L$ and $a = A/L$ result from using the new spatial scale. A few more simplifications are needed:

- The damping is small, so that $\zeta = \varepsilon b$.

[1] In [7] they show that this hypothesis is compatible with the parameters we previously introduced, performing several experiments. We simply assume that this is the actual situation, since doing that goes beyond the scope of this work.

- The stepping frequencies are similar to the bridge's resonance frequency, so that we can write $\Omega_i/\Omega_0 = 1+\varepsilon\omega_i$. This is the interesting case from a physical point of view, since the wobbling is clearly related to the resonance between the bridge's vibrations and walkers' frequencies, as already stated.
- Redefinition of the origin of phases: $\theta_i \to \theta_i - \tau$, $\Psi \to \Psi - \tau$.

The resulting equations are:

$$\frac{d^2x}{d\tau^2} + x = \varepsilon\left[\langle\sin(\tau+\theta_i)\rangle - 2b\frac{dx}{d\tau}\right],$$
$$\frac{d\theta_i}{d\tau} = \varepsilon\left[\omega_i + a\sin(\Psi-\theta_i+\alpha)\right], \qquad i=1,\dots,N.$$

We can take advantage of having the small parameter ε multiplying the right hand side of the previous equation. If $\varepsilon = 0$ the solutions would be $x(\tau) = a\sin(\tau+\Psi)$, $\theta_i(\tau) =$ constant, $i = 1,\dots,N$. Thus, we suppose that the amplitude a, the bridge's phase Ψ and the walkers' phases, θ_i, vary slowly on a time scale $\tau = O(1/\varepsilon)$, and use the averaging method, as explained in [4]. Defining $T = \varepsilon\tau$, we are left with the following equations:

$$\frac{da}{dT} = -ba - \frac{1}{2}\langle\sin(\Psi-\theta_i)\rangle,$$
$$a\frac{d\Psi}{dT} = -\frac{1}{2}\langle\cos(\Psi-\theta_i)\rangle,$$
$$\frac{d\theta_i}{dT} = \omega_i + a\sin(\Psi-\theta_i+\alpha),\ \ i=1,\dots,N.$$

Next we shall look for stationary solutions. We remark that the previous averaged equations remain unchanged when redefining the origin of angles, so that we can seek for stationary solutions with any phase we want. It is convenient to set $\Psi = -\pi/2$. Therefore, if we also impose stationary solutions with constant amplitude, the previous equations, in the case $\alpha = \pi/2$, are transformed into

$$\begin{aligned} ba &= \frac{1}{2}\langle\cos(\theta_i)\rangle, \\ 0 &= \langle\sin(\theta_i)\rangle, \\ \frac{d\theta_i}{dT} &= \omega_i - a\sin(\theta_i),\ \ i=1,\dots,N. \end{aligned} \tag{7}$$

Now it is easy to see that (7) are the equations defining the stationary solutions in the Kuramoto model, as reported in [6], if we make the following assumptions:

- $a = K\rho$, where, in the Kuramoto model looking for stationary solutions:
  - K is the coupling constant of the biological oscillators.
  - $\rho = \sum_{k=1}^{N} e^{i\theta_k}$ is the order parameter (it gives us the location in polar coordinates of the center of mass).
- $2ab = \rho$.
- Limit $N \to \infty$, so that the stepping frequencies of the walkers are distributed according to $g(\omega)$, a unimodal a symmetric distribution centered at zero (such that the real frequencies are distributed around the resonant frequency of the bridge, as previously indicated).

The last equation in (7) is solved assuming $a =$constant, and then a is obtained from the first and second equations (consistency relations). Following [6], the second equation is satisfied if $g(\omega)$ fulfills the previous conditions, and tells us that walkers with stepping frequencies $|\omega_j| > a$ do not amplify the bridge's vibrations. Then, using $K = (2b)^{-1}$ and $a = K\rho$ the other consistency equation can be written as

$$2ab = \int_{-\pi/2}^{\pi/2} a\cos^2(\theta) g(a\sin\theta) d\theta.$$

It admits two different solutions.

- $a = 0$: motionless bridge, there are not shynchronized walkers.
- $a > 0$, solution of $2b = \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(a\sin\theta) d\theta$. Such a solution only exists if b is less than a critical value b_c (the opposite situation to that of the Kuramoto model, since now $K = (2b)^{-1}$). Moreover, it can be shown that this is the stable solution in that case (the bifurcation is supercritical). The critical value b_c can be obtained taking the limit $a \to 0^+$ in the previous transcendental equation, so that

$$b_c = \frac{\pi}{4} g(0).$$

Recalling the definitions of b and ε we finally arrive at the critical crowd size of (6).

III Results and discussion

III.I Basic results

Two of the most important magnitudes to be computed are the oscillations amplitude $A(t)$ (already introduced in Section II) and the order parameter $R(t)$ shown in equations (8) and (9) for increasing values of N,

$$A(t) = \sqrt{X(t)^2 + \left(\frac{1}{\Omega_0}\frac{dX(t)}{dt}\right)^2}, \tag{8}$$

$$R(t) = \frac{1}{N}\left|\sum_{j=1}^{N} \exp[i\theta_j(t)]\right|. \tag{9}$$

Figure 1 depicts the time evolution of such magnitudes. The amplitude and the order parameter figures show an approximated critical crowd size N_c of 150, the same found in [7]. Before N_c the order parameter fluctuates around a small value, which corresponds to the desynchronized crowd case: the phases θ_i are randomly distributed in the interval $[0, 2\pi)$ and they are not coherent. On the other hand, for number of walkers above N_c both the amplitude and the order parameter show a growing tendency despite the fluctuations in the latter: the walkers synchronize their steps and thus apply an increasing lateral force to the bridge. Finally, the system reaches a steady state in which the order parameter and the amplitude are constant.

Let us compare the numerical result of N_c obtained from Figure 1 to the theoretical value predicted by equation 6 using the parameters included in Section II. Since $P(\Omega_0)$ is a Gaussian distribution evaluated at the mean value Ω_0, we have $P(\Omega_0) = 1/(\sqrt{2\pi}\sigma)$, where $\sigma = 0.63$ rad/s. The predicted critical crowd size turns out to be the same as the one estimated graphically:

$$N_c = 149$$

In reference [1] it is proposed another method of predicting the critical value of N by varying a parameter β (which is essentially our C from equation (2)). The equations to be solved are:

$$M\ddot{y} + 2M\varepsilon\dot{y} + Ky = \bar{F}\sum_{i=1}^{N} \cos(\theta_i), \tag{10}$$

$$\dot{\theta}_i = \Omega_i - \bar{\beta}\ddot{y}cos(\theta_i). \tag{11}$$

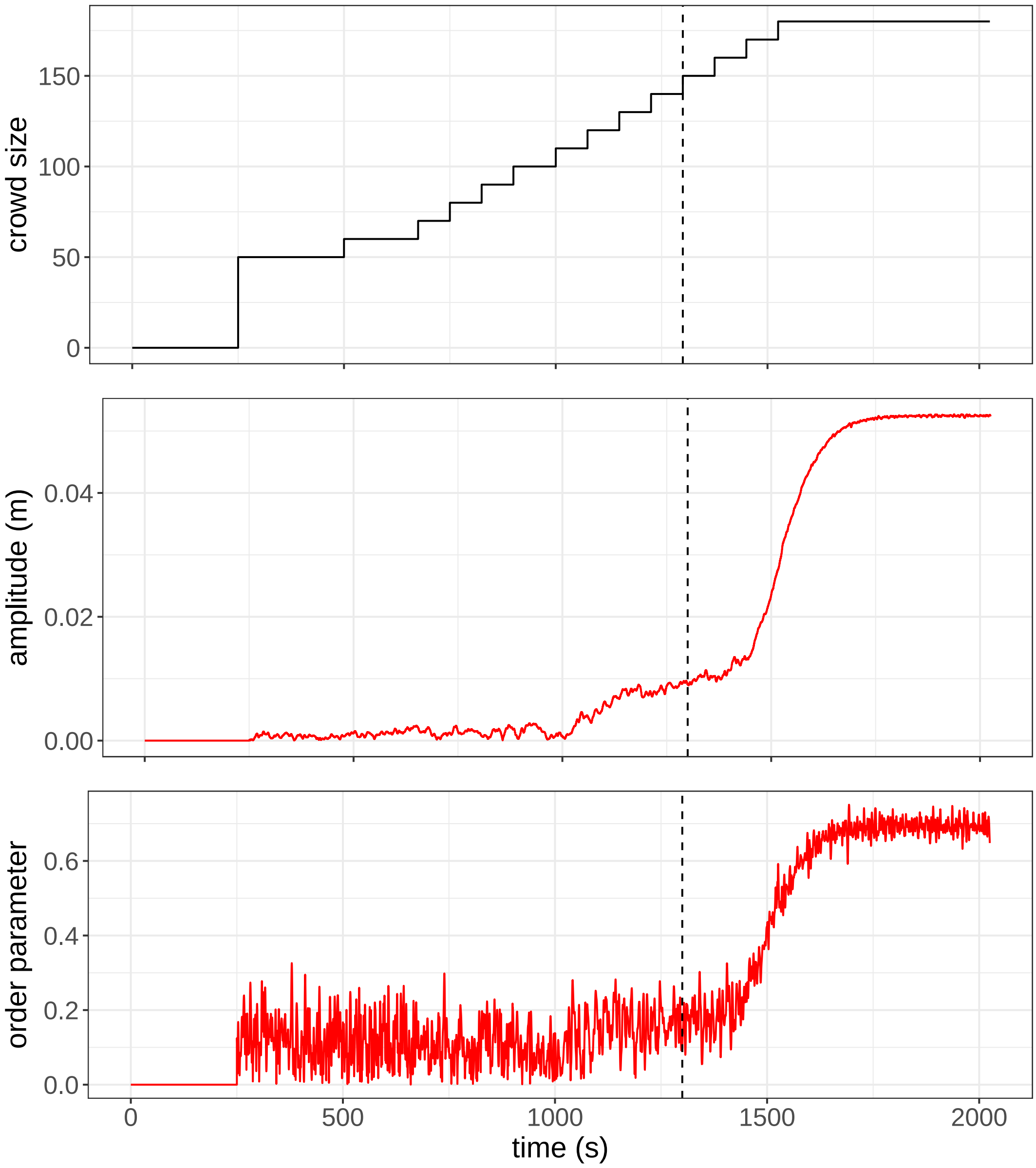

Figure 1: Results of the numerical experiment performed to study the instability on the Millenium Bridge. The number of pedestrians on the bridge is slowly increased (adding 10 walkers each time) according to the first graph. The wobbling amplitude, (8), and the order parameter, (9) are measured every integration step (second and third graphs). The dashed line corresponds to the theoretical prediction of the critical crowd size, (6).

A comparison between equations (2) and (11) reveals that the new parameter $\bar{\beta}$ that accounts for the walkers sensibility to the bridge's oscillations is simply $\bar{\beta} = CA$ (with a proper redefinition of the phases Ψ and α). Also, by comparing equations (1) and (10) we get $\varepsilon = B/(2M)$. The final difference with our initial model is the probability distribution assumed for the walking frequencies Ω_i: instead of a Gaussian with mean $\Omega_0 = \sqrt{K/M}$ it is assumed a Cauchy-Lorentz distribution with width Δ and the same mean Ω_0, which is symmetrical as well:

$$P(\Omega) = \frac{1}{\pi}\frac{\Delta}{(\Omega - \Omega_0)^2 + \Delta^2}$$

Once we can associate the new parameters ε and β with our initial model, it can be deduced from a linear analysis of equations (10) and (11) that the critical crowd size N_c is given by (more details of the derivation can be found in [1]):

$$N_c = \frac{8M\Delta\varepsilon}{\bar{F}\bar{\beta}\Omega_0} \tag{12}$$

Where $\bar{\beta} = CA$, being $C = 16\,\mathrm{m}^{-1}\mathrm{s}^{-1}$ and $A = 0.05$ the mean amplitude of the oscillations as estimated from Figure 1. Finally, if we substitute $\bar{F} = G = 30$ N, $\Omega_0 = 6.47$ rad/s and $\Delta = 0.466$ rad/s (the same Cauchy distribution used in [1]) we get the following value of N_c:

$$N_c = 132.$$

We did not expect the same result as the one obtained before $N_c \approx 150$ since the two models are slightly different regarding the coupling between the bridge and the walkers and the choice of the stepping frequencies distribution $P(\Omega)$. However this last result can be made more similar to our previous N_c by choosing a width Δ such that the peak of the Cauchy-Lorentz distribution coincides with the peak of the previous Gaussian distribution. This width is $\Delta = 0.5$ and hence the new value of N_c deduced from (12) is:

$$N_c = 141.$$

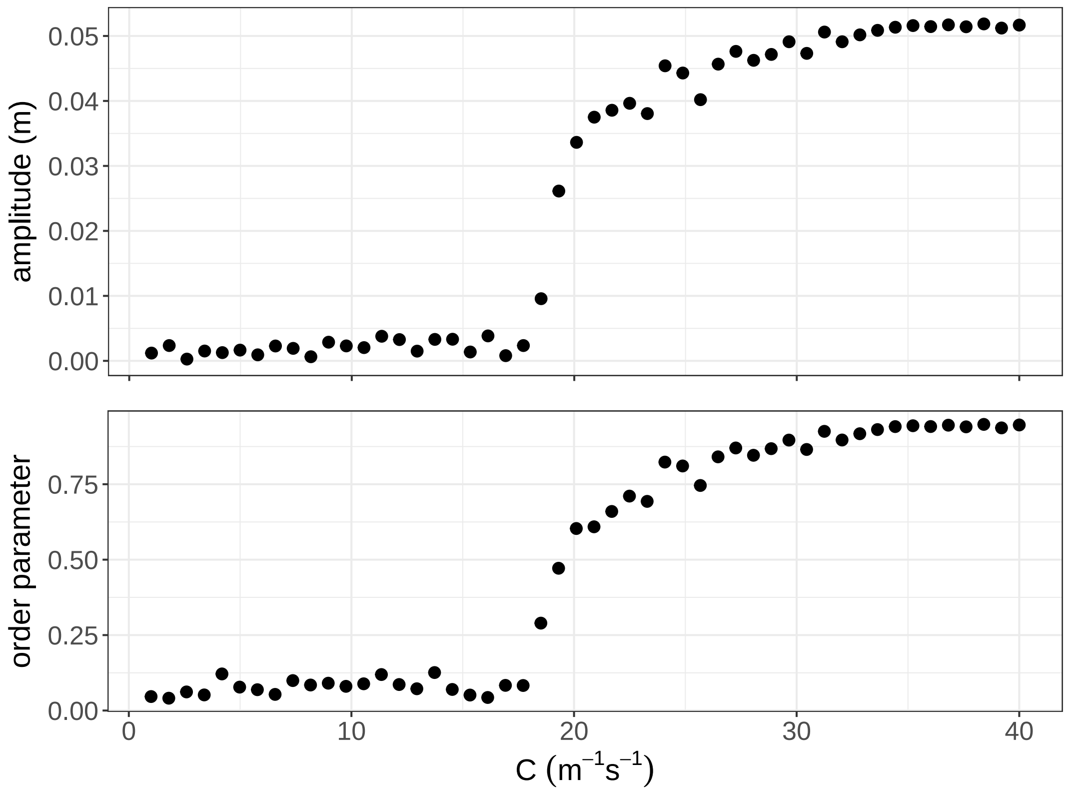

Figure 2: Results of the numerical experiment performed to study the instability on the Millenium Bridge. The amplitude of the vibrations and the order parameter are computed for different values of the coupling constant C and a constant $N = 130$.

On the other hand, Figure 2 depicts the dependence of the amplitude and the order parameter on C with a fixed number of walkers $N = 130$. There is a clear critical value at $C = 18\,\mathrm{m}^{-1}\mathrm{s}^{-1}$ that defines a phase transition between a disordered crowd movement ($C < C_c$) and a synchronized collective movement ($C > C_c$). Let us recall equation 2:

$$\frac{d\theta_i}{dt} = \Omega_i + CA\sin(\Psi - \theta_i + \alpha) \qquad i = 1, \ldots, N$$

For small values of C the first term of the equation above dominates over the second coupling term and therefore each pedestrian has its own random frequency Ω_i. As C is increased the pedestrians become more influenced by the bridge vibrations and the coupling term overcomes the random initial frequencies Ω_i.

III.II Additional results

In Subsection II.I we showed (using the well-known solution of the Kuramoto model) that a non-zero wobbling amplitude exists if $b < b_c$, where $b \propto B$ if the other parameters are fixed. Indeed, in [7] they also indicate that a supercritical bifurcation occurs at b_c, since the zero solution loses its stability if $b < b_c$.

We may attempt to numerically check if this is the situation. To do that we keep constant the rest of the parameters (the same values as before), we define $N = 130$ and we change the bridge's damping B. For each value of B we measure the wobbling amplitude, (8), and the order parameter, (9), after the steady state has been reached. The results are those shown in Figure 3.

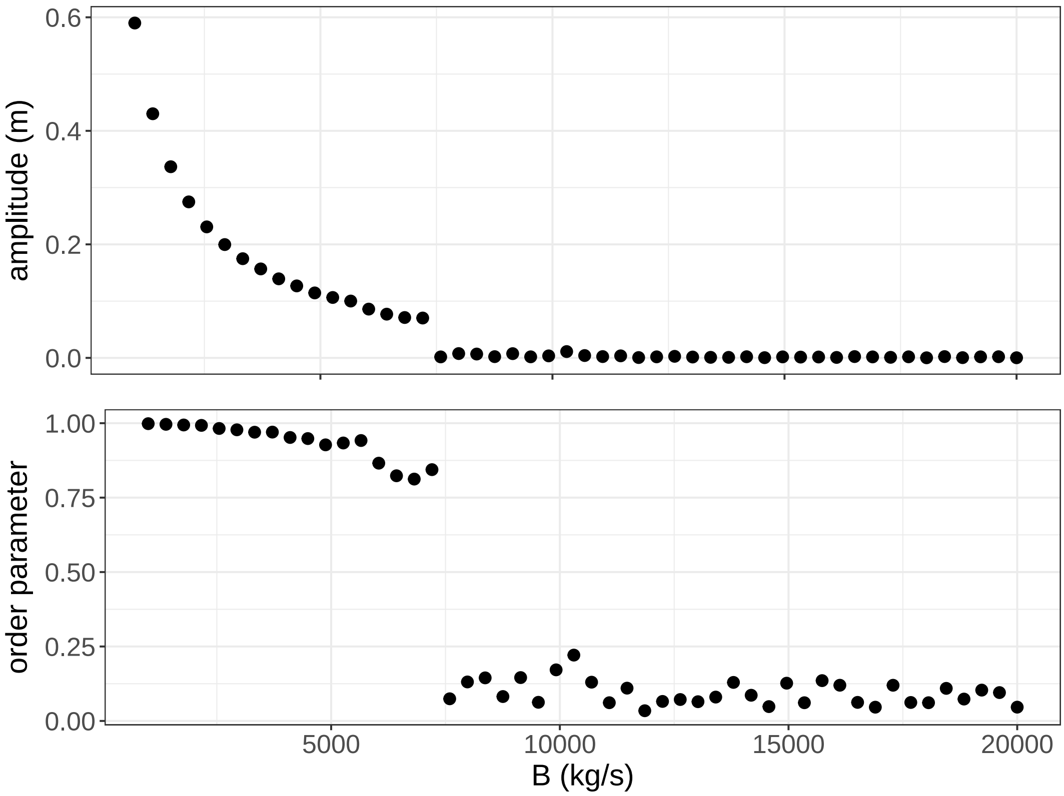

Figure 3: Results of the numerical experiment performed to study the instability on the Millenium Bridge. The amplitude of the vibrations and the order parameter are computed for different values of the bridge's damping B.

Looking at Figure 3 we see that a supercritical bifurcation seems to occur at a critical value B_c. Since $N = 130$, this value is of course smaller than $1.1 \cdot 10^4$ kg/s (the damping used to obtain Figure 1). However, a first inspection reveals a quite different behavior from that previously shown. First of all, our numerical results indicate that the wobbling amplitude diverges when the damping B tends to zero. At the same time, the order parameter is close to one, so that the walkers are almost perfectly synchronized.

This new behavior can be understood recalling the basic equation (1). In the limit $B \to 0$ the solution of (1) if $G = 0$ (pedestrians do not have any effect on the bridge's vibrations) is a pure Fourier mode of frequency Ω_0. Thus, if $G \neq 0$ the walkers can synchronize their stepping frequencies with Ω_0, resulting in a perfect synchronization ($R \to 1$) and a pure resonance ($A \to \infty$). On the other hand, if $B \neq 0$ the solution of (1) in the absence of walkers is no longer a single Fourier mode (it is modulated by an exponential decay, so that more frequencies contribute). Thus, the perfect synchrony is not possible, and a finite wobbling amplitude (more realistic, as we already saw) is obtained. These different solutions depending on the bridge's damping are in rough agreement with Figure 3.

Finally, we can also comment on the empirical law reported in [2], $F = kV$, where F is the force exerted by the pedestrians and V the velocity of the bridge's vibrations. In [7] they show that the aforementioned law is in agreement with the model we are studying here from a very intuitive point of view. For the sake of simplicity we will skip the calculations (they are described in the aforementioned reference), so that we only comment on a few interesting aspects. The force F is found to be proportional to the order parameter, R, and this shows the importance of synchronization in the wobbling phenomenon. On the other hand, it can also be shown that the velocity V is proportional to the amplitude A. Thus, the empirical law $F = kV$ is simply the proportionality between A and R, that can be qualitatively checked in Figure 1. In [7] they focus on studying if equations (1) and (2) predict this law, and they found that indeed, it is true, with the exception of slowly varying factors. However, the functional dependence of the proportionality constant k is not thoroughly studied. Without performing any formal analysis, we can infer from our numerical results that k does not depend on the coupling constant C, but it must depend on the bridge's damping B. Of course a more detailed study is needed to confirm these predictions, but they seem to be true from a very qualitative point of view.

IV Conclusions

To conclude, we shall remark the simplicity of the model studied in this document. In fact, the governing equations are simply a damped and forced harmonic oscillator equation (for the dynamics of the bridge) and the most simple possibility (from a physical point of view) for the temporal evolution of the walkers' phases. The resulting set of equations is too complicated to be formally studied, but we saw that making an analogy with the Kuramoto model for biological oscillators it is possible to obtain a few accurate predictions (for example, the different stationary solutions) and also a good understanding of the phenomenon.

On the other hand, we also remark the usefulness of numerical techniques when simulating this kind of problems (a large set of coupled differential equations). Even if we could not obtain theoretical predictions we would be able to estimate some quantities (for example, the characteristic length of the bridge's vibrations, or a less accurate critical crowd size) only looking at the results of the numerical simulations. In this sense, we shall assume that they are correct, since these predictions are compatible with the theoretical ones. Moreover, we also checked the differences with a more advanced model, which includes a different coupling between the bridge and the walkers. To conclude we shall add that the numerical results are also useful when understanding some important characteristics of the phenomenon, such as the strong connection between shynchrony and resonance.

References

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A Method of averaging

We briefly explain how the method of averaging is used in the particular case we are dealing with, following [4]. Our equation is of the form

$$\frac{d^2x}{d\tau^2} + x = \varepsilon g(x, d_\tau x, \tau),$$

where $g(x, d_\tau x, \tau) = \langle \sin(\tau + \theta_i) \rangle - 2bd_\tau x$. To obtain the averaged equations for a and Ψ (where $x(\tau) = a\sin(\tau + \Psi)$) we assume the walkers' phases to be constant, since their derivative with respect to τ is of order $O(\varepsilon)$, that is, they depend only on the slow time $T = \varepsilon\tau$.

Now, proceeding as explained in [4] (see, for example, the example of the Van der Pol oscillator) the equations for a and Ψ can be written as

$$\begin{aligned}
\frac{da}{d\tau} &= \varepsilon \cos(\tau + \Psi) g(a\sin(\tau + \Psi), a\cos(\tau + \Psi), \tau),\\
\frac{d\Psi}{d\tau} &= \frac{1}{r}\cos(\tau + \Psi) g(a\sin(\tau + \Psi), a\cos(\tau + \Psi), \tau).
\end{aligned}$$

The method of averaging tells us that

$$\begin{aligned}
\frac{da}{dT} &= \frac{1}{2\pi}\int_0^{2\pi} \cos(s + \Psi) g(a\sin(s + \Psi), a\cos(s + \Psi), s) ds,\\
\frac{d\Psi}{d\tau} &= \frac{1}{2\pi a}\int_0^{2\pi} \sin(s + \Psi) g(a\sin(s + \Psi), a\cos(s + \Psi), s) ds.
\end{aligned}$$

Now we compute the required integrals:

$$\begin{aligned}
I_1 = &\int_0^{2\pi} \cos(s + \Psi)\left[\langle \sin(s + \theta_i) \rangle - 2ab\cos(s + \Psi)\right] ds\\
= &-2ab\int_0^{2\pi} \cos^2(s + \Psi) ds + \left\langle \int_0^{2\pi} \cos(s + \Psi)\sin(s + \theta_i) ds \right\rangle\\
= &-2ab\int_0^{2\pi} \frac{1 + \cos(2(s + \Psi))}{2} ds + \left\langle \int_0^{2\pi} \frac{\sin(2s + \Psi + \theta_i) - \sin(\Psi - \theta_i)}{2} ds \right\rangle,
\end{aligned}$$

where we have used the well-known formulas for $\cos^2(\alpha)$ and $\sin(\alpha - \beta)$. If we also use that the integral of $\cos(2(s - \Psi))$ and $\cos(\Psi - \theta_i - 2s)$ in $[0, 2\pi]$ are zero, we are left with

$$I_1 = -2\pi ab - \pi\langle \sin(\Psi - \theta_i) \rangle.$$

Proceeding in a similar fashion:

$$\begin{aligned}
I_2 = &\int_0^{2\pi} \sin(s + \Psi)\left[\langle \sin(s + \theta_i) \rangle - 2ab\cos(s - \Psi)\right] ds\\
= &-2ab\int_0^{2\pi} \sin(s + \Psi)\cos(s + \Psi) ds + \left\langle \int_0^{2\pi} \sin(s + \Psi)\sin(s + \theta_i) ds \right\rangle\\
= &-2ab\int_0^{2\pi} \frac{\sin(2(s + \Psi))}{2} ds + \left\langle \int_0^{2\pi} \frac{\cos(2s + \Psi + \theta_i) - \cos(\Psi - \theta_i)}{2} ds \right\rangle\\
= &-\pi\langle \cos(\Psi - \theta_i) \rangle.
\end{aligned}$$

Thus we arrive at the desired formulas for a and Ψ:

$$\begin{aligned}
\frac{da}{dT} &= -ba - \frac{1}{2}\langle \sin(\Psi - \theta_i) \rangle,\\
a\frac{d\Psi}{dT} &= -\frac{1}{2}\langle \cos(\Psi - \theta_i) \rangle.
\end{aligned}$$

B Code

Next we include the code bridge.R, used to perform the numerical experiments explained in this document.

```
library(deSolve)
#
#Millenium bridge
#Strogatz, Eckhardt and Ott model
#
#Parameters of the model
M=1.13e5;B=1.1e4;K=4.73e6;G=30;C=16;alpha=pi/2
w0=6.47;sdw=0.63
#
#Function defining the dynamical system
f=function(t,u,p){
  with(
    as.list(c(u,p)),{
      NN=length(u)
      NNN=length(p)
      du=numeric(NN)
      du[1]=u[2]
      du[2]=-p[2]*u[2]/p[1]-p[3]*u[1]/p[1]+p[4]/p[1]*sum(sin(u[3:
         NN]))
      du[3:NN]=p[7:NNN]+p[5]*u[1]*cos(alpha-u[3:NN])+p[5]*u[2]*
         sin(alpha-u[3:NN])/p[6]
      #
      list(du)
    }
  )
}
#
ini=c(0,0,runif(50,-pi,pi))
A=c(0);R=c(0)
t_end=c(250,175,75,75,75,100,75,75,75,75,75,75,75,500)
j=1
for(N in seq(50,180,10)){
  NN=N+2
  #
  #Parameters
  w=rnorm(N,mean=w0,sd=0.63)
  p=c(M,B,K,G,C,w0,w)
  #
  #Times
  times=seq(0,t_end[j],0.01)
  #
  #Solution
  sol=ode(y=ini,times=times,func=f,parms=p)
  x=sol[,2]
  v=sol[,3]
  A=c(A,sqrt(x*x+v*v/(w0*w0)))
  theta=sol[,4:length(sol[1,])]
  R=c(R,sqrt(apply(cos(theta),1,"sum")^2+apply(sin(theta),1,"sum"
     )^2)/N)
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```
  #
  for(i in 1:NN){
    ini[i]=sol[length(sol[,1]),(i+1)]
  }
  ini=c(ini,runif(10,-pi,pi))
  j=j+1
}
t=seq(1,length(A),1)*sum(t_end)/length(A)
t=c(seq(0,249,1),t+250)
A=c(rep(0,250),A)
R=c(rep(0,250),R)
#
#Write the results
write.table(data.frame(t,A,R),file="figures_paper.txt",row.names=
    FALSE,col.names=FALSE)
#
#Changing b and assessing the final results
M=1.13e5;K=4.73e6;G=30;C=16;alpha=pi/2
w0=6.47;sdw=0.63
b=seq(1.0e3,2e4,length.out=50)
#
#Set N large (N=180)
N=130
#
#Times
times=seq(0,1000,1)
m=length(times)
A_v=numeric(length(b))
R_v=numeric(length(b))
#
j=1
for(B in b){
  #
  #Initial state
  ini=c(0,0,runif(N,-pi,pi))
  #
  #Parameters
  w=rnorm(N,mean=w0,sd=sdw)
  p=c(M,B,K,G,C,w0,w)
  #
  sol=ode(y=ini,times=times,func=f,parms=p)
  x=sol[,2]
  v=sol[,3]
  A=sqrt(x*x+v*v/(w0*w0))
  theta=sol[,4:length(sol[1,])]
  R=sqrt(apply(cos(theta),1,"sum")^2+apply(sin(theta),1,"sum")^2)
     /N
  A_v[j]=mean(tail(A),n=100)
  R_v[j]=mean(tail(R),n=100)
  j=j+1
}
#
#Write the results
```

```
write.table(data.frame(b,A_v,R_v),file="eval_b.txt",row.names=
    FALSE,col.names=FALSE)
#
#Changing C and assessing the final results
M=1.13e5;B=1.1e4;K=4.73e6;G=30;alpha=pi/2
w0=6.47;sdw=0.63
c=seq(1,40,length.out=50)
#
#Set N large (N=180)
N=130
#
#Times
times=seq(0,1000,1)
m=length(times)
A_v=numeric(length(c))
R_v=numeric(length(c))
#
j=1
for(C in c){
  #
  #Initial state
  ini=c(0,0,runif(N,-pi,pi))
  #
  #Parameters
  w=rnorm(N,mean=w0,sd=sdw)
  p=c(M,B,K,G,C,w0,w)
  #
  sol=ode(y=ini,times=times,func=f,parms=p)
  x=sol[,2]
  v=sol[,3]
  A=sqrt(x*x+v*v/(w0*w0))
  theta=sol[,4:length(sol[1,])]
  R=sqrt(apply(cos(theta),1,"sum")^2+apply(sin(theta),1,"sum")^2)
     /N
  A_v[j]=mean(tail(A),n=100)
  R_v[j]=mean(tail(R),n=100)
  j=j+1
}
#
write.table(data.frame(c,A_v,R_v),file="eval_c.txt",row.names=
    FALSE,col.names=FALSE)
```