Effects of increasing crowds on the Millenium Bridge

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What caused the bridge oscillations?

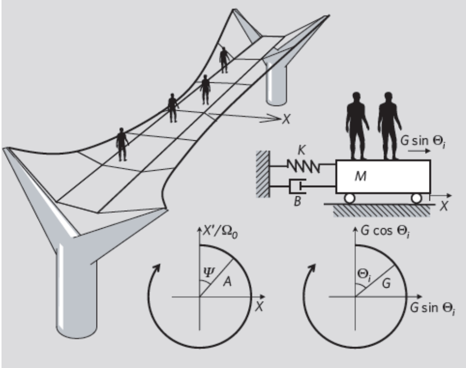
- Growing side to side oscillations were observed as the number of walkers increased.
- Architectural explanation: innovative design of the bridge.
- Oscillations are controlled by fluid-viscous dampers and tuned mass dampers.

Belykh, I. et al. Nature Commun. https://doi.org/10.1038/s41467-021-27568-y (2021)

What caused the bridge oscillations?

Physical explanation:

- Pedestrians: coupled biological oscillators
  $\longrightarrow$ Kuramoto model (neurons, heart cells...)
- Bridge: weakly damped and forced oscillator.



Coupled oscillators

$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N}\sin\theta_i \tag{1}$$

$$\frac{d\theta_i}{dt} = \Omega_i + CA\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \dots, N \tag{2}$$

- (1) models the bridge as a weakly damped and driven oscillator: M is the mass, B the damping, K the stiffness and $G\sin\theta_i$ the external force of each of the N walkers.
- (2) models the effects of the bridge's oscillation on each pedestrian's steps.
- A, Ψ are defined such that $X = A\sin(\Psi)$ and $dX/dt = A\Omega_0\cos(\Psi)$, where $\Omega_0 = \sqrt{K/M}$.

Kuramoto Model

Equation (2) is indeed a transformation of the most popular form of the Kuramoto model:

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{C}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{3}$$

For that we define an average amplitude and phase A and Ψ in terms of the order parameter:

$$Ae^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{4}$$

If we multiply (4) by $e^{-i\theta_i}$, we take the imaginary part and we use (3), we recover equation (2):

$$\frac{d\theta_i}{dt} = \Omega_i + CA\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \ldots, N$$

Numerical integration

$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N} \sin\theta_i$$
$$\frac{d\theta_i}{dt} = \Omega_i + CA\sin(\Psi - \theta_i + \alpha), \qquad i = 1, \ldots, N$$

Use $\sin(\alpha_1 + \alpha_2)$ formula, $X = A\sin(\Psi)$, $dX/dt = \Omega_0 A\cos(\Psi)$:

$$\Downarrow$$

$$\frac{dX}{dt} = Y$$
$$M\frac{dY}{dt} + BY + KX = G\sum_{i=1}^{N}\sin(\theta_i)$$
$$\frac{d\theta_i}{dt} = \Omega_i + CX\cos(\alpha - \theta_i) + \frac{C}{\Omega_0}Y\sin(\alpha - \theta_i)$$

Integration method: 4th order Runge-Kutta method, library deSolve, software R

Numerical details

$$\frac{dX}{dt} = Y$$
$$M\frac{dY}{dt} + BY + KX = G\sum_{i=1}^{N}\sin(\theta_i)$$
$$\frac{d\theta_i}{dt} = \Omega_i + CX\cos(\alpha - \theta_i) + \frac{C}{\Omega_0}Y\sin(\alpha - \theta_i)$$

- Parameters: $M = 1.13 \times 10^5$ kg, $B = 1.10 \times 10^4$ kg/s, $K = 4.73 \times 10^6$ kg$/s^2$, $G = 30$ kgm$/s^2$, $C = 16$ m^{-1}s^{-1}, $\alpha = \pi/2$.
- Ω_i are distributed according to a Gaussian with mean $\Omega_o = 6.47$ rad/s and $\sigma = 0.63$ rad/s.
- Initially, θ_i are randomly chosen in $[0, 2\pi)$.

Theoretical critical crowd size

Scale analysis and redefinition of the parameters:

$\tau = \Omega_0 t$, $\Omega_0 = \sqrt{K/M}$ bridge's resonant frequency

$L_1 = NG/K$, $L_2 = \Omega_0/C$ spatial scales, $X = \sqrt{L_1 L_2} x$

$\varepsilon = \sqrt{L_1/L_2}$ can be considered a small parameter

Additional assumptions:

The damping is small: $\zeta = B/\sqrt{4MK}$, $\zeta = \varepsilon b$

Stepping frequencies near Ω_0, $\Omega_i/\Omega_0 = 1 + \varepsilon \omega_i$

Redefinition of phases: $\theta_i \to \theta_i - \tau$, $\Psi \to \Psi - \tau$

$$\frac{d^2 x}{d\tau^2} + x = \varepsilon \left[\langle \sin(\tau + \theta_i) \rangle - 2b \frac{dx}{d\tau} \right],$$

$$\frac{d\theta_i}{d\tau} = \varepsilon \left[\omega_i + a \sin(\Psi - \theta_i + \alpha) \right], \qquad i = 1, \ldots, N.$$

Taking advantage of the Kuramoto model

Write $x = a\sin(\Psi + \tau)$ and use the averaging method, $T = \varepsilon\tau$

$$\frac{da}{dT} = -ba - \frac{1}{2}\langle \sin(\Psi - \theta_i)\rangle, \quad a\frac{d\Psi}{dT} = -\frac{1}{2}\langle \cos(\Psi - \theta_i)\rangle,$$
$$\frac{d\theta_i}{dT} = \omega_i + a\sin(\Psi - \theta_i + \alpha), \; i = 1, \ldots, N.$$

Look for stationary solutions with $\Psi = -\pi/2$, $\alpha = \pi/2$.

Equivalence with Kuramoto model if $a = K\rho$, $2ab = \rho$ and $g(\omega)$ unimodal symmetric distribution centered at zero.

Two different solutions:

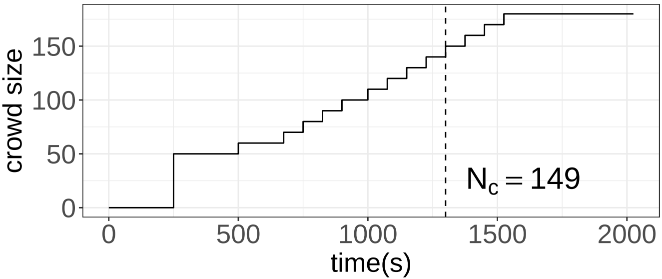
$a = 0$

$a > 0$ solution of $2b = \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(a\sin\theta) d\theta$, only if

$$N > N_c = N_c = \frac{4\zeta}{\pi}\left(\frac{K}{GCP(\Omega_0)}\right)$$

Numerical experiment

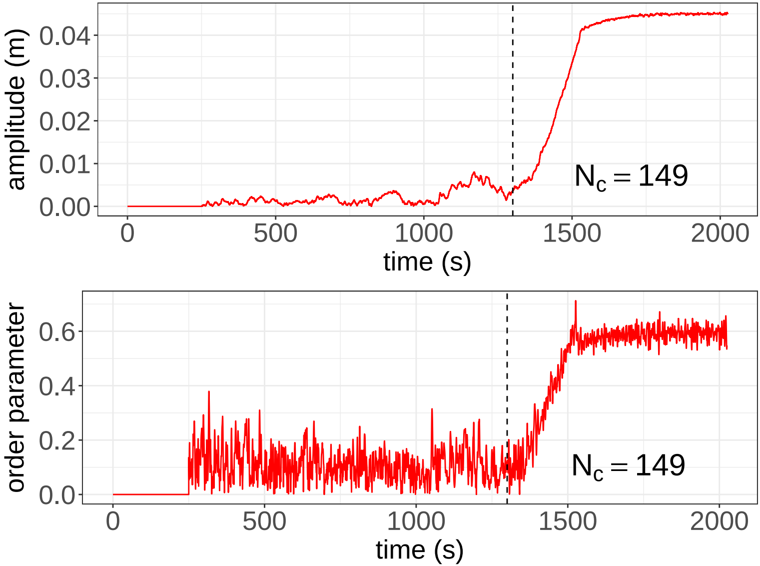
- Realistic simulation: more and more people on the bridge
- Initial conditions for each crowd size?
  The final state for the previous one



Measure the **wobbling amplitude** and the **order parameter**

$$A(t) = \sqrt{X(t)^2 + \left(\frac{1}{\Omega_0}\frac{dX(t)}{dt}\right)^2}, \quad R(t) = \frac{1}{N}\left|\sum_{j=1}^{N} \exp[i\theta_j(t)]\right|.$$

Amplitude and order parameter (I)



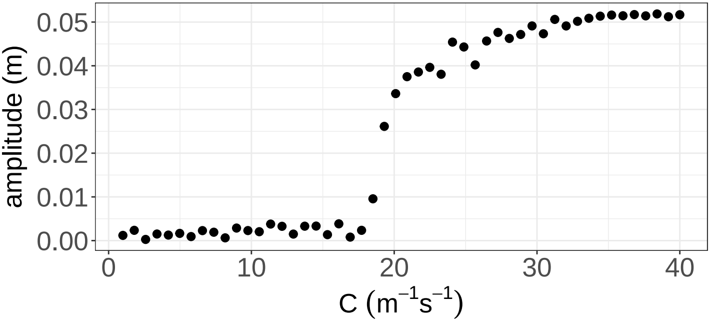
Amplitude and order parameter (II)

Wobbling and synchrony are strongly related.

Observations:

- Other models that use a Cauchy distribution for Ω_i and the same mean Ω_o give $N_c = 132$ and $N_c = 142$ depending on the width of the distribution.
- Empirical law $F = kV$, where
  - F force exerted by pedestrians (prop. to R)
  - V velocity of bridge's vibrations (prop. to A)

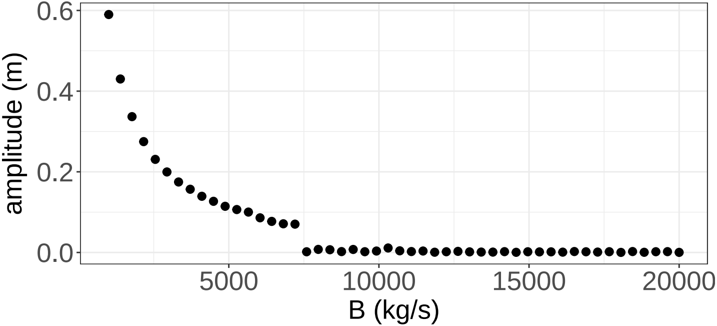
Changing the coupling constant



Observations

- Supercritical bifurcation: zero solution loses stability
- Two different phases: ordered and disordered

Changing the damping



Observations:

- Supercritical bifurcation: zero solution loses stability
- The divergence reflects the pure resonant behavior in absence of damping ($B = 0$)

Conclusions

Strength of the model:

- Simplicity
- Interdisciplinary approach

Main findings:

- Wobbling and synchrony are related phenomena
- Realistic, despite its simplicity
  - It can be handled theoretically (Kuramoto model)
  - Rough estimation of the critical crowd size

Strogatz, S. H., Abrams, D. M., McRobie, A., Eckhardt, B., & Ott, E. (2005). Crowd synchrony on the Millennium Bridge. Nature, 438(7064), 43-44.